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Doubling properties and unique continuation at the boundary for elliptic operators with singular magnetic fields

by

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Abstract. Let u be a solution to a second order elliptic equation with singular magnetic fields, vanishing continuously on an open subset Γ of the boundary of a Lipschitz domain. An elementary proof of the doubling property for u^2 over balls centered at some points near Γ is presented. Moreover, we get the unique continuation at the boundary of Dini domains for elliptic operators.

1. Introduction and statement of results. The unique continuation problem has been receiving increasing attention from workers in both partial differential equations and mathematical physics. In the survey paper on Schrödinger semigroups [S] Simon formulated the following conjecture: Let $V \in K_n^{\mathrm{loc}}$, the local Kato class of potentials. Then the Schrödinger operator $H = -\Delta + V$ has the *unique continuation property* (UCP). By this it is meant that given any connected open subset $\Omega \subset \mathbb{R}^n$, the only solution u of $Hu = 0$ in Ω vanishing in an open subset $\Omega_0 \subset \Omega$ is $u \equiv 0$. It was proved in [SS] that if $V^2 \in K_n^{\mathrm{loc}}$, then the Schrödinger operator H has the UCP. A recent outstanding result of Fabes, Garofalo and Lin's [FGL] proves that if $V \in K_n^{\mathrm{loc}}$ and is monotone radial, then H has the UCP.

On the other hand, the following unique continuation question was raised in [L], [AEK] and [AE]: If u is a harmonic function in a connected Lipschitz domain Ω, vanishing continuously on an open subset Γ of the boundary $\partial\Omega$ and whose normal derivative vanishes on a subset of Γ of positive surface measure, does it follow that u is identically zero in Ω? In [AEK], it was shown that this holds for convex domains. In [AE] and [KN], it was proven that the answer is affirmative for Dini domains, and thus, in particular, for

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$C^{1,\alpha}$ domains with $\alpha > 0$, while it is also pointed out in [AE] that the result can be generalized to solutions to elliptic operators with Lipschitz second order coefficients and bounded lower order coefficients in $C^{1,\alpha}$ domains with $\alpha > 0$. The key step in [AE] is to prove that u satisfies a doubling type condition on the boundary. In this paper, we prove doubling properties for solutions of $Hu = 0$ (Theorems 1.1 and 1.2), and give an affirmative answer to related unique continuation questions for the Schrödinger operator H in Dini domains (Theorem 1.3).

In fact, we consider second order elliptic operators of the following form:

$$Lu(X) = -\sum_{j,k=1}^{n} D_j(a_{jk}(X)D_k u(X)) + \sum_{j=1}^{n} W_j(X)D_j u(X) + V(X)u(X) \tag{1.1}$$

in some domains Ω, where $A(X) = (a_{jk}(X))_{j,k=1}^n$ is a real symmetric matrix function, $D_j = \partial/\partial x_j - ib_j$ is a differential operator, $b(X) = (b_j(X))_{j=1}^n$ and $W(X) = (W_j(X))_{j=1}^n$ are real-valued vector fields, and $V(X) = V^{\mathrm{R}}(X) + iV^{\mathrm{I}}(X)$ is a complex-valued function.

Throughout this paper we use the notation

$$D = \nabla - ib, \qquad D^* = \nabla + ib, \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right),$$

$$B = (b_{jk})_{j,k=1}^n, \qquad b_{jk}(X) = \frac{\partial b_j}{\partial x_k} - \frac{\partial b_k}{\partial x_j},$$

$$H^m_{\mathrm{loc}}(\Omega) = \{u \in L^2_{\mathrm{loc}}(\Omega) : \nabla^\alpha u \in L^2_{\mathrm{loc}}(\Omega),\ |\alpha| \le m\}.$$

Note that $\overline{Du} = D^* \overline{u}$ for a complex-valued function u.

To state our main results, we first recall the definition of the Kato class K_n^{loc}.

Definition 1.1. We say a measurable function $g \in L^1_{\mathrm{loc}}(\Omega)$ belongs to the *Kato class* $K_n^{\mathrm{loc}}(\Omega)$ if $\lim_{r\to 0} \eta(r; g\chi_{\Omega_0}) = 0$ for every bounded subdomain Ω_0 of Ω. Here

$$\eta(r; g) = \sup_{X\in\mathbb{R}^n} \int_{B_r(X)} \frac{|g(Y)|}{|X-Y|^{n-2}}\, dY$$

where $B_r(X) = \{Y \in \mathbb{R}^n : |Y - X| \le r\}$ is the ball in $\mathbb{R}^n$.

In this paper, the assumptions on A, b, W and V are the following.

Assumption (A). *For any $X_0 \in \bar{\Omega}$, there exists a $\lambda > 1$ such that, for every $X \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n$,*

$$\lambda^{-1}|\xi|^2 \le A(X)\xi\cdot\xi \le \lambda|\xi|^2; \tag{1.2}$$

and there exists a nondecreasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\lim_{r\to 0} f(r) = 0$, *and for every* $X \in B_1(X_0) \cap \Omega$,

$$|\nabla A(X)| \le \frac{f(|X - X_0|)}{|X - X_0|}, \qquad |A(X) - A(X_0)| \le f(|X - X_0|), \tag{1.3}$$

and

$$|W(X)| \le \frac{f(|X - X_0|)}{|X - X_0|}, \qquad |\nabla \cdot W(X)| \le \frac{f(|X - X_0|)}{|X - X_0|^2}. \tag{1.4}$$

ASSUMPTION (B). *For any* $X_0 \in \bar{\Omega}$, *we have*

(1) $b \in L^4_{\mathrm{loc}}(\Omega)$, $\nabla \cdot b \in L^2_{\mathrm{loc}}(\Omega)$, $|b|^2 \in K^{\mathrm{loc}}_n(\Omega)$,
$(|X - X_0||B|)^2 \in K^{\mathrm{loc}}_n(\Omega)$;

(2) $V^{\mathrm{R}} \in K^{\mathrm{loc}}_n(\Omega)$, $(|X - X_0| \cdot |V^{\mathrm{I}}|)^2 \in K^{\mathrm{loc}}_n(\Omega)$,
$2\dfrac{X - X_0}{|X - X_0|} V^{\mathrm{R}} + |X - X_0| \nabla V^{\mathrm{R}} \in K^{\mathrm{loc}}_n(\Omega)$.

We remark that if $V \in K^{\mathrm{loc}}_n(\Omega)$ and is a monotone radial real function, then it satisfies Assumption (B).

In this paper, we always denote by

$$\triangle_r(Q) = B_r(Q) \cap \partial\Omega \quad \text{and} \quad T_r(Q) = B_r(Q) \cap \Omega$$

a surface ball and a Carleson region for the boundary point $Q \in \partial\Omega$. Take $Q_0 \in \partial\Omega$. Let $\eta_0(r; g) = \eta(r; g\chi_{T_3(Q_0)})$ and

$$\eta_0(r) = \eta_0(r; V^{\mathrm{R}}) + \eta_0\left(r; 2\,\frac{X - X_0}{|X - X_0|}\, V^{\mathrm{R}} + |X - X_0| \nabla V^{\mathrm{R}}\right) \\ + \eta_0(r; (|X - X_0| V^{\mathrm{I}})^2)^{1/2} + \eta_0(r; (|X - X_0| \cdot |B|)^2)^{1/2};$$

we also put

$$\theta_0(r) = \eta_0(r) + f(r),$$

and denote by r_* a small number defined in Definition 3.1.

The main results of this work are the following doubling properties near the boundary.

THEOREM 1.1. *Let* Ω *be a Lipschitz domain, and* L *be an operator as in* (1.1) *satisfying Assumptions* (A) *and* (B) *with* $\int_0^1 (\theta_0(r)/r)\,dr = M < \infty$, *and let* $u \in H^2_{\mathrm{loc}}(\Omega)$ *be a solution to* $Lu = 0$ *in* Ω *vanishing on* $\triangle_3(Q_0)$ *for some* $Q_0 \in \partial\Omega$. *Suppose that there exist a positive number* r_0, $0 < r_0 < r_*$, *and a point* $X_0 \in B_1(Q_0) \cap \bar{\Omega}$ *such that* $A(X_0) = I$, *the unit matrix, and*

$$A(Q)(Q - X_0) \cdot \nu(Q) \ge 0 \quad \textit{for a.e. } Q \textit{ in } B_{2r_0}(X_0) \cap \partial\Omega \tag{1.5}$$

where $\nu(Q)$ is the outward unit normal vector at $Q \in \partial\Omega$. Then

$$\int\limits_{B_{2r}(X_0)\cap\Omega} |u(X)|^2\,dX \le 2^{C(r_0)} \int\limits_{B_r(X_0)\cap\Omega} |u(X)|^2\,dX$$

for all $0 < r < r_0$, where $C(r_0)$ is a constant independent of X_0 and r.

THEOREM 1.2. *Assume the same conditions as in the above theorem, but without $\int_0^1 (\theta_0(r)/r)\,dr < \infty$. Then there exist absolute constants C_1, C_2 and K independent of $0 < r < r_0$ and $X_0 \in B_1(Q_0) \cap \bar{\Omega}$ such that*

$$\int\limits_{B_{2r}(X_0)\cap\Omega} |u(X)|^2\,dX \le \exp\left(\frac{C_1}{r^{C_2\theta_0(Kr_0)}}\right) \int\limits_{B_r(X_0)\cap\Omega} |u(X)|^2\,dX.$$

The proof of the two theorems above will use a Rellich type identity near the boundary and the variational method; the original idea goes back to Garofalo and Lin [GL1] who dealt with the equation $\operatorname{div}(A\nabla u) = 0$. We will give the proofs in Section 3. From the two theorems, it is easy to deduce the following boundary unique continuation for Dini domains (Corollaries 1.1 and 1.2), and the inner unique continuation in any domain $\Omega \subset \mathbb{R}^n$ (Corollaries 1.3 and 1.4).

COROLLARY 1.1. *Let Ω be a connected Dini domain in $\mathbb{R}^n$ and $u \in H^2_{\mathrm{loc}}(\Omega)$ be a solution in Ω to $Lu = 0$ vanishing continuously on an open subset Γ of $\partial\Omega$, where L is an operator as in* (1.1) *with Lipschitz coefficients A and satisfies Assumptions* (A), (B) *and $\int_0^1 (\theta_0(r)/r)\,dr < \infty$. Assume that for some point Q in Γ and for every $m > 0$,*

$$\int\limits_{T_r(Q)} |u(X)|^2\,dX = O(r^m), \qquad r \to 0.$$

Then u must be identically zero in Ω.

COROLLARY 1.2. *Let Ω be a connected C^1 domain in $\mathbb{R}^n$ and $u \in H^2_{\mathrm{loc}}(\Omega)$ be a solution in Ω to $Lu = 0$ vanishing continuously on an open subset Γ of $\partial\Omega$, where L is an operator as in* (1.1) *and satisfies Assumptions* (A) *and* (B). *Assume that for some point Q in Γ, there are positive constants K and β such that*

$$\int\limits_{T_r(Q)} |u(X)|^2\,dX = O(\exp(-K/r^\beta)), \qquad r \to 0.$$

Then u must be identically zero in Ω.

COROLLARY 1.3. *Suppose Assumptions* (A) *and* (*B*) *hold, $\int_0^1 (\theta_0(r)/r)\,dr < \infty$, and Ω is a connected domain in $\mathbb{R}^n$. Then L has the strong inner unique continuation property*: *if $u \in H^2_{\mathrm{loc}}(\Omega)$ is a solution to $Lu = 0$ and* *satisfies, for some* $X_0 \in \Omega$ *and every* $m > 0$,

$$\int\limits_{B_r(X_0)} |u(X)|^2\, dX = O(r^m), \qquad r \to 0,$$

then $u \equiv 0$ *in* Ω.

COROLLARY 1.4. *Suppose Assumptions* (A) *and* (B) *hold, and* Ω *is a connected domain in* $\mathbb{R}^n$. *Then* L *has the inner unique continuation property: if* $u \in H^2_{\mathrm{loc}}(\Omega)$ *is a solution to* $Lu = 0$ *and satisfies, for some* $X_0 \in \Omega$ *and* $K, \varepsilon > 0$,

$$\int\limits_{B_r(X_0)} |u(X)|^2\, dX = O(\exp(-K/r^{\varepsilon})), \qquad r \to 0,$$

then $u \equiv 0$ *in* Ω.

In particular, L *has the weak inner unique continuation property: if* u *vanishes on a subdomain* Ω_0 *of* Ω, *then* $u \equiv 0$ *in* Ω.

These corollaries improve the previous results of [AE], [AEK], [FGL], [GL2] and [Ku]. The arguments for their proofs have already been given in [AE] and [FGL]; in Section 4 we outline the proofs of Corollaries 1.1 and 1.2 for the sake of completeness.

We also consider the unique continuation at the boundary, and obtain the following results.

THEOREM 1.3. *Let* Ω *be a Dini domain in* $\mathbb{R}^n$, *and let* L *be an operator as in* (1.1) *with Lipschitz second order coefficients, satisfying Assumptions* (A) *and* (B) *and* $\int_0^1 (\theta_0(r)/r)\, dr < \infty$. *If* $u \in H^2_{\mathrm{loc}}(\Omega)$ *is a solution in* Ω *to* $Lu = 0$ *vanishing continuously on an open subset* Γ *of* $\partial\Omega$ *and whose normal derivative vanishes on a subset of* Γ *with positive surface measure, then* u *must be identically zero in* Ω.

Since $Lu = -\operatorname{div}(A\nabla u) + (W + 2iAb)\cdot\nabla u + i\operatorname{div}(Ab)u + Ab\cdot bu + Vu$, we can apply the results of [AE] for a Dini domain. But they require stronger conditions: $|W|, |Ab|, \operatorname{div}(Ab), |V| \in L^\infty_{\mathrm{loc}}$, while our method only requires, for instance, $\operatorname{div}(Ab) \in L^2_{\mathrm{loc}}$.

COROLLARY 1.5. *Let* Ω *be a Dini domain, and* $\widetilde{L}$ *a real elliptic operator of the form*

$$\widetilde{L}u = -\operatorname{div}(A\nabla u) + W\cdot\nabla u + Vu, \tag{1.6}$$

where A *satisfies* (1.2) *and the Lipschitz condition;* W *satisfies* (1.4) *or* $|W| \in L^\infty_{\mathrm{loc}}(\Omega)$; $V \in K_n^{\mathrm{loc}}(\Omega)$ *and* $2\frac{X-X_0}{|X-X_0|}V + |X - X_0|\nabla V \in K_n^{\mathrm{loc}}(\Omega)$; *and let* $\int_0^1 (\theta_0(r)/r)\, dr < \infty$. *Then* $\widetilde{L}$ *has the unique continuation property at the boundary: if* $u \in H^2_{\mathrm{loc}}(\Omega)$ *is a nonconstant solution to* $\widetilde{L}u = 0$ *vanish-*

ing continuously on an open subset Γ of $\partial\Omega$, then the surface measure of $\{Q \in \Gamma : |\nabla u| = 0\}$ *is zero.*

REMARK 1.1. Let $X_0 = 0$, $B_1 = B_1(0)$ and $V(X) = |X|^{-l}\big|\log|X|\big|^{-m}$, $l > 0$, $m \in \mathbb{R}$. Then (see [Ku])

$$\begin{aligned} V \in K_n^{\mathrm{loc}}(B_1) &\Leftrightarrow (m > 1,\, l = 2) \text{ or } (m \in \mathbb{R},\, l < 2);\\ V^2 \in K_n^{\mathrm{loc}}(B_1) &\Leftrightarrow (m > 1/2,\, l = 1) \text{ or } (m \in \mathbb{R},\, l < 1);\\ 2\frac{X}{|X|}V + |X|\nabla V \in K_n^{\mathrm{loc}}(B_1) &\Leftrightarrow (m > 0,\, l = 2) \text{ or } (m \in \mathbb{R},\, l < 2). \end{aligned}$$

So in this example the condition $V^2 \in K_n^{\mathrm{loc}}$ is the strongest. In fact if the assumption on V is replaced by $|V|^2 \in K_n^{\mathrm{loc}}$ the theorems are valid (see Remark 3.2 below).

REMARK 1.2. By using an approximation argument we can show unique continuation theorems similar to those above even for H^1_{loc}-solutions.

In this paper, the letter C always denotes positive constants which may depend on λ, n, the K_n^{loc} norm and the Lipschitz character of Ω but may change at different occurrences. The notation $h = O(f)$ means that $|h| \le C|f|$ for some constant C.

2. Kato potentials and auxiliary lemmas. In this section, we recall some notations and lemmas concerning the Kato class which will be useful in this paper.

LEMMA 2.1. *Let Ω be a Lipschitz domain, and assume $g \in K_n^{\mathrm{loc}}(\Omega)$, and $u \in H^1_{\mathrm{loc}}(\Omega)$ vanishes continuously on $B \cap \partial\Omega$, where $B = B_r(X_0)$ for some $X_0 \in \bar{\Omega}$ and $r > 0$. Then there exists a dimensional constant C_n, independent of r, X_0 and u, such that*

$$\int_{B\cap\Omega} |g|\cdot|u|^2\,dX \le C_n\eta(r; g\chi_{B\cap\Omega})\Big(\int_{B\cap\Omega}|\nabla u|^2\,dX + \frac{1}{r}\int_{\partial B\cap\Omega}|u|^2\,d\sigma\Big), \tag{2.1}$$

$$\int_{B\cap\Omega} |g|\cdot|u|^2\,dX \le C_n\eta(r; g\chi_{B\cap\Omega})\Big(\int_{B\cap\Omega}|Du|^2\,dX + \frac{1}{r}\int_{\partial B\cap\Omega}|u|^2\,d\sigma\Big). \tag{2.2}$$

Proof. Arguing as in Lemma 1.1 of [FGL], we get (2.1) for all $r > 0$, $g \in K_n^{\mathrm{loc}}(\Omega)$ and $u \in C^\infty(\Omega)$ vanishing on $B \cap \partial\Omega$. By density, (2.1) also holds for $u \in H^1_{\mathrm{loc}}(\Omega)$ vanishing on $B \cap \partial\Omega$.

On the other hand, it is not difficult to see that

$$|Du|^2 = \big|\nabla|u|\big|^2 + \Big|bu - \frac{u_1\nabla u_2 - u_2\nabla u_1}{u}\Big|^2 \ge \big|\nabla|u|\big|^2, \tag{2.3}$$

where $u_1 = \mathrm{Re}(u)$, $u_2 = \mathrm{Im}(u)$. This yields (2.2).

LEMMA 2.2. *Let Ω be a Lipschitz domain with $X_0 \in \bar{\Omega}$, and assume that $u \in H^1_{\mathrm{loc}}(\Omega)$ vanishes continuously on $B_r(X_0) \cap \partial\Omega$. Then*

$$\int_{B_r(X_0)\cap\Omega} \frac{|u(X)|^2}{|X-X_0|^2}\,dX \le 4\Big(\int_{B_r(X_0)\cap\Omega} |Du|^2\,dX + \frac{1}{r}\int_{\partial B_r(X_0)\cap\Omega} |u|^2 d\sigma\Big) \tag{2.4}$$

for all r, X_0 and u.

Proof. This is a variation of Heisenberg's uncertainty principle (see [RS]). We observe that it is not restrictive to assume that $X_0 = 0$. From the divergence theorem,

$$\begin{aligned}\int_{B_r\cap\Omega} \frac{|u(X)|^2}{|X|^2}\,dX &= \int_{B_r\cap\Omega} |u|^2 \operatorname{div}\Big(\frac{X}{|X|^2}\Big)\,dX \\ &= -2\int_{B_r\cap\Omega} |u|\nabla|u| \cdot \frac{X}{|X|^2}\,dX + \frac{1}{r}\int_{\partial B_r\cap\Omega} |u|^2\,d\sigma \\ &\le 2\Big(\int_{B_r\cap\Omega} \frac{|u|^2}{|X|^2}\,dX\Big)^{1/2}\Big(\int_{B_r\cap\Omega} \big|\nabla|u|\big|^2\,dX\Big)^{1/2} + \frac{1}{r}\int_{\partial B_r\cap\Omega} |u|^2\,d\sigma.\end{aligned}$$

This inequality, Cauchy's inequality and (2.3) yield (2.4).

Using Lemma 2.1 we can deduce the following Caccioppoli inequality ([Ke]).

LEMMA 2.3. *Let Ω be a Lipschitz domain with $Q_0 \in \partial\Omega$ and L be an operator as in* (1.1) *satisfying Assumptions* (A) *and* (B). *Suppose $u \in H^1_{\mathrm{loc}}(\Omega)$ is a solution to $Lu = 0$ vanishing on $\triangle_3(Q_0)$. Then there exist constants C and $0 < r_* < 1$ such that for all $0 < r < r_*$ and $X_0 \in B_1(Q_0) \cap \bar{\Omega}$,*

$$\int_{B_r(X_0)\cap\Omega} |\nabla u|^2\,dX \le \frac{C}{r^2}\int_{B_{2r}(X_0)\cap\Omega} |u|^2\,dX. \tag{2.5}$$

Proof. Take $0 < r < 1$, and let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a real function, $\phi \equiv 1$ on $B_r(X_0)$, $\operatorname{supp}\phi \subset B_{2r}(X_0)$, $|\nabla\phi| \le C/r$. Since $u \equiv 0$ on $B_{2r}(X_0) \cap \partial\Omega$, we have $u\phi^2 \in H^1_0(B_{2r}(X_0) \cap \Omega)$. Thus

$$\int_{B_{2r}(X_0)\cap\Omega} [ADu \cdot D^*(\overline{u}\phi^2) + W \cdot Du\overline{u}\phi^2 + Vu\overline{u}\phi^2]\,dX = 0.$$

By assumptions and Hölder's inequality we have

$$\int\limits_{B_{2r}(X_0)\cap\Omega} |\nabla u|^2\phi^2\,dX \le C_{\lambda,n} \int\limits_{B_{2r}(X_0)\cap\Omega} |u|^2|\nabla\phi|^2\,dX + C_{\lambda,n} \int\limits_{B_{2r}(X_0)\cap\Omega} ((V^{\mathrm{R}})^- + |b|^2 + |W|^2)|u\phi|^2\,dX$$

where $C_{\lambda,n}$ is a constant depending only λ and n. Thus, from (2.1) in Lemma 2.1, one can see that

$$\int\limits_{B_{2r}(X_0)\cap\Omega} |\nabla u|^2\phi^2\,dX \le C_{\lambda,n} \int\limits_{B_{2r}(X_0)\cap\Omega} |u|^2|\nabla\phi|^2\,dX + C_{\lambda,n}(\eta_0(2r;(V^{\mathrm{R}})^- + |b|^2) + f^2(2r)) \int\limits_{B_{2r}(X_0)\cap\Omega} |\nabla u|^2\phi^2\,dX. \tag{2.6}$$

Taking $0 < r_* < 1$ such that $C_{\lambda,n}(\eta_0(2r;(V^{\mathrm{R}})^- + |b|^2) + f^2(2r)) \le 1/2$ for all $r \in (0, r_*)$, we then get (2.5). The lemma is proved.

3. Doubling property. The purpose of this section is to establish the doubling property.

Definition 3.1. Let $Q_0 \in \partial\Omega$, $C_1 = \max\{2\lambda, C_n, C_{\lambda,n}\}$, where C_n is a dimensional constant of Lemma 2.1, and $C_{\lambda,n}$ is the constant in (2.6). Set

$$r_* = \max\{0 < r < 1 : \eta_0(2r;(V^{\mathrm{R}})^- + |b|^2) + f^2(2r) \le (2C_1)^{-1}\}.$$

Without loss of generality, we may assume $X_0 = 0$ is the origin and write $B_r = B_r(0)$. Thus condition (1.5) can be rewritten as

$$A(Q)Q\cdot\nu(Q) \ge 0 \quad \text{for a.e. } Q \text{ in } B_{2r_0}\cap\partial\Omega \tag{3.1}$$

We now consider the function μ and vector field β defined as

$$\mu(X) = A(X)X\cdot X/|X|^2, \qquad \beta(X) = A(X)X/\mu(X).$$

From Assumption (A) we have, with $|X| = r$,

$$\lambda^{-1} \le \mu(X) \le \lambda, \quad |\nabla\mu(X)| \le O(f(r)/r), \quad \mu(X) = 1 + O(f(r)), \tag{3.2}$$

$$|\beta(X)| = O(r), \quad \operatorname{div}(AX) = n + O(f(r)), \quad (\partial/\partial x_j)\beta_k = \delta_{jk} + O(f(r)), \tag{3.3}$$

where the constants depend only on λ and n. For u as in Theorem 1.1 and $0 < r < 2$, we introduce the following functions:

$$\begin{aligned} I_1(r) &= \int\limits_{B_r\cap\Omega} ADu\cdot\overline{Du}\,dX, & I_2(r) &= \int\limits_{B_r\cap\Omega} \mathrm{Re}(W\cdot Du\overline{u})\,dX,\\ I_3(r) &= \int\limits_{B_r\cap\Omega} V^{\mathrm{R}}|u|^2\,dX, & I(r) &= I_1(r) + I_2(r) + I_3(r),\\ H(r) &= \int\limits_{\partial B_r\cap\Omega} \mu|u|^2\,d\sigma, & N(r) &= \frac{rI(r)}{H(r)}. \end{aligned} \tag{3.4}$$

Differentiating $H(r)$ we have, from (3.2),

$$H'(r) = \left(\frac{n-1}{r} + O\left(\frac{f(r)}{r}\right)\right)H(r) + 2\,\mathrm{Re}\int_{\partial B_r\cap\Omega} \mu\overline{u}\,\frac{\partial u}{\partial r}\,d\sigma.$$

We note that u is a solution to $-D(ADu) + W\cdot Du + Vu = 0$ and u vanishes on $B_r \cap \partial\Omega$. A direct computation yields

$$\begin{aligned}
I(r) &= \int_{B_r\cap\Omega} \mathrm{div}(\overline{u}ADu)\,dX - i\int_{B_r\cap\Omega}\{\mathrm{Im}(W\cdot Du\overline{u}) + V^{\mathrm{I}}|u|^2\}\,dX \\
&= \int_{\partial B_r\cap\Omega} \overline{u}\,\frac{\partial u}{\partial \nu_A}\,d\sigma - i\int_{\partial B_r\cap\Omega} |u|^2 b\cdot A\nu\,d\sigma \\
&\quad - i\int_{B_r\cap\Omega}\{\mathrm{Im}(W\cdot Du\overline{u}) + V^{\mathrm{I}}|u|^2\}\,dX
\end{aligned}$$

where we have used the divergence theorem, and where $\partial u/\partial\nu_A$ is the conormal derivative.

Remark 3.1. Using Lemmas 2.1, 2.2, Assumptions (A) and (B), and the fact that $u \in H^2_{\mathrm{loc}}(\Omega)$, we can deduce the integrability of each integrand above.

Since the conormal derivative on ∂B_r is given by $\partial u/\partial\nu_A = A\nabla u\cdot X/|X|$, and $\alpha = AX/|X| - \mu X/|X|$ is a tangential vector field on ∂B_r with $|\,\mathrm{div}(\alpha(X))| = O(f(r)/r)$, and noting that $I(r)$ is real-valued, we obtain from the divergence theorem and (3.2) the following identity:

$$\begin{aligned}
I(r) &= \mathrm{Re}\int_{\partial B_r\cap\Omega} \overline{u}\,\frac{\partial u}{\partial\nu_A}\,d\sigma \\
&= \mathrm{Re}\int_{\partial B_r\cap\Omega} \mu\overline{u}\,\frac{\partial u}{\partial r}\,d\sigma + \mathrm{Re}\int_{\partial B_r\cap\Omega} \overline{u}\nabla u\cdot\alpha\,d\sigma \\
&= \mathrm{Re}\int_{\partial B_r\cap\Omega} \mu\overline{u}\,\frac{\partial u}{\partial r}\,d\sigma + \frac{1}{2}\int_{\partial B_r\cap\Omega} \nabla(|u|^2)\cdot\alpha\,d\sigma \\
&= \mathrm{Re}\int_{\partial B_r\cap\Omega} \mu\overline{u}\,\frac{\partial u}{\partial r}\,d\sigma + O\left(\frac{f(r)}{r}\right)H(r)
\end{aligned}$$

Thus

$$H'(r) = 2I(r) + \left[\frac{n-1}{r} + O\left(\frac{f(r)}{r}\right)\right]H(r). \tag{3.5}$$

Lemma 3.1. *For every $0 < r < 1$, there exists an absolute constant $C_{\lambda,n} > 0$ depending only on λ and n such that*

$$|I_2(r)| + |I_3(r)| \le C_{\lambda,n}\theta_0(r)\left(\frac{H(r)}{r} + I_1(r)\right); \tag{3.6}$$

and for all $0 < r < r_*$,

$$I_1(r) \le 2\left(\frac{H(r)}{r} + I(r)\right). \tag{3.7}$$

Proof. By Hölder's inequality, Assumption (A) and Lemma 2.2 we have

$$\begin{aligned}|I_2(r)| &\le \Big(\int_{B_r\cap\Omega} |W|^2|u|^2\,dX\Big)^{1/2}\Big(\int_{B_r\cap\Omega} |Du|^2\,dX\Big)^{1/2}\\ &\le 2\lambda f(r)\sqrt{\frac{H(r)}{r} + I_1(r)}\,\sqrt{I_1(r)} \le 2\lambda f(r)\left(\frac{H(r)}{r} + I_1(r)\right),\end{aligned}$$

and from Lemma 2.1,

$$|I_3(r)| \le \int_{B_r\cap\Omega} |V^{\mathrm{R}}|\cdot|u|^2\,dX \le C_n\eta_0(r;V^{\mathrm{R}})\left(\frac{H(r)}{r} + I_1(r)\right),$$

whence (3.6) follows. Moreover,

$$\begin{aligned}I(r) &= I_1(r) + I_2(r) + I_3(r)\\ &\ge I_1(r) - [2\lambda f(r) + c_n\eta_0(r;(V^{\mathrm{R}})^-)]\left(\frac{H(r)}{r} + I_1(r)\right),\end{aligned}$$

which implies (3.7).

Lemma 3.2. *For every* $r \in (0, r_*)$, $H(r) > 0$ *unless* $u \equiv 0$ *in* $B_r \cap \Omega$.

Proof. Assume that $H(r) = 0$ for a certain r sufficiently small. Then

$$I(r) = \mathrm{Re}\int_{\partial B_r\cap\Omega} \overline{u}\,\frac{\partial u}{\partial\nu_A}\,d\sigma = 0.$$

This and (3.7) imply $I_1(r) = 0$, and so we obtain $|Du(X)| = 0$ for a.e. $X \in B_r \cap \Omega$. Since $\big|\nabla|u|\big| \le |Du|$ a.e., $|u|$ is constant in $B_r \cap \Omega$. Thus, $H(r) = 0$ implies $u \equiv 0$ in $B_r \cap \Omega$.

Our next task is to consider the differentiation of the functions $I(r)$ and $N(r)$. Our argument is based on the following identity:

Lemma 3.3. *For every* $0 < r < 1$, *we have*

$$\begin{aligned}&\int_{\partial B_r\cap\Omega} ADu\cdot\overline{Du}\,d\sigma \\ &\quad= 2\int_{\partial B_r\cap\Omega} \frac{1}{\mu}\,|ADu\cdot\nu|^2\,d\sigma + \frac{1}{r}\int_{B_r\cap\partial\Omega} \frac{AQ\cdot\nu\,A\nu\cdot\nu}{\mu}\,|\nabla u\cdot\nu|^2\,d\sigma\end{aligned} \tag{3.8}$$

$$+\left[\frac{n-2}{r}+O\left(\frac{f(r)}{r}\right)\right]\int\limits_{B_r\cap\Omega} ADu\cdot\overline{Du}\,dX$$

$$-\frac{2}{r}\operatorname{Re}\int\limits_{B_r\cap\Omega}\beta\cdot\overline{Du}\,[W\cdot Du+Vu]\,dX$$

$$+\frac{2}{r}\operatorname{Im}\int\limits_{B_r\cap\Omega}\beta_l b_{lk}a_{jk}D_ju\overline{u}\,dX.$$

Proof. Observing that

$$(\partial/\partial x_k)\,\overline{D_ju}-(\partial/\partial x_j)\,\overline{D_ku}=i[b_{jk}\overline{u}+b_j(\partial\overline{u}/\partial x_k)-b_k(\partial\overline{u}/\partial x_j)],$$

we have the following Rellich identity:

$$\begin{aligned}\operatorname{div}(\beta ADu\cdot\overline{Du})&-2\operatorname{div}(\beta\cdot\overline{Du}\,ADu)\\ &=\operatorname{div}(\beta)ADu\cdot\overline{Du}+\beta_l(\partial a_{jk}/\partial x_l)D_ju\,\overline{D_ku}\\ &\quad-2(\partial\beta_l/\partial x_k)\,\overline{D_lu}\,a_{jk}D_ju-2\beta_l\overline{D_lu}\,D_k(a_{jk}D_ju)\\ &\quad-2i\beta_lb_{lk}a_{jk}D_ju\overline{u}-2i\beta\cdot bADu\cdot\overline{Du}\\ &\quad+\beta_la_{jk}(\overline{D_ku}(\partial(D_ju)/\partial x_l)-D_ju(\partial(\overline{D_ku})/\partial x_l)).\end{aligned}$$

Taking real parts, we have

$$\begin{aligned}\operatorname{div}(\beta ADu\cdot\overline{Du})&-2\operatorname{div}(\beta\cdot\overline{Du}\,ADu)\\ &=\operatorname{div}(\beta)ADu\cdot\overline{Du}+\beta_l(\partial a_{jk}/\partial x_l)D_ju\,\overline{D_ku}\\ &\quad-2(\partial\beta_l/\partial x_k)\,\overline{D_lu}\,a_{jk}D_ju-2\operatorname{Re}\{\beta_l\,\overline{D_lu}\,D_k(a_{jk}D_ju)\}\\ &\quad+2\operatorname{Im}\{\beta_lb_{lk}a_{jk}D_ju\overline{u}\}.\end{aligned}\tag{3.9}$$

We recall that $\beta\cdot\nu=r$ on ∂B_r and $\beta\cdot\overline{Du}\,ADu\cdot\nu=\frac{r}{\mu}|ADu\cdot\nu|^2$ on ∂B_r. Also since $u=0$ and then $Du=\nabla u$, $\nabla u=(\nabla u\cdot\nu)\nu$ almost everywhere on $B_r\cap\partial\Omega$, we have

$$\begin{aligned}\beta\cdot\nu ADu\cdot\overline{Du}=\beta\cdot\overline{Du}\,ADu\cdot\nu&=\beta\cdot\nu A\nu\cdot\nu|\nabla u\cdot\nu|^2\\ &=\frac{AQ\cdot\nu A\nu\cdot\nu}{\mu}\,|\nabla u\cdot\nu|^2\end{aligned}$$

on $B_r\cap\partial\Omega$. Therefore, integrating over $B_r\cap\Omega$ the Rellich–Nečas identity (3.9), we obtain (3.8).

As remarked above, the existence of the integrals in (3.8) follows from Lemmas 2.1, 2.2 and Assumptions (A) and (B). Now we introduce the quantities

$$F(r) = 2 \int_{\partial B_r \cap \Omega} \frac{1}{\mu} |ADu \cdot \nu|^2 \, d\sigma, \tag{3.10}$$
$$+ \frac{1}{r} \int_{B_r \cap \partial\Omega} \frac{AQ \cdot \nu \, A\nu \cdot \nu}{\mu} |\nabla u \cdot \nu|^2 \, d\sigma,$$

$$J(r) = -2 \,\mathrm{Re} \int_{B_r \cap \Omega} \beta \cdot \overline{Du} \, [W \cdot Du + Vu] \, dX \tag{3.11}$$
$$+ 2 \,\mathrm{Im} \int_{B_r \cap \Omega} \beta_l b_{lk} a_{jk} D_j u \overline{u} \, dX.$$

Then (3.8) can be rewritten as

$$I_1'(r) = F(r) + \left[\frac{n-2}{r} + O\left(\frac{f(r)}{r}\right)\right] I_1(r) + \frac{1}{r} J(r). \tag{3.12}$$

In order to deal with the term $J(r)$, we use the fact that

$$\mathrm{Re}(\beta \cdot \overline{Du} \, V^{\mathrm{R}} u) = \mathrm{Re}(\beta \cdot \nabla \overline{u} V^{\mathrm{R}} u) = \tfrac{1}{2} \beta \cdot \nabla(|u|^2) V^{\mathrm{R}};$$

then from the divergence theorem and (3.3) we obtain

$$\mathrm{Re} \int_{B_r \cap \Omega} \beta \cdot \overline{Du} \, V u \, dX = \frac{1}{2} \int_{B_r \cap \Omega} \beta \cdot \nabla(|u|^2) V^{\mathrm{R}} \, dX + \mathrm{Im} \int_{B_r \cap \Omega} \beta \cdot \overline{Du} \, V^{\mathrm{I}} u \, dX$$
$$= -\frac{1}{2} \int_{B_r \cap \Omega} \mathrm{div}(\beta V^{\mathrm{R}}) |u|^2 \, dX + \frac{r}{2} \int_{\partial B_r \cap \Omega} V^{\mathrm{R}} |u|^2 \, d\sigma + \mathrm{Im} \int_{B_r \cap \Omega} \beta \cdot \overline{Du} \, V^{\mathrm{I}} u \, dX$$
$$= -\frac{n + O(f(r))}{2} I_3(r) - \frac{1}{2} \int_{B_r \cap \Omega} (\beta \cdot \nabla V^{\mathrm{R}}) |u|^2 \, dX$$
$$+ \frac{r}{2} I_3'(r) + O(1) \int_{B_r \cap \Omega} |X| \cdot |V^{\mathrm{I}}| \cdot |Du| \cdot |u| \, dX.$$

Using Assumptions (A), (B), Lemma 2.1 and Hölder's inequality, we then get the following estimates:

$$\frac{1}{r} J(r) = \frac{n - 2 + O(f(r))}{r} I_3(r) - I_3'(r) \tag{3.13}$$
$$+ \frac{1}{r} \int_{B_r \cap \Omega} (2V^{\mathrm{R}} + \beta \cdot \nabla V^{\mathrm{R}}) |u|^2 \, dX + \frac{O(1)}{r} \int_{B_r \cap \Omega} |X| \cdot |W| \cdot |Du|^2 \, dX$$
$$+ \frac{O(1)}{r} \int_{B_r \cap \Omega} (|X| \cdot |V^{\mathrm{I}}| + |X| \cdot |B|) |Du| \cdot |u| \, dX$$
$$\geq \frac{n - 2 + O(f(r))}{r} I_3(r) - I_3'(r) - \frac{C\theta_0(r)}{r} \left(\frac{H(r)}{r} + I_1(r)\right)$$

with a positive constant C independent of r.

REMARK 3.2. If $|V|^2 \in K_n^{\mathrm{loc}}(\Omega)$, then by Lemma 2.1 we can get, directly from (3.11),

$$J(r) \geq (n-2+O(f(r)))I_3(r) - rI_3'(r) - C\theta_0(r)\left(\frac{H(r)}{r} + I_1(r)\right).$$

Proceeding in the standard way, we will show that there exists a constant $C > 0$ such that the frequency function $Z(r) = N(r)+1$ satisfies

$$Z'(r) \geq -C\,\frac{\theta_0(r)}{r}\,Z(r) \quad \text{for all } 0 < r < r_0, \tag{3.14}$$

First recall that

$$I_2'(r) = \operatorname{Re} \int\limits_{\partial B_r \cap \Omega} W \cdot Du\overline{u}\, d\sigma = \frac{1}{2} \int\limits_{\partial B_r \cap \Omega} W \cdot \nabla(|u|^2)\, d\sigma;$$

then the divergence theorem implies

$$|I_2'(r)| \leq \frac{Cf(r)}{r^2}\, H(r) \tag{3.15}$$

with a constant C independent of $0 < r < 1$.

Using (3.12), (3.13), (3.15) and Lemma 3.1, we have

$$\begin{aligned} I'(r) &= I_1'(r) + I_2'(r) + I_3'(r) \\ &\geq F(r) + \frac{n-2}{r}\, I(r) - \frac{C\theta_0(r)}{r}\left(I_1(r) + \frac{H(r)}{r}\right) \\ &\geq F(r) + \frac{n-2}{r}\, I(r) - \frac{C\theta_0(r)}{r}\left(I(r) + \frac{H(r)}{r}\right). \end{aligned}$$

By the above inequality, (3.5), and the quotient rule we obtain

$$\begin{aligned} Z'(r) &= \frac{I(r)H(r) + rI'(r)H(r) - rI(r)H'(r)}{H(r)^2} \\ &\geq \frac{rF(r)H(r) - 2rI(r)^2}{H(r)^2} - C\,\frac{\theta_0(r)}{r}\, Z(r) \end{aligned} \tag{3.16}$$

with an absolute constant $C > 0$ independent of $r \in (0, r_*)$.

On the other hand, recalling the definition of $F(r)$ and

$$I(r) = \operatorname{Re} \int\limits_{\partial B_r \cap \Omega} A\nabla u \cdot \nu\overline{u}\, d\sigma = \operatorname{Re} \int\limits_{\partial B_r \cap \Omega} ADu \cdot \nu\overline{u}\, d\sigma,$$

we see by Hölder's inequality that $F(r)H(r) - 2I(r)^2 \geq 0$ if $A(Q)Q \cdot \nu(Q) \geq 0$. Hence the desired differential inequality (3.14) holds, which yields the following monotonicity of $Z(r)$:

LEMMA 3.4. *Let L be an operator as in* (1.1) *satisfying Assumptions* (A) *and* (B), *and u a solution to $Lu = 0$ in Ω vanishing on $\triangle_3(Q_0)$. With notations as above*, *if condition* (1.5) *in Theorem* 1.1 *holds for $X_0 = 0$ and*

$0 < r_0 < r_*$, and $A(0) = I$, then there exists an absolute constant $C > 0$ such that

$$Z(r)\exp\left\{-C\int_r^{r_0}\frac{\theta_0(t)}{t}\,dt\right\}$$

is nondecreasing in $r \in (0, r_0)$. Moreover:

(i) *If $\int_0^1(\theta_0(r)/r)\,dr < \infty$, then $N(r) \le C(r_0)$ for all $r \in (0, r_0)$.*
(ii) *In general, for every $r \in (0, r_0)$,*

$$N(r) \le \frac{C_1(r_0)}{r^{C_2(r_0)}\theta_0(r_0)}$$

where $C(r_0), C_1(r_0)$ and $C_2(r_0)$ are bounded constants independent of r.

Proof. From (3.14) above,

$$\frac{d}{dr}\log Z(r) \ge -C\,\frac{\theta_0(r)}{r} \quad \text{for all } 0 < r < r_0, \tag{3.17}$$

which shows that $Z(r))\exp\{-C\int_r^{r_0}(\theta_0(t)/t)\,dt\}$ is nondecreasing. Further, we integrate (3.17) between r and r_0 to get

$$\frac{Z(r)}{Z(r_0)} \le \exp\left\{C\int_r^{r_0}\frac{\theta_0(t)}{t}\,dt\right\},$$

which yields the assertion.

This lemma and (3.5) imply Theorems 1.1 and 2.2 by a standard argument. For the details see [FGL] and [AEK].

4. Unique continuation at the boundary

Proof of Corollary 1.1. Let $Q \in \triangle_1(Q_0)$ and $\triangle_3(Q_0) \subseteq \Gamma$. Using the linear change of variable $X = SY + Q$, where S is a nonsingular matrix satisfying $A(Q) = S^tS$, we can assume that $Q = 0$ and $A(0) = I$, and that Ω is the set of points $Y = (y, y_n)$ in the unit cylindrical body of $\mathbb{R}^n$ such that $y_n > \varphi(y)$, where φ is a Lipschitz function in $\mathbb{R}^{n-1}$ satisfying $\varphi(0) = 0$ and $|\nabla\varphi(y) - \nabla\varphi(0)| \le \varrho(|y|)$ for all $y \in \mathbb{R}^{n-1}$, where ϱ is a Dini function. From the mean value theorem we get

$$y\nabla\varphi(y) - \varphi(y) \ge -2|y|\varrho(|y|) \quad \text{for all } y \in \mathbb{R}^{n-1}. \tag{4.1}$$

We consider the change of variables $Y = \Psi(X) = (x, x_n + 3|X|\widetilde{\varrho}(|X|))$, where

$$\widetilde{\varrho}(r) = (\log 2)^{-2}\int_r^{2r}\frac{1}{t}\int_t^{2t}\frac{\varrho(s)}{s}\,ds\,dt;$$

Ψ defines a C^1-diffeomorphism in a neighborhood of zero. A calculation shows that the function $w = u \circ \Psi$ is an H^2_{loc}-solution in $\widetilde{\Omega} = \Psi^{-1}(\Omega)$ to

$$\widetilde{L}w = -\widetilde{D} \cdot (\widetilde{A}\widetilde{D}w) + \widetilde{W}\widetilde{D}w + \widetilde{V}w = 0$$

with $w = 0$ on $\triangle_{r_1}$ for some $0 < r_1 < 1$, where

$$\begin{aligned}
\widetilde{A}(X) &= \det J\Psi(X) J\Psi^{-t}(X) A \circ \Psi(X) J\Psi^{-1}(X),\\
\widetilde{b}(X) &= J\Psi(X) b \circ \Psi(X), \qquad \widetilde{D} = \nabla + \widetilde{b},\\
\widetilde{W}(X) &= \det J\Psi(X) J\Psi^{-t}(X) W \circ \Psi(X),\\
\widetilde{V}(X) &= \det J\Psi(X) V \circ \Psi(X),
\end{aligned}$$

where $J\Psi(X)$ denotes the Jacobian of $\Psi(X)$. Then it is not difficult to show that the operator $\widetilde{L}$ satisfies Assumptions (A) and (B) with $C_1\theta_0(C_2 r)$ replacing $\theta_0(r)$, where C_1 and C_2 depend on n and λ. Moreover, for $P \in \triangle_{r_1}$ and taking r_1 smaller if necessary, we can see from (4.1) that

$$\begin{aligned}
&\widetilde{A}(P)P \cdot \nu(P)\\
&\qquad \geq \frac{1}{2}\,|P|\theta(|P|) + \det J\Psi(P) J\Psi(P)^{-t}(A \circ \Psi(P) - I) J\Psi(P)^{-1} P \cdot \nu(P)\\
&\qquad = \frac{1}{2}\,|P|\theta(|P|) + O(|P|^2) \geq 0,
\end{aligned}$$

and $\widetilde{A}(0) = I$. Thus, Theorem 1.1 implies the doubling property for w and as a consequence for u, which implies that u cannot vanish to infinite order. This proves the corollary.

Proof of Corollary 1.2. Using the notation of the proof of Corollary 1.1 and putting $\varrho = \varepsilon$, we can find $r(\varepsilon) > 0$ for each $\varepsilon > 0$ such that

$$y\nabla\varphi(y) - \varphi(y) \geq -2\varepsilon|y| \quad \text{for } |y| \leq r(\varepsilon).$$

Proceeding as in the proof of Corollary 1.1, we deduce that $\widetilde{A}(P)P \cdot \nu(P) \geq 0$ on $\triangle_{r(\varepsilon)}$. At this point, the argument for (3.14) and (3.5) in the proof of Theorem 1.1 implies that the functions $H(r)$ and $Z(r)$ associated with $w, \widetilde{L}$ and $\widetilde{\Omega}$ satisfy the differential inequalities

$$Z'(r) \geq -\frac{C\varepsilon}{r}\,Z(r), \qquad r\frac{d}{dr}\log H(r) \leq 2Z(r) + C\varepsilon$$

for all $r \leq r(\varepsilon)$, where C depends only on n and λ. Integrating these inequalities we obtain

$$H(r) \geq H(r(\varepsilon))\left[\frac{r}{r(\varepsilon)}\right]^{C\varepsilon} \exp\left(-\frac{2}{C\varepsilon}\left[\frac{r}{r(\varepsilon)}\right]^{C\varepsilon} Z(r(\varepsilon))\right)$$

for all $r < r(\varepsilon)$. But the assumption on u in Corollary 1.2 implies that $H(r) \leq C\exp(-r^{-\beta/2})$ for all $r \leq r(\varepsilon)$. From this, and taking $\varepsilon > 0$ such that $C\varepsilon \le \beta/2$, we see that $H(r(\varepsilon))$ must be equal to zero. Thus, u must be identically zero on an open neighborhood of zero, which proves the corollary.

Our ultimate aim is to establish the unique continuation at the boundary, Theorem 1.3. Before doing that we need to prove the following lemmas.

LEMMA 4.1. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ with $Q_0 \in \partial\Omega$, and let u be a nonconstant solution in $T_3(Q_0)$ to $Lu = 0$ vanishing continuously on $\triangle_3(Q_0)$, where L is an elliptic operator as in* (1.1) *satisfying Assumptions* (A) *and* (B). *Then there exist constants C such that*, *for any $Q \in \triangle_1(Q_0)$ and all $0 < r < 1$*,

$$(4.2)\qquad \Big\{ \int_{\triangle_r(Q)} \left| \frac{\partial u(Q)}{\partial \nu_A} \right|^2 d\sigma \Big\}^{1/2} \le C r^{-(n+3)/2} \int_{T_{2r}(Q)} |u(X)|\, dX.$$

Proof. Without loss of generality we may assume $Q = 0$ and $A(0) = I$. Let β denote a vector field supported in T_{2r}, $0 < r < 1$, with $|\nabla\beta| \le r^{-1}$, $\beta\cdot\nu \ge C$ on $\triangle_r$ for some positive constant C depending on the Lipschitz character of Ω, and $\beta\cdot\nu \ge 0$ on $\triangle_{2r}$ (see [G]). Recalling the Rellich–Nečas identity (3.9) and integrating over T_{2r}, we get

$$(4.3)\qquad \int_{B_{2r}\cap\partial\Omega} \beta\cdot\nu\, A\nu\cdot\nu \left|\frac{\partial u}{\partial\nu}\right|^2 d\sigma$$
$$= O(1) \int_{B_{2r}\cap\Omega} |Du|^2\, dX + 2\,\mathrm{Re} \int_{B_{2r}\cap\Omega} \beta\cdot\overline{Du}\,[W\cdot Du + Vu]\, dX$$
$$- 2\,\mathrm{Im} \int_{B_{2r}\cap\Omega} \beta_l b_{lk} a_{jk} D_j u \overline{u}\, dX.$$

Arguing as for (3.13), from (4.3) and Lemmas 2.1, 2.2 we obtain

$$C_\lambda \int_{\triangle_r} \left|\frac{\partial u}{\partial\nu}\right|^2 d\sigma \le \frac{C}{r} \int_{T_{2r}} |Du|^2\, dX \le \frac{C}{r} \int_{T_{2r}} |\nabla u|^2\, dX$$

with constants C_λ and C independent of $r < 1$. This proves the lemma.

LEMMA 4.2 ([AE]). *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, L an operator as in* (1.1) *satisfying Assumptions* (A), (B) *and the condition $\int_0^1 (\theta_0(r)/r)\, dr < \infty$. For each $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that if $Q \in \partial\Omega$, $0 < r < 1$, and u is a solution to $Lu = 0$ on $T_{2r}(Q)$ vanishing continuously on $\triangle_{2r}(Q)$, then*

$$\int_{T_r(Q)} |u|\, dX \le C(\varepsilon) r^2 \int_{\triangle_{2r}(Q)} \left|\frac{\partial u}{\partial\nu}\right| d\sigma + \varepsilon \int_{T_{2r}(Q)} |u|\, dX.$$

This lemma follows from the same arguments used in [AE].

Finally we turn to

Proof of Theorem 1.3. Without loss of generality we may assume that $\Gamma = \triangle_6(Q_0)$. We let $u \in H^2_{\mathrm{loc}}(\Omega)$ be a solution to $Lu = 0$ as in Theorem 1.3 and $Q \in \triangle_1(Q_0)$ denote a density point of the set $E = \{Q \in \triangle_1(Q_0) : \nabla u(Q) = 0\}$ whose surface measure is positive; that is,

$$\lim_{r\to 0} \frac{\sigma(\triangle_r(Q) \cap E)}{\sigma(\triangle_r(Q))} = 1. \tag{4.4}$$

By Lemma 4.2, Hölder's inequality and Lemma 4.1, for all $m > 0$ we have

$$\int_{T_r(Q)} |u|\, dX \le \left[C_m \left(\frac{\sigma(\triangle_r(Q) \setminus E)}{\sigma(\triangle_r(Q))} \right)^{1/2} + 2^{-m} \right] \int_{T_{4r}(Q)} |u|\, dX$$

with a constant C_m independent of r. Thus from (4.4), using the doubling property (Theorem 1.1) and choosing m large enough, we find that for all $\varepsilon > 0$ there exists $r(\varepsilon) > 0$ such that

$$\int_{T_r(Q)} |u|\, dX \le \varepsilon \int_{T_r(Q)} |u|\, dX \qquad \text{for all } 0 < r < r(\varepsilon),$$

and this is well known to imply that u vanishes to infinite order at Q.

Hence, using Corollary 1.1, we finish the proof of Theorem 1.3.

Remark 4.1. Inspecting the proof shows that the assumptions on W may be replaced with $|W| \in L^\infty_{\mathrm{loc}}(\Omega)$. In particular, Corollary 1.5 follows from Theorem 1.3.

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