STUDIA MATHEMATICA 151 (1) (2002)

Regularity of stopping times of diffusion processes in Besov spaces

by

Xicheng Zhang (Wuhan and Lisboa)

Abstract. We prove that the exit times of diffusion processes from a bounded open set Ω almost surely belong to the Besov space $B^\alpha_{p,q}(\Omega)$ provided that $p\alpha<1$ and $1\le q<\infty$.

1. Introduction and statement of results. Recently, many authors devoted their efforts to the study of stopping times. In [1], Airault, Malliavin and Ren studied the smoothness of stopping times of diffusion processes in Wiener space. In [8], Pedersen and Peskir computed the expectation of the Azéma–Yor stopping times. In [4], Knight and Maisonneuve gave two characterizations of stopping times via martingales and Markov processes. On the other hand, in [2], [3], Boufoussi and Roynette studied the regularity of Brownian local time L^x_t as a function of $x\in\mathbb{R}$, and they proved that it is almost everywhere in Besov–Orlicz spaces on $\mathbb{R}$. Motivated by their work, we study the smoothness of stopping times regarded as a function of starting points in Besov spaces. We emphasize that in [1] the authors proved that for an elliptic diffusion process, the exit time from an open set is in the fractional Sobolev spaces D^p_α provided that $p\alpha<1$. In the case of Brownian motion, they also showed that the result is almost optimal. Here we borrow some methods from [1] to prove our main result.

For any $0<\alpha<1$, $p>1$ and $q\ge 1$, we use $B^\alpha_{p,q}$ to denote the usual Besov spaces in $\mathbb{R}^d$, and the norm in $B^\alpha_{p,q}$ is denoted by $\|\cdot\|_{\alpha,p,q}$. We refer to [10, p. 189] for the detailed definition. The Besov spaces over an arbitrary domain Ω are defined as restriction of the corresponding spaces over $\mathbb{R}^d$ to Ω. That is to say,

$$\|f\|_{B^\alpha_{p,q}(\Omega)} := \inf_{g|_\Omega=f,\, g\in B^\alpha_{p,q}(\mathbb{R}^d)} \|g\|_{\alpha,p,q}. \tag{1}$$

When Ω is a bounded C^2 domain, another norm equivalent to (1) is given

2000 *Mathematics Subject Classification*: 60H07, 60G05, 46E35.
Key words and phrases: Besov spaces, stopping times.

by (cf. [10, p. 324])

$$\|f\|^*_{B^\alpha_{p,q}(\Omega)} = \|f\|_{L^p(\Omega)} + \bigg(\int_{\mathbb{R}^d} \frac{\|f(x+h)-f(x)\|^q_{L^p(\Omega_h)}}{|h|^{d+\alpha q}}\,dh\bigg)^{1/q}, \tag{2}$$

where $\Omega_h = \Omega \cap \{x \in \mathbb{R}^d : x+h \in \Omega\}$ and $|\cdot|$ is the usual norm in $\mathbb{R}^d$.

Let (W, H, μ) be the classical Wiener space. W and H respectively stand for the completions of $C_0^\infty([0,\infty), \mathbb{R}^d)$ with respect to the norms

$$\|w\|_W = \sup_{t\ge 0} \frac{|w(t)|}{1+t} \quad \text{and} \quad \|w\|_H = \Big(\int_0^\infty |w'(t)|^2\,dt\Big)^{1/2},$$

μ is the Wiener measure.

In this context, we consider the following diffusion process:

$$\begin{cases} dX_i(t,x) = \sum_{k=1}^d \sigma_{k,i}(X(t,x))dw_k(t) + b_i(X(t,x))dt, & i = 1,\dots,d,\\ X(0,x) = x \end{cases} \tag{3}$$

where $x \in \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are $C^2_{\rm b}$ functions, and $w(\cdot)$ is the standard d-dimensional Wiener process. The second order elliptic differential operator A on $\mathbb{R}^d$ associated with this diffusion process is given by

$$A = \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(x)\partial^2_{i,j} + \sum_{i=1}^d b_i(x)\partial_i,$$

where $a = \sigma\sigma^T \in C^2_{\rm b}$.

Let Ω be a bounded connected open set in $\mathbb{R}^d$ with C^2 boundary (or $\partial\Omega$ is a regularly imbedded C^2 submanifold of $\mathbb{R}^d$). It is well known that there is a function (cf. [5, p. 59]) $\varrho : \mathbb{R}^d \to \mathbb{R}$ satisfying:

(i) ϱ is C^2;
(ii) $\{x \in \mathbb{R}^d : \varrho(x) < 0\} = \Omega$;
(iii) $\nabla\varrho(x) \neq 0$ on $\partial\Omega = \{x \in \mathbb{R}^d : \varrho(x) = 0\}$, where ∇ stands for gradient.

ϱ is called a *defining function* for Ω. From the definition of ϱ, it is not hard to find that there exist two strictly positive constants C_1 and ε such that

$$C_1 \le |\nabla\varrho(x)| \quad \forall x \in \overline{\Omega}_\varepsilon, \tag{4}$$

where $\Omega_\varepsilon := \{x : -\varepsilon < \varrho(x) < \varepsilon\}$ is a bounded set and $\overline{\Omega}_\varepsilon$ is the closure of Ω_ε.

For $x \in \Omega$, we define the exit time as follows:

$$\tau_x(w) := \inf\{t \ge 0 : X(t,w,x) \notin \Omega\} = \inf\{t \ge 0 : \varrho(X(t,w,x)) = 0\}. \tag{5}$$

For fixed $T > 0$, setting

$$\tau_x^T(w) = \min\{\tau_x(w), T\},$$

we prove the following result:

THEOREM 1. *In addition to the conditions* $\sigma, b \in C_{\mathrm{b}}^2$, *assume also that there exist two positive constants* C_2, C_3 *such that*

$$C_2|\xi|^2 \le \xi a \xi^T \le C_3|\xi|^2 \tag{6}$$

for all $\xi \in \mathbb{R}^d$. *Then for any* $p \ge 1$ *and* $0 < \alpha < 1$, *there exists a constant* $C = C(T, p, \alpha)$ *such that*

$$E|\tau_x^T(w) - \tau_y^T(w)|^p \le C|x - y|^\alpha \quad \forall x, y \in \Omega. \tag{7}$$

Furthermore, in the case of one-dimensional Brownian motion, for $x < 1$, *set* $\tau_x(w) = \inf\{t \ge 0 : w_t + x = 1\}$ *and* $\tau_x^T(w) = \tau_x(w) \wedge T$; *then for any* $a < 1$, *there exists a constant* $C = C(T, a, p)$ *such that*

$$E|\tau_x^T(w) - \tau_y^T(w)|^p \ge C|x - y| \quad \forall a < x, y < 1. \tag{8}$$

From this theorem we easily deduce the following results:

COROLLARY 2. *For almost all* $w \in W$, *if* $p\alpha < 1$ *and* $1 \le q < \infty$, *then* $\tau_x^T(w) \in B_{p,q}^\alpha(\Omega)$.

Set $f(x) = E(\tau_x^T)$. Since $|f(x) - f(y)| \le E|\tau_x^T(w) - \tau_y^T(w)| \le C|x - y|^\alpha$, we obtain

COROLLARY 3. *For* $0 < \alpha < 1$ *and* $1 < p < \infty$, *we have* $f \in B_{p,q}^\alpha(\Omega)$.

2. Proof of Theorem 1. Henceforth, we make a convention: C denotes a positive constant whose value may change in different occurrences.

First of all, from equation (3), we observe that for $m > 1$,

$$E|X(s,x) - X(t,y)|^m \le C(|x - y|^m + |s - t|^{m/2}) \quad \text{for all } s, t \in [0, T],\ x, y \in \mathbb{R}^d \text{ (cf. [9])}.$$

By the Kolmogorov criterion (cf. [9]), if we take $m > (d + 3)/(1 - \alpha)$, then

$$\max_{0 \le s \le T} |X(s, w, x)) - X(s, w, y)| \le B(w)|x - y|^{\alpha + 1/m}, \tag{9}$$

where $E|B(w)|^m < \infty$.

By condition (i), we know that

$$|\nabla \varrho(x)| < C_4 \quad \forall x \in \Omega_\varepsilon \cup \Omega. \tag{10}$$

For $z \in \partial\Omega$, if we define

$$\eta_z^\varepsilon(w) := \inf\{t : X(t, w, z) \notin \Omega_\varepsilon\} = \inf\{t : |\varrho(X(t, w, z))| = \varepsilon\},$$

then for $s < [\tau_x^T(w) + \eta^\varepsilon_{X(\tau_x^T(w), w, x)}(w)] \wedge \tau_y^T(w)$, we have

$$X(s, w, x) \in \Omega_\varepsilon \cup \Omega, \qquad X(s, w, y) \in \Omega.$$

Thus by the mean value theorem and (9), (10), we have

$$
\begin{aligned}
(11)\quad |\varrho(X(s,w,x)) - \varrho(X(s,w,y))| &\le |\nabla \varrho(\eta(w))| \cdot |X(s,w,x) - X(s,w,y)| \\
&\le C_4 B(w)|x-y|^{\alpha+1/m}.
\end{aligned}
$$

where $\eta(w) \in \Omega_\varepsilon \cup \Omega$.

Now we estimate the $\mu\{|\tau_x^T(w) - \tau_y^T(w)| > \lambda\}$ for $\lambda < T$. Note that if $\tau_y^T(w) > \tau_x^T(w)$ for some $w \in W$, then

$$\tau_y^T(w) = \tau_x^T(w) + \inf\{s : \varrho(X(\tau_x^T(w) + s, w, y)) = 0\}.$$

Hence

$$\{\tau_y^T(w) - \tau_x^T(w) > \lambda\} \subset \{\max_{0 \le s \le \lambda} \varrho(X(\tau_x^T(w) + s, w, y)) < 0,\ \tau_x^T(w) < T - \lambda\}.$$

On the other hand, setting $u(s) = \tau_x^T(w) + s \wedge \eta^\varepsilon_{X(\tau_x^T(w),w,x)}(w) < \tau_x^T(w) + s$, from (11), we find that

$$\max_{0 \le s \le \lambda} |\varrho(X(u(s), w, x)) - \varrho(X(u(s), w, y))| \le C_4 B(w)|x-y|^{\alpha+1/m} \quad \text{a.s.}$$

Consequently, we have

$$
\begin{aligned}
&\mu\{\tau_y^T(w) - \tau_x^T(w) > \lambda\} \\
&\quad \le \mu\{\max_{0 \le s \le \lambda} \varrho(X(\tau_x^T(w) + s, w, y)) < 0,\ \tau_x^T(w) < T - \lambda\} \\
&\quad = \mu\{\max_{0 \le s \le \lambda} \varrho(X(\tau_x^T(w) + s, w, y)) < 0,\ \tau_x^T(w) < T - \lambda, \\
&\qquad\quad \max_{0 \le s \le \lambda} \varrho(X(u(s), w, x)) - \max_{0 \le s \le \lambda} \varrho(X(u(s), w, y)) \le B(w)|x-y|^{\alpha+1/m}\} \\
&\quad \le \mu\{\max_{0 \le s \le \lambda} \varrho(X(u(s), w, x)) < B(w)|x-y|^{\alpha+1/m},\ \tau_x^T(w) < T - \lambda\} \\
&\quad \le \mu\{\max_{0 \le s \le \lambda} \varrho(X(u(s), w, x)) < B(w)|x-y|^{\alpha+1/m},\ \tau_x^T(w) < T - \lambda, \\
&\qquad\quad B(w) \le |x-y|^{-1/m}\} + \mu\{B(w) \ge |x-y|^{-1/m}\} \\
&\quad \le \mu\{\max_{0 \le s \le \lambda} \varrho(X(u(s), w, x)) < |x-y|^{\alpha},\ \tau_x^T(w) < T - \lambda\} \\
&\qquad\quad + \mu\{B(w) \ge |x-y|^{-1/m}\} \\
&\quad \le \mu\{\max_{0 \le s \le \lambda} \varrho(X(\tau_x(w) + s \wedge \eta^\varepsilon_{X(\tau_x(w),w,x)}(w), w, x)) < |x-y|^{\alpha}\} \\
&\qquad\quad + \mu\{B(w) \ge |x-y|^{-1/m}\} \\
&\quad := I_1 + I_2.
\end{aligned}
$$

The Chebyshev inequality yields

$$I_2 \le E|B(w)|^m \cdot |x-y| \le C|x-y|.$$

By the strong Markov property, we continue to get

$$I_1 = \mu\{\max_{0\le s\le\lambda} \varrho(X(s\wedge \eta^{\varepsilon}_{X(\tau_x(w),w,x)}(w), \theta_{\tau_x(w)}(w), X(\tau_x(w), w, x))) < |x-y|^{\alpha}\}$$

$$= \int_{\partial\Omega} \mu\{\max_{0\le s\le\lambda} \varrho(X(s\wedge \eta^{\varepsilon}_{z}(w), \theta_{\tau_x(w)}(w), z)) < |x-y|^{\alpha}\} \times \mu\{X(\tau_x(w), w, x) \in dz\}$$

where $\theta_{\tau_x(w)}(w)(s) = w(s+\tau_x(w)) - w(\tau_x(w))$.

On the other hand, for fixed $z\in\partial\Omega$, by the Ito formula ([6] or [7]), there exists an abstract Brownian motion $b(s) = b^z(s)$ such that

$$\varrho(X(t,z)) = \int_0^t [(\nabla\varrho)^T a \nabla\varrho]^{1/2}(X(s,z))\, db(s) + \int_0^t A\varrho(X(s,z))\, ds.$$

Set

$$\xi(t,z) = \int_0^t [(\nabla\varrho)^T a \nabla\varrho]^{1/2}(X(s\wedge\eta^{\varepsilon}_z,z))\, db(s) + \int_0^t A\varrho(X(s\wedge\eta^{\varepsilon}_z,z))\, ds.$$

Then for $t<\eta^{\varepsilon}_z$, we have

$$\varrho(X(t,w,z)) = \xi(t,w,z).$$

By the Girsanov theorem, $\xi(t,z)$ is a martingale under the new probability $d\widehat{\mu} = M\, d\mu$ where

$$M = \exp\Big[-\int_0^T (A\varrho)(X(s\wedge\eta^{\varepsilon}_z,z))\, db(s) - \frac{1}{2}\int_0^T (A\varrho(X(s\wedge\eta^{\varepsilon}_z,z)))^2\, ds\Big].$$

The increasing process of the martingale $\xi(t,z)$ is

$$\beta(t,z) = \int_0^t [(\nabla\varrho)^T a \nabla\varrho](X(s\wedge\eta^{\varepsilon}_z,z))\, ds.$$

From (4), (6) and (10), it is easy to see that

$$C_5 t \le \beta(t,z) \le C_6 t.$$

By a change of clock, we deduce that $\gamma_t(w) := \xi(\beta_t^{-1}, z)$ is a Brownian motion under $\widehat{\mu}$, starting at zero. Hence

$$\widehat{\mu}\{\max_{0\le s\le t} \gamma_s < h\} = 2\int_0^h \frac{e^{-x^2/(2t)}}{(2\pi t)^{1/2}}\, dx \le Cht^{-1/2}.$$

So we have

$$\widehat{\mu}\{\max_{0\le s\le\lambda}\varrho(X(s\wedge\eta_z^\varepsilon(w),w,z))<|x-y|^\alpha\}=\widehat{\mu}\{\max_{0\le s\le\lambda}\xi(s,z)<|x-y|^\alpha\}$$
$$\le\widehat{\mu}\{\max_{0\le s\le C_5\lambda}\gamma_s<|x-y|^\alpha\}\le C|x-y|^\alpha\lambda^{-1/2}.$$

Thus, by the Hölder inequality, for any $q>1$, we have

$$\mu\{\max_{0\le s\le\lambda}\varrho(X(s\wedge\eta_z^\varepsilon(w),w,z))<|x-y|^\alpha\}$$
$$\le C\Big(\int M^{q/(q-1)}d\widehat{\mu}\Big)^{(q-1)/q}|x-y|^{\alpha/q}\lambda^{-1/(2q)}.$$

Since $M\in\bigcap_p L^p$, we see that for any $0<\lambda<T$,

$$\mu\{\max_{0\le s\le\lambda}\varrho(X(s\wedge\eta_z^\varepsilon(w),w,z))<|x-y|^\alpha\}\le C|x-y|^{\alpha/q}\lambda^{-1/(2q)}.$$

Lastly, $I_1\le C|x-y|^{\alpha/q}\lambda^{-1/(2q)}$. So

$$E|\tau_x^T(w)-\tau_y^T(w)|^p=\int_{\mathbb{R}}p\lambda^{p-1}\mu\{|\tau_x^T(w)-\tau_y^T(w)|>\lambda\}\,d\lambda$$
$$\le C|x-y|^{\alpha/q}\int_0^T p\lambda^{p-1}\lambda^{-1/(2q)}\,d\lambda\le C|x-y|^{\alpha/q}.$$

In view of the arbitrariness of $q>1$, we let q tend to 1, and (7) follows.

Next we look at the second part of the theorem. Assume that $x<y$ and $T/3<\lambda<T/2$; we obviously have

$$\{\tau_x^T(w)-\tau_y^T(w)>\lambda\}=\{\tau_x(w)-\tau_y(w)>\lambda\}\cap\{\tau_y(w)<T-\lambda\}.$$

Thanks to the independence of $\tau_x(w)-\tau_y(w)$ and $\tau_y(w)$ (cf. [11, p. 165, Problem 8.22]), we have

$$\begin{aligned}\mu\{\tau_x^T(w)-\tau_y^T(w)>\lambda\}\\&=\mu\{\tau_x(w)-\tau_y(w)>\lambda\}\mu\{\tau_y(w)<T-\lambda\}\\&=\mu\{\max_{0\le s\le\lambda}w(s+\tau_y)-w(\tau_y)<y-x\}\mu\{\tau_y(w)<T-\lambda\}\\&=\mu\{\max_{0\le s\le\lambda}w(s)<y-x\}\mu\{\tau_y(w)<T-\lambda\}\\&=\frac{2}{\sqrt{2\pi\lambda}}\int_0^{y-x}e^{-r^2/(2\lambda)}\,dr\cdot\frac{2}{\sqrt{2\pi(T-\lambda)}}\int_{1-y}^{\infty}e^{-r^2/(2(T-\lambda))}\,dr\\&\ge C(y-x).\end{aligned}$$

Consequently,

$$E|\tau_x^T(w)-\tau_y^T(w)|^p\ge C|x-y|.\ \blacksquare$$

Acknowledgements. I would like to express my hearty thanks to Professor Jiagang Ren for his valuable suggestions. I would also like to express my deep thanks to the referee for his/her useful advice.

References

[1] H. Airault, P. Malliavin and J. Ren, *Smoothness of stopping times of diffusion processes*, J. Math. Pures Appl. 78 (1999), 1069–1091.

[2] B. Boufoussi, *Régularité du temps local brownien dans les espaces de Besov–Orlicz*, Studia Math. 118 (1996), 145–156.

[3] B. Boufoussi et B. Roynette, *Le temps local brownien appartient p.s. à l'espace de Besov* $\mathcal{B}^{1/2}_{p,\infty}$, C. R. Acad. Sci. Paris. Sér. I Math. 312 (1993), 843–848.

[4] F. B. Knight and B. Maisonneuve, *A characterization of stopping times*, Ann. Probab. 22 (1994), 1600–1606.

[5] S. G. Krantz, *Function Theory of Several Complex Variables*, Pure Appl. Math., Wiley, 1982.

[6] P. Malliavin, *Asymptotic of the Green's function of a Riemannian manifold and Ito's stochastic integral*, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 381–383.

[7] —, *Géométrie différentielle stochastique*, Presses Univ. Montréal, 1978.

[8] J. L. Pedersen and G. Peškir, *Computing the expectation of the Azéma–Yor stopping times*, Ann. Inst. Henri Poincaré Probab. Statist. 34 (1998), 265–276.

[9] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, 1991.

[10] H. Triebel, *Interpolation Theory*, *Function Spaces*, *Differential Operators*, North-Holland, 1978.

[11] A. D. Wentzell, *A Course in the Theory of Stochastic Process*, translated from the Russian by S. Chomet, McGraw-Hill, 1981.

Department of Mathematics
Huazhong University of Science and Technology
Wuhan, Hubei 430074, P.R. China
E-mail: xczhang@hust.edu.cn

Grupo de Fisica-Matematica
Universidade de Lisboa
Av. Prof. Gama Pinto, 2
1649-003 Lisboa, Portugal
E-mail: xczhang@cii.fc.ul.pt

Received January 3, 2001
Revised version December 13, 2001 (4664)