STUDIA MATHEMATICA 151 (1) (2002)

Bounds for quotients in rings of formal power series with growth constraints

by

VINCENT THILLIEZ (Lille)

Abstract. In rings Γ_M of formal power series in several variables whose growth of coefficients is controlled by a suitable sequence $M = (M_l)_{l\ge 0}$ (such as rings of Gevrey series), we find precise estimates for quotients F/Φ, where F and Φ are series in Γ_M such that F is divisible by Φ in the usual ring of all power series. We give first a simple proof of the fact that F/Φ belongs also to Γ_M, provided Γ_M is stable under derivation. By a further development of the method, we obtain the main result of the paper, stating that the ideals generated by a given analytic germ in rings of ultradifferentiable germs are closed provided the generator is homogeneous and has an isolated singularity in $\mathbb{R}^n$. The result is valid under the aforementioned assumption of stability under derivation, and it does not involve (non-)quasianalyticity properties.

1. Introduction and statement of results. Let $M = (M_l)_{l\ge 0}$ be a non-decreasing sequence of positive numbers, with $M_0 = 1$. We consider the set Γ_M of those formal power series $F = \sum_{J\in\mathbb{N}^n} F_J X^J$, in n indeterminates $(X_1, \dots, X_n) = X$, with complex coefficients, for which there exist positive constants C_1 and C_2, depending on F, such that the estimate

$$|F_J| \le C_1 C_2^j M_j \tag{1}$$

holds for every integer j and every multi-index J of length j. In what follows, we shall always make the following essential assumption:

(2) the sequence M is logarithmically convex.

Then the set Γ_M becomes a ring for the usual operations on power series. Of course, the ring of convergent power series is a subring of Γ_M; it coincides with Γ_M if and only if $\sup_{l\ge 1}(M_l)^{1/l} < \infty$. A classical example of a ring Γ_M is obtained by putting $M_l = l!^\alpha$, where α is some positive number; Γ_M is then the ring of Gevrey series of order α well known in the theory of

2000 *Mathematics Subject Classification*: 32B05, 13F25, 26E10.

Key words and phrases: formal power series, division theorems, analytic and ultradifferentiable function germs.

differential equations (either ordinary or partial); see for instance [6], [9], [14], [15] and the many references therein.

In [2], Chaumat and Chollet have given a necessary and sufficient condition for the ring $\varGamma_M$ to be noetherian: namely, the existence of a positive constant A such that

$$M_{l+1} \leq A^{l+1} M_l \quad \text{for every } l \in \mathbb{N}. \tag{3}$$

We shall say that a sequence M satisfying (2) and (3) is *admissible*. Condition (3) actually amounts to saying that $\varGamma_M$ is stable under formal derivation. The aforementioned work of Chaumat and Chollet relies on a delicate adaptation of the Weierstrass division theorem in the setting of rings $\varGamma_M$ (however, a much simpler proof has recently been given by Mouze [7]). Another important consequence of these results can be stated as follows:

THEOREM 1. *Assume that the sequence M is admissible. Let $\varPhi$ and F be two elements of the ring $\varGamma_M$ such that F is divisible by $\varPhi$ in the ring $\mathbb{C}[[X]]$ of all formal power series. Then the quotient $F/\varPhi$ belongs to $\varGamma_M$.*

In [12] and [13], this property plays an important role in the study of ideals generated by a real-analytic germ in rings of ultradifferentiable germs of functions at the origin in $\mathbb{R}^n$. As a starting point of the present article, we shall give a direct "elementary" proof of Theorem 1. The task is not superfluous, since our particular method of proof will, in fact, lead us to a new result on ideals of ultradifferentiable germs.

Let $C^\infty(\mathbb{R}^n, 0)$ denote the ring of C^∞ function germs at the origin in $\mathbb{R}^n$. Being given an admissible sequence M, the Carleman class of germs $C_M(\mathbb{R}^n, 0)$ is defined (with the usual confusion between germs and representatives) as the set of those elements f of $C^\infty(\mathbb{R}^n, 0)$ for which one can find a neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^n$ and positive constants C_3 and C_4, depending on f, such that the estimate

$$|D^J f(x)| \leq C_3 C_4^j j! M_j \tag{4}$$

holds for every point x in $\mathcal{U}$ and every multi-index J of length j. The classes $C_M(\mathbb{R}^n, 0)$ yield classical scales of regularity between the analytic and the C^∞ ones. The sequence M measures, in some sense, the defect of analyticity of their elements. To this end, one usually has to deal with various growth properties of M. Assuming that M is admissible ensures that $C_M(\mathbb{R}^n, 0)$ has a few fundamental properties, namely that it is a local algebra, stable under composition and derivation, with its maximal ideal generated by the coordinate functions. A much stronger assumption would be the existence of positive constants A and B such that

$$M_{j+k} \leq A^{j+k} M_j M_k \quad \text{for every } (j,k) \in \mathbb{N}^2, \tag{5}$$

$$\sum_{j\geq l}\frac{M_j}{(j+1)M_{j+1}}\leq B\frac{M_l}{M_{l+1}}\quad\text{for every } l\in\mathbb{N}. \tag{6}$$

When the sequence M satisfies (2), (5) and (6), it is said to be *strongly regular*. For instance, the Gevrey sequences $M_l = l!^\alpha$ ($\alpha > 0$) are strongly regular. At this point, it is worth being said that the Taylor series at 0 of any element of $C_M(\mathbb{R}^n, 0)$ is clearly an element of Γ_M, and that a C_M version of Borel's extension theorem (stating conversely that any element of Γ_M is the Taylor series of some C_M germ) holds for strongly regular sequences, but fails as soon as (6) is not satisfied; see [8]. Condition (6) is usually called *strong non-quasianalyticity*, by comparison with the classical non-quasianalyticity condition of Denjoy–Carleman: $C_M(\mathbb{R}^n, 0)$ is said to be *non-quasianalytic* if it contains non-trivial flat germs, which happens if and only if

$$\sum_{j\geq 0}\frac{M_j}{(j+1)M_{j+1}}<\infty.$$

We also recall a few basic definitions about topology in Carleman classes: for any smoothly bounded neighborhood $\mathcal{U}$ of 0, define $\|f\|_{\mathcal{U},C_4}$ as the smallest C_3 for which (4) holds in $\mathcal{U}$. Let $C_{M,C_4}(\overline{\mathcal{U}})$ be the Banach space given by those elements f of $C^\infty(\mathcal{U})$ having finite norm $\|f\|_{\mathcal{U},C_4}$, and denote by $C_M(\overline{\mathcal{U}})$ the inductive limit of $C_{M,C_4}(\overline{\mathcal{U}})$ as C_4 increases.

Now let φ be a real-analytic function germ at the origin in $\mathbb{R}^n$, and let $I_{\varphi,M}$ be the ideal generated by φ in $C_M(\mathbb{R}^n, 0)$. We shall say that $I_{\varphi,M}$ is *closed* if any element of $C_M(\mathbb{R}^n, 0)$ which, for some suitable neighborhood $\mathcal{U}$ of 0, belongs to the closure of the ideal generated by φ in $C_M(\overline{\mathcal{U}})$, actually belongs to $I_{\varphi,M}$ (in other words, it belongs to $\varphi C_M(\overline{\mathcal{V}})$ for some neighborhood $\mathcal{V}$ of 0, maybe smaller than $\mathcal{U}$). Replacing $C_M(\overline{\mathcal{U}})$ by $C^\infty(\mathcal{U})$, a similar definition could be given in $C^\infty(\mathbb{R}^n, 0)$, but not usefully, since it is known, by famous results of Łojasiewicz and Malgrange ([5], Theorem VI.1.1), that the ideal generated by φ in $C^\infty(\mathcal{U})$ is always closed for the usual C^∞ topology. On the contrary, Chaumat and Chollet have observed in [3] that the corresponding statement is generally false in the C_M setting. It is thus natural to find for which germs φ the ideal $I_{\varphi,M}$ is closed. First results on the subject appear in [12] and [13], as byproducts of more general division problems. In particular, the arguments of [13], Theorem 4.2, can be adapted to yield the following useful proposition, which relates closedness of ideals to a special sort of division estimates.

PROPOSITION 1. *Let φ be a germ of real-analytic function at the origin in $\mathbb{R}^n$ and let M be an admissible sequence. A sufficient condition for closedness of the ideal generated by φ in $C_M(\mathbb{R}^n, 0)$ is that the following property $(\mathcal{P})$ be satisfied*:

($\mathcal{P}$) *If g is a germ in $C^\infty(\mathbb{R}^n, 0)$ such that φg belongs to $C_M(\mathbb{R}^n, 0)$, then g belongs in fact to $C_M(\mathbb{R}^n, 0)$.*

If M is strongly regular, property ($\mathcal{P}$) is also necessary for the ideal to be closed.

For the reader's convenience, we shall sketch the proof in Section 4. We just mention now that property ($\mathcal{P}$) will enable us to obtain the main result of the article, which can be stated as follows.

Theorem 2. *Let φ be a homogeneous polynomial whose only critical point in $\mathbb{R}^n$ is 0. Then, for any admissible sequence M, the ideal generated by φ in $C_M(\mathbb{R}^n, 0)$ is closed.*

It is important to notice that, besides the homogeneity, the condition on critical points is essential here: we shall describe, at the end of the article, an example in $\mathbb{R}^3$ showing that the theorem is no longer true if this assumption is omitted. We also mention that the conclusion of Theorem 2 was already known in the very special case of a strongly regular sequence M and a positive-definite homogeneous polynomial φ: in this situation, it can indeed be deduced from Theorem 2.6 in [12] since, for a homogeneous φ, being positive- (or negative-) definite amounts to saying that the set X_φ of real zeros of φ is reduced to $\{0\}$. Notice that the critical point 0 is then automatically isolated in $\mathbb{R}^n$, by virtue of the Euler identity $\varphi(x) = (\deg \varphi)^{-1}\langle \nabla\varphi(x), x\rangle$. Therefore, in the particular setting of homogeneous polynomials, Theorem 2 is much more general than the statement implied by [12].

Theorem 2 should also be compared with another result. In [13], we have been able to obtain, in the strongly regular case, a necessary and sufficient condition for closedness of the ideal generated in $C_M(\mathbb{R}^2, 0)$ by a real-analytic germ φ of two variables. It turns out that this condition is always satisfied by homogeneous germs, even if they have non-isolated real critical points. Therefore, in the 2-dimensional setting, *any* homogeneous polynomial generates a closed ideal in $C_M(\mathbb{R}^2, 0)$. But, as we have already observed, this is generally false in higher dimensions if the assumption on critical points is omitted.

Thus, one of the main interests of Theorem 2 lies in the fact that, without any restriction on the dimension, it gives a positive result for a class of generators φ whose real zero set X_φ is not reduced to a single point as in [12]. Another important feature of the theorem is that it works under weak assumptions on M. In particular, it does not involve non-quasianalyticity properties, contrarily to [12] and [13] whose computations and tools rely heavily on the strong regularity of M. In fact, for many questions of differential analysis in C_M classes, the strongly regular case is currently (and by far) the best understood. We emphasize that Theorem 2 is a first result of closedness for ideals in a more general situation.

The paper is organized as follows. In Section 2, we describe elementary lemmas which will enable us to give a simple proof of Theorem 1. Section 3 is devoted to proving Theorem 1, in a slightly more general form (Theorem 3). The last section is devoted to Theorem 2. As in [13], the most delicate part of the proof (Proposition 2) consists in showing first that property $(\mathcal{P})$ holds when restricted to the zero set X_φ. Nevertheless, the method used here to get this division estimate is definitely different from the arguments of [13], which are based on Puiseux's theorem. In the present paper, the homogeneity and the isolated singularity of φ allow us to proceed in the spirit of Theorem 1, using bounds on quotients of formal power series. This is the reason why both results are closely related.

NOTATIONS. For any point ζ in $\mathbb{C}^n$ and any positive r, let $B(\zeta, r)$ denote the open euclidean ball in $\mathbb{C}^n$, with center ζ and radius r. Let $\mathbb{S}$ be the unit sphere $\partial B(0,1)$ in $\mathbb{C}^n$. For any multi-index $J = (j_1, \ldots, j_n)$ in $\mathbb{N}^n$, we denote by the corresponding lowercase letter j the length $j_1 + \ldots + j_n$ of J. For $v = (v_1, \ldots, v_n)$, we put $v^J = v_1^{j_1} \ldots v_n^{j_n}$ and we denote by D_v^J, or D^J if it causes no confusion, the operator $\partial^j/\partial v_1^{j_1} \ldots \partial v_n^{j_n}$. For any holomorphic polynomial P in $\mathbb{C}^n$, we denote by $|P|_\infty$ the supremum of the moduli of the coefficients of P. Let l be a non-negative integer. With every formal power series F, we associate its homogeneous part of degree l, that is, the polynomial $F^{(l)}$ given by

$$F^{(l)}(z) = \sum_{J \in \mathbb{N}^n;\, j=l} F_J z^J \quad \text{for every } z \in \mathbb{C}^n.$$

For any function h smooth in a neighborhood of a point a in $\mathbb{R}^n$, we denote by $T_a^l h$ the Taylor polynomial of degree l of h at a. Finally, if E is a compact subset of $\mathbb{C}^n$, we denote by $\|\cdot\|_E$ the supremum norm for bounded functions on E.

2. Basic tools. Our approach to division estimates in Γ_M requires two polynomial lemmas stated below. The second one involves a bit of pluripotential theory; this is the only point where the section is not fully elementary.

LEMMA 1. *For any holomorphic polynomial P in $\mathbb{C}^n$, one has*

$$(1/\sqrt{n})^{\deg P} |P|_\infty \le \|P\|_{\mathbb{S}} \le (2^n)^{\deg P} |P|_\infty.$$

Proof. The first inequality can be obtained by applying the Cauchy formula on the polydisc $D = \{z \in \mathbb{C}^n : |z_i| < 1/\sqrt{n} \text{ for } i = 1, \ldots, n\}$. It is then enough to see that $\overline{D}$ is contained in the closed unit ball of $\mathbb{C}^n$ and that we have therefore $\|P\|_{\overline{D}} \le \|P\|_{\mathbb{S}}$ by the maximum principle. To get the second inequality, one can simply use the rough majorization of $\|P\|_{\mathbb{S}}$ by the sum of the moduli of the coefficients of P and notice that the number of monomials of degree at most m is majorized by $(m+1)^n$, hence by 2^{nm}. ∎

LEMMA 2. *For any real $\varepsilon > 0$, there exists a positive constant $C_{n,\varepsilon}$, depending only on n and ε, such that, for any holomorphic polynomial P in $\mathbb{C}^n$ and any point ζ on $\mathbb{S}$, one has*

$$\|P\|_{\mathbb{S}} \leq (C_{n,\varepsilon})^{\deg P} \|P\|_{\mathbb{S}\cap\overline{B}(\zeta,\varepsilon)}.$$

Proof. The lemma is essentially a particular case of the Bernstein–Walsh –Siciak inequality, but we shall include some details for the reader's convenience. Let E be a compact subset of $\mathbb{C}^n$ and let ϕ_E denote Siciak's extremal function, which with each z in $\mathbb{C}^n$ associates $\phi_E(z) = \sup\{|Q(z)|^{1/\deg Q} : Q \in \mathcal{P}_E\}$, where $\mathcal{P}_E$ is the set of all non-constant holomorphic polynomials satisfying $\|P\|_E \leq 1$. If E is the closure of some open subset of $\mathbb{S}$, it is non-pluripolar in $\mathbb{C}^n$ (this can be seen in the following way: assume that there exists a function u plurisubharmonic in $\mathbb{C}^n$ such that $u \equiv -\infty$ on E, pick a finite family $\varrho_1, \ldots, \varrho_N$ of unitary transformations such that $\bigcup_{1\leq i\leq N} \varrho_i(E) = \mathbb{S}$, and put $\widetilde{u} = \sum_{1\leq i\leq N} u \circ \varrho_i^{-1}$. The function $\widetilde{u}$ would be plurisubharmonic in $\mathbb{C}^n$ and satisfy $\widetilde{u} \equiv -\infty$ on $\mathbb{S}$, hence a contradiction, by the maximum principle). Therefore, ϕ_E is bounded on compact subsets of $\mathbb{C}^n$; see e.g. [11], Section 3. Choosing a point ζ_0 in $\mathbb{S}$ and using the above fact with $E = \mathbb{S}\cap\overline{B}(\zeta_0, \varepsilon)$, one easily gets the desired inequality at the particular point $\zeta = \zeta_0$, with $C_{n,\varepsilon} = \|\phi_E\|_{\mathbb{S}}$. The same result for any other point ζ of $\mathbb{S}$ immediately follows by considering polynomials $P \circ \varrho$, where ϱ is a unitary transformation such that $\varrho(\zeta_0) = \zeta$. Notice, finally, that another proof of Lemma 2 can be found in [1], Example 1.1, in a more general perspective. ∎

Finally, we shall use a simple estimate in the ring $\mathbb{C}[[T]]$ of formal power series in one indeterminate T on $\mathbb{C}$. Let $U = \sum_{j\geq 0} u_j T^j$ and $V = \sum_{j\geq 0} v_j T^j$ be two such series, and let $W = \sum_{j\geq 0} w_j T^j$ denote their product. Assuming that U is non-zero, we denote its order by ν, in such a way that we have

$$u_0 = \ldots = u_{\nu-1} = 0 \quad \text{and} \quad u_\nu \neq 0.$$

We put $\Delta_0(U) = 1$ and, for every integer $k \geq 1$,

$$\Delta_k(U) = \sup_{1\leq j\leq k} \left|\frac{u_{\nu+j}}{u_\nu}\right|^{1/j}.$$

LEMMA 3. *With the above notations, we have $w_0 = \ldots = w_{\nu-1} = 0$ and, for any integer $j \geq 0$, we have the equality*

$$v_j = \frac{1}{u_\nu}\Big(\sum_{i=0}^{j} \lambda_{ji}(U) w_{\nu+i}\Big) \tag{7}$$

where the coefficients $\lambda_{ji}(U)$ *satisfy*

$$|\lambda_{ji}(U)| \le (2\Delta_{j-i}(U))^{j-i} \quad \text{for } 0 \le i \le j. \tag{8}$$

Proof. The vanishing of $w_0, \ldots, w_{\nu-1}$ is obvious. In order to get (7) and (8), we start from the equality $v_j = u_\nu^{-1}(w_{\nu+i} - \sum_{k=0}^{j-1} u_{\nu+j-k}v_k)$, which is a trivial consequence of the assumption $W = UV$ and of the definition of ν. We derive easily the expression (7) for v_j, where the $\lambda_{ji}(U)$ are given inductively by

$$\lambda_{jj}(U) = 1 \quad \text{and} \quad \lambda_{ji}(U) = -\sum_{k=i}^{j-1} \frac{u_{\nu+j-k}}{u_\nu}\lambda_{ki}(U) \quad \text{for } 0 \le i \le j-1. \tag{9}$$

Now, observe that

$$\left|\frac{u_{\nu+j-k}}{u_\nu}\right| \le (\Delta_{j-k}(U))^{j-k} \le (\Delta_{j-i}(U))^{j-k} \quad \text{for } 0 \le i \le k \le j. \tag{10}$$

From (9) and (10), it is easy to prove (8) by induction on j. ■

3. Formal estimates. As announced, we now give some bounds for quotients in Γ_M. Notice that Theorem 1 of Section 1 is just a corollary of the result below: putting $G = F/\Phi$ in the statement and assuming moreover that M satisfies (3), it is clear that F/Φ belongs to Γ_M.

THEOREM 3. *Suppose the sequence* M *only satisfies* (2). *Let* Φ *and* G *be two formal power series in* $\mathbb{C}[[X]]$. *Put* $F = \Phi G$ *and assume that*:

- *the series* Φ *is non-zero and belongs to the ring* Γ_M;
- *the series* F *belongs to* Γ_M.

Then, *for any multi-index* J,

$$|G_J| \le C_1(C\sup(1, C_2))^{j+\nu}M_{j+\nu},$$

where the constants C_1 *and* C_2 *are associated with* F *by* (1), *the number* ν *is the order of* Φ *and* C *is a positive constant depending only of* Φ.

Proof. By Borel's classical extension theorem, we can find two functions $\widetilde{\Phi}$ and $\widetilde{G}$ of class C^∞ in $\mathbb{C}^n$ such that, for any multi-indices J and K in $\mathbb{N}^n$,

$$\begin{aligned} \frac{1}{J!K!}D_z^J D_{\bar z}^K \widetilde{\Phi}(0) &= \begin{cases} \Phi_J & \text{if } K = 0, \\ 0 & \text{otherwise}, \end{cases} \\ \frac{1}{J!K!}D_z^J D_{\bar z}^K \widetilde{G}(0) &= \begin{cases} G_J & \text{if } K = 0, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{11}$$

Pick a point z on $\mathbb{S}$ and consider the product $\widetilde{\Phi}(tz)\widetilde{G}(tz)$, viewed as a function of one real variable t. Taking Taylor series at 0 and using (11), we obtain, in $\mathbb{C}[[T]]$, the identity

$$\sum_{j\geq 0} w_j(z)T^j = \Big(\sum_{j\geq 0} u_j(z)T^j\Big)\Big(\sum_{j\geq 0} v_j(z)T^j\Big) \tag{12}$$

with $u_j(z) = \Phi^{(j)}(z)$, $v_j(z) = G^{(j)}(z)$ and $w_j(z) = F^{(j)}(z)$ for any integer $j \geq 0$ and any z in $\mathbb{S}$. The assumption on F, together with Lemma 1, ensures that

$$|w_j(z)| \leq C_1(8^n C_2)^j M_j \quad \text{for any } j \in \mathbb{N} \text{ and } z \in \mathbb{S}, \tag{13}$$

where C_1 and C_2 are associated with F by (1). For the same reason, there exist positive constants C_5 and C_6, depending only on Φ, such that

$$|u_j(z)| \leq C_5 C_6^j M_j \quad \text{for any } j \in \mathbb{N} \text{ and } z \in \mathbb{S}. \tag{14}$$

Denote by U_z the series $\sum_{j\geq 0} u_j(z)T^j$. By definition of the order ν of the series Φ, we have $u_0(z) = \ldots = u_{\nu-1}(z) = 0$ for any z of $\mathbb{S}$, whereas the homogeneous polynomial u_ν is not identically zero. Let ζ_0 be a point of $\mathbb{S}$ satisfying $u_\nu(\zeta_0) \neq 0$. For a sufficiently small $\varepsilon > 0$, there exists a constant $\eta > 0$, depending only on Φ, such that $|u_\nu| \geq \eta$ on $\mathbb{S} \cap \overline{B}(\zeta_0, \varepsilon)$. Taking into account the estimate (14) and the fact that $(M_{j+\nu})^{1/j}$ increases with j by virtue of (2), it is then easily seen that there exists a positive constant C_7, depending only on Φ, such that, with the notations of Lemma 3,

$$\Delta_k(U_z) \leq C_7 (M_{k+\nu})^{1/k} \quad \text{for any } k \in \mathbb{N} \text{ and } z \in \mathbb{S} \cap \overline{B}(\zeta_0, \varepsilon).$$

Using (8), we thus have

$$|\lambda_{ji}(U_z)| \leq (2C_7)^{j-i} M_{j-i+\nu} \quad \text{for } 0 \leq i \leq j \text{ and } z \in \mathbb{S} \cap \overline{B}(\zeta_0, \varepsilon). \tag{15}$$

By virtue of (13) and (15), applying Lemma 3 to the product of series (12) yields the estimate

$$|v_j(z)| \leq \eta^{-1} \sum_{i=0}^{j} (2C_7)^{j-i} M_{j-i+\nu} C_1 (8^n C_2)^{i+\nu} M_{i+\nu} \tag{16}$$

for any integer $j \geq 0$ and any point z of $\mathbb{S} \cap \overline{B}(\zeta_0, \varepsilon)$. The logarithmic convexity of M also easily implies $M_{j-i+\nu} M_{i+\nu} \leq M_\nu M_{j+\nu}$ for $0 \leq i \leq j$. Combining this estimate and (16), we obtain, for suitable constants C_8 and C_9, depending only on Φ, the majorization

$$\|G^{(j)}\|_{\mathbb{S}\cap \overline{B}(\zeta_0,\varepsilon)} \leq C_8 C_1 (C_9 \sup(1, C_2))^{j+\nu} M_{j+\nu} \quad \text{for any } j \in \mathbb{N}. \tag{17}$$

From (17), it is enough to apply Lemma 2 first, then Lemma 1, to complete the proof of Theorem 3. ■

Remark. Incidentally, Theorem 3 proves that Γ_M is a local ring whose maximal ideal is given by the set of those elements of Γ_M without constant term. Indeed, if Φ has a non-zero constant term, that is, if $\nu = 0$, Theorem 3 implies that Φ is invertible in Γ_M. This result can be seen as an old part of the folklore on rings Γ_M, under various assumptions on M. In fact, most of the classical papers deal with functions rather than formal series, but some of them (see e.g. [10], Theorem 13 and its Corollary) use majorizations which also apply in the formal case. This actually goes back to the work of Gevrey (Section I.2 of [4]).

4. Estimates for ultradifferentiable germs. In what follows, we always work within the assumptions of Theorem 2; in other words the sequence M satisfies (2) and (3) and we consider some real homogeneous polynomial φ having no other critical point than 0 in $\mathbb{R}^n$. The polynomial φ can obviously be viewed as the restriction to $\mathbb{R}^n$ of a holomorphic polynomial on $\mathbb{C}^n$; we can thus consider the set Z_φ of complex zeros of φ in $\mathbb{C}^n$. We denote by X_φ the set $Z_\varphi \cap \mathbb{R}^n$ of its real zeros. We remark that no particular assumption is made on non-real critical points of φ.

Now, as explained in the introduction, we have to show that property $(\mathcal{P})$ holds. Let g be a germ in $C^\infty(\mathbb{R}^n, 0)$ such that φg belongs to $C_M(\mathbb{R}^n, 0)$. Put $f = \varphi g$. For any point a in a neighborhood of 0, we have the identity of formal power series

$$F_a = \Phi_a G_a, \tag{18}$$

with

$$F_a = \sum_{J \in \mathbb{N}^n} \frac{1}{J!} D^J f(a) X^J, \qquad \Phi_a = \sum_{J \in \mathbb{N}^n} \frac{1}{J!} D^J \varphi(a) X^J,$$

$$G_a = \sum_{J \in \mathbb{N}^n} \frac{1}{J!} D^J g(a) X^J.$$

The assumption on f implies that F_a belongs to Γ_M. Besides this, Φ_a is just the polynomial $\varphi(X + a)$. Theorem 1 shows therefore that one can find two positive numbers $C(a)$ and $D(a)$, depending on a, such that, for any multi-index J,

$$|D^J g(a)| \leq C(a) D(a)^j j! M_j. \tag{19}$$

However, it is not difficult to see that the estimates found in the proof of Theorem 3 are by no means sufficient to deduce that $C(a)$ and $D(a)$ are uniform with respect to a. Thus, starting from (19), we have to show that one can actually replace them by uniform constants. This will be done in several steps below. Before this, we sketch, as announced, a proof for Proposition 1 of Section 1.

Proof of Proposition 1. We show first that property $(\mathcal{P})$ is sufficient for closedness. Since the C_M topology is clearly stronger than the C^∞ topology, every germ f belonging to the C_M closure of the ideal $\varphi C_M(\overline{\mathcal{U}})$ also belongs to the C^∞ closure of the ideal $\varphi C^\infty(\mathcal{U})$, hence to $\varphi C^\infty(\mathcal{U})$ itself, as explained in the introduction. Writing $f = \varphi g$ and applying $(\mathcal{P})$, we infer that f belongs to $I_{\varphi,M}$, which is therefore closed. Conversely, assume that $I_{\varphi,M}$ is closed and let g be a C^∞ germ such that φg belongs to $C_M(\mathbb{R}^n, 0)$. We see, using Theorem 1 (as for proving (19) above), that the Taylor series of g at any point a of a suitable neighborhood $\mathcal{U}$ of 0 belongs to Γ_M; in other words, φg belongs formally to the ideal $\varphi C_M(\overline{\mathcal{U}})$. If we also assume that the sequence M is strongly regular, we can then use the C_M version of Whitney's spectral theorem [2] to derive that φg belongs to the closure of $\varphi C_M(\overline{\mathcal{U}})$ in $C_M(\overline{\mathcal{U}})$, hence to $I_{\varphi,M}$. Therefore, g belongs to $C_M(\mathbb{R}^n, 0)$ and property $(\mathcal{P})$ is established. ∎

Before going further, we have to state now a technical lemma, to be used twice in what follows (namely, in the proofs of Proposition 2 and Theorem 2).

Lemma 4. *There exist a neighborhood $\mathcal{V}$ of 0 in $\mathbb{R}^n$ and two constants C_{10} and C_{11}, not depending on a, such that*

$$(20)\qquad |D^K(f - \varphi T_a^l g)(x)| \le C_{10} C(a)(C_{11} D(a))^l k! M_l |x - a|^{l+1-k},$$

$$(21)\qquad |D^K T_a^l g(x)| \le C_{10} C(a)(C_{11} D(a))^l k! M_l,$$

for any a and x in $\mathcal{V}$, any integer $l \ge 0$ and any multi-index K of length $k \le l+1$.

Proof. Using the Taylor formula, we notice first that

$$(22)\qquad |D^K(f - T_a^l f)(x)| \le C_{12} C_{13}^l k! M_{l+1} |x - a|^{l+1-k},$$

for some suitable C_{12}, C_{13} depending only on n and on the constants C_3, C_4 associated with f by (4). On the other hand, if we put $m = \deg \varphi$, a direct computation yields

$$(23)\qquad (T_a^l f - \varphi T_a^l g)(x) = \sum_{p=l+1}^{l+m} \sum_{q=0}^{l} (\Phi_a^{(p-q)} G_a^{(q)})(x - a).$$

By the Leibniz formula, for each term on the right-hand side of (23), we have

$$(24)\qquad D^K(\Phi_a^{(p-q)} G_a^{(q)}) = \sum_{H+I=K} \frac{K!}{H!I!} (D^H \Phi_a^{(p-q)})(D^I G_a^{(q)}).$$

Now we estimate each term on the right-hand side of (24). Since φ is a polynomial and $\Phi_a = \varphi(X + a)$, one immediately gets the rough majorization

$$(25)\qquad |D^H \Phi_a^{(p-q)}(x - a)| \le C_{14} |x - a|^{p-q-h}$$

for any p, q and H, with a suitable constant C_{14} depending only on φ. Now recall that

$$(26)\qquad G_a^{(q)}(x - a) = \sum_{J \in \mathbb{N}^n : j = q} \frac{1}{J!} D^J g(a)(x - a)^J.$$

The derivative $D^I(x-a)^J$ is zero except when $i_s \le j_s$ for $s=1,\dots,n$. In this situation, one has explicitly $D^I(x-a)^J = (J!/(J-I)!)(x-a)^{J-I}$ and $J!/(J-I)! \le 2^j I! \le 2^j i!$. Hence, in any case,

$$|D^I(x-a)^J| \le 2^j i! |x-a|^{j-i} \tag{27}$$

for all multi-indices I and J. Using (19), (26), (27) and the simple estimate $j! \le n^j J!$, we derive

$$|D^I G_a^{(q)}(x-a)| \le C(a)(C_{15}D(a))^q i! M_q |x-a|^{q-i} \tag{28}$$

for some C_{15} depending only on n. By (24), (25) and (28), and since $h+i=k$, we get

$$|D^K(\Phi_a^{(p-q)}G_a^{(q)})(x-a)| \le C_{14}C(a)(C_{15}D(a))^q C_{16}^k k! M_q |x-a|^{p-k}, \tag{29}$$

with a constant C_{16} depending only on n. In (23), one always has $p \ge l+1$, hence $|x-a|^{p-k} \le |x-a|^{l+1-k}$ since we can assume $|x-a| \le 1$ for any x and a. Taking also into account the conditions $q \le l$ and $k \le l+1$, we see that (23) and (29) yield the majorization

$$|D^K(T_a^l f - \varphi T_a^l g)(x)| \le C_{17}C(a)(C_{18}D(a))^l k! M_l |x-a|^{l+1-k}, \tag{30}$$

for some suitable C_{17}, C_{18}, and maybe after having replaced $D(a)$ by $\sup(1, D(a))$. Together with (22) and property (3) of the sequence M, this yields (20). Finally, the proof of (21) goes along the same lines as (28). ∎

We state now a key step in the proof of Theorem 2. This proposition describes division estimates restricted to the real zero set X_φ. It is thus comparable to Proposition 2.6 of [13].

PROPOSITION 2. *There exist constants C_{19} and C_{20} such that, for any point a of $\mathcal{V} \cap X_\varphi$ and any multi-index J,*

$$|D^J g(a)| \le C_{19} C_{20}^j j! M_j.$$

In other words, for $a \in \mathcal{V} \cap X_\varphi$, the estimate (19) *holds with $C(a) = C_{19}$ and $D(a) = C_{20}$.*

Proof. For any integer $l \ge 0$, put $f_l = f - \varphi T_0^l g$ and $h_l = g - T_0^l g$, so that $f_l = \varphi h_l$. Instead of (18), we shall use the identity

$$F_{l,a} = \Phi_a H_{l,a} \tag{31}$$

with

$$F_{l,a} = \sum_{J \in \mathbb{N}^n} \frac{1}{J!} D^J f_l(a) X^J \quad \text{and} \quad H_{l,a} = \sum_{J \in \mathbb{N}^n} \frac{1}{J!} D^J h_l(a) X^J.$$

Proceeding now as in the proof of Theorem 3, we derive from (31) the formal identity

$$\sum_{j\geq 0} w_j(l,a,z)T^j = \Big(\sum_{j\geq 0} u_j(a,z)T^j\Big)\Big(\sum_{j\geq 0} v_j(l,a,z)T^j\Big) \tag{32}$$

with $u_j(a,z) = \Phi_a^{(j)}(z)$, $v_j(l,a,z) = H_{l,a}^{(j)}(z)$ and $w_j(l,a,z) = F_{l,a}^{(j)}(z)$ for any integer $j \geq 0$, any point a of $\mathcal{V} \cap X_\varphi$ and any point z in $\mathbb{S}$. Denote by $U_{a,z}$ the series $\sum_{j\geq 0} u_j(a,z)T^j$. The homogeneity of φ and the assumption of isolated real critical point at 0 show that we have, at any point a of $X_\varphi \setminus \{0\}$, the estimate $|\nabla\varphi(a)| \geq C_{21}|a|^{m-1}$ for $m = \deg\varphi$ and for a constant C_{21} depending only on φ. Consider the point $\zeta_a = |\nabla\varphi(a)|^{-1}\nabla\varphi(a)$ on $\mathbb{S}\cap\mathbb{R}^n$. We then have

$$|\Phi_a^{(1)}(z)| \geq \frac{1}{2}C_{21}|a|^{m-1} \quad \text{for any } a \in X_\varphi\setminus\{0\} \text{ and } z \in \mathbb{S}\cap B(\zeta_a, 1/2). \tag{33}$$

We also have, using once again the homogeneity of the polynomial φ,

$$\|\Phi_a^{(p)}\|_{\mathbb{S}} \leq C_{22}|a|^{m-p} \quad \text{for any } p \in \mathbb{N}, \tag{34}$$

with a constant C_{22} depending only on φ. Thus, for any point a of $\mathcal{V} \cap X_\varphi \setminus \{0\}$, the order ν of the series $U_{a,z}$ equals 1 and, with the notations of Lemma 3, we have, by virtue of (33) and (34), the inequality

$$\Delta_k(U_{a,z}) \leq C_{23}|a|^{-1} \quad \text{for any } k \in \mathbb{N} \text{ and } z \in \mathbb{S}\cap B(\zeta_a, 1/2)$$

with a constant C_{23} depending only on φ. Together with (8), this yields the crucial estimate

$$|\lambda_{ji}(U_{a,z})| \leq \left(\frac{2C_{23}}{|a|}\right)^{j-i} \quad \text{for } 0 \leq i \leq j,\ a \in \mathcal{V}\cap X_\varphi\setminus\{0\}$$
$$\text{and } z \in \mathbb{S}\cap B(\zeta_a, 1/2).$$

Therefore, applying Lemma 3 to the product of series (32) shows that, for any integer $j \geq 0$ and any point a in $\mathcal{V}\cap X_\varphi \setminus \{0\}$,

$$|v_j(l,a,z)| \leq \sum_{i=0}^{j}(2C_{23}/|a|)^{j-i}|w_{i+1}(l,a,z)| \quad \text{for } z \in \mathbb{S}\cap B(\zeta_a, 1/2). \tag{35}$$

Now, using the inequality (20) of Lemma 4 with a replaced by 0 and x replaced by a, we obtain, for every multi-index K of length $k \leq l+1$,

$$|D^K f_l(a)| \leq C_{24}C_{25}^l k! M_l |a|^{l+1-k},$$

with $C_{24} = C_{10}C(0)$ and $C_{25} = C_{11}D(0)$. As a consequence, using the elementary estimate $k! \leq n^k K!$ and Lemma 1, and putting $k = i+1$, we have

$$|w_{i+1}(l,a,z)| \leq C_{26}C_{27}^l M_l |a|^{l-i} \quad \text{for } i \leq l,\ a \in \mathcal{V}\cap X_\varphi\setminus\{0\} \text{ and } z \in \mathbb{S}.$$

Combining this estimate with (35) yields finally

$$\|H_{l,a}^{(j)}\|_{\mathbb{S}\cap\overline{B}(\zeta_a,1/2)} \le C_{28}C_{29}^l M_l |a|^{l-j} \quad \text{for } j \le l \text{ and } a \in \mathcal{V}\cap X_\varphi \setminus \{0\}, \tag{36}$$

with some suitable constants C_{28} and C_{29}. Now, applying Lemma 2 first, then Lemma 1, and recalling that $J! \le j!$, we see that (36) amounts to

$$|D^J(g - T_0^l g)(a)| \le C_{30}C_{31}^l j! M_l |a|^{l-j}$$

for any multi-index J of length $j \le l$ and any a in $\mathcal{V} \cap X_\varphi \setminus \{0\}$. The proposition follows by choosing $l = j$, since, by virtue of the inequality (21) of Lemma 4, we also have

$$|D^J T_0^j g(a)| \le C_{32}C_{33}^j j! M_j$$

for some suitable C_{32} and C_{33}. ∎

Starting from the estimates on X_φ which have been obtained in Proposition 2, we now have to get estimates in a whole neighborhood of 0. As in [13], this can be done with the help of geometric information on the relative position of Z_φ and X_φ. Such information is obtained in Lemmas 5 and 6 and Proposition 3 below. We shall use the following classical abbreviation: if $A(x)$ and $B(x)$ are two positive functions of x on a set X, one writes $A(x) \lesssim B(x)$ (or, equivalently, $B(x) \gtrsim A(x)$) if one can find a real number C, not depending on x, such that $A(x) \le CB(x)$ for any $x \in X$. The notation $A(x) \approx B(x)$ means $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$ simultaneously.

LEMMA 5. *There exists a neighborhood $\mathcal{W}$ of $\mathbb{S} \cap X_\varphi$ in $\mathbb{C}^n$ such that*

$$d(z, \mathbb{S} \cap Z_\varphi) \approx |\varphi(z)| + |z|^2 - 1 \quad \textit{for } z \in \mathcal{W}.$$

Proof. By an obvious compactness argument, it is enough to obtain the estimate in a suitable neighborhood of a given point a of $\mathbb{S} \cap X_\varphi$. To this end, we make the standard identification between the point $z = (x_1 + iy_1, \ldots, x_n + iy_n)$ of $\mathbb{C}^n$ and the point $(x_1, y_1, \ldots, x_n, y_n)$ of $\mathbb{R}^{2n}$. We denote by I the corresponding inclusion $\mathbb{R}^n \to \mathbb{R}^{2n}$ and by J the multiplication by i, that is, $J(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n)$. Now we consider $u_1(z) = \operatorname{Re}\varphi(z)$, $u_2(z) = \operatorname{Im}\varphi(z)$ and $u_3(z) = |z|^2 - 1$. Using the fact that a is real and φ has real coefficients, it is then easy to check that the gradients of u_1, u_2, u_3 (viewed as functions on $\mathbb{R}^{2n}$) are given, at the point a, by

$$\nabla u_1(a) = I\biggl(\frac{\partial\varphi}{\partial x_1}(a), \ldots, \frac{\partial\varphi}{\partial x_n}(a)\biggr),$$
$$\nabla u_2(a) = J(\nabla u_1(a)) \quad \text{and} \quad \nabla u_3(a) = I(2a).$$

None of these three gradients vanishes, since a belongs to $\mathbb{S} \cap \mathbb{R}^n$, hence is not a critical point for φ. Moreover, $\nabla u_j(a)$ is orthogonal to $\nabla u_k(a)$ for $j \ne k$: this claim is obvious for $(j,k) = (1,2)$ and $(j,k) = (2,3)$, whereas for

$(j,k)=(1,3)$, it is a consequence of the fact that

$$\langle \nabla u_1(a), \nabla u_3(a)\rangle = 2\sum_{l=1}^{n} \frac{\partial \varphi}{\partial x_l}(a)a_l = 2(\deg\varphi)\varphi(a) = 0$$

as φ is homogeneous and a belongs to X_φ. By these properties of u_1, u_2, u_3, we can, in particular, find C^∞ functions $u_4,\dots,u_{2n}$ such that $(u_1,\dots,u_{2n})$ is a local real coordinate system in a neighborhood of a. In this system, $\mathbb{S}\cap Z_\varphi$ is the real submanifold given by $u_1=u_2=u_3=0$ and therefore $d(z,\mathbb{S}\cap Z_\varphi)\approx |u_1(z)|+|u_2(z)|+|u_3(z)|\approx |\varphi(z)|+|z|^2-1$. ∎

Lemma 6. *Maybe after shrinking the neighborhood $\mathcal{W}$ of Lemma* 5, *we have*

$$d(x,\mathbb{S}\cap X_\varphi)\approx |\varphi(x)|+|x|^2-1 \quad \textit{for } x\in\mathbb{R}^n\cap\mathcal{W}.$$

Proof. Use the same arguments as in Lemma 5, but in a purely real context, and observe that in $\mathbb{R}^n$, the submanifold $\mathbb{S}\cap X_\varphi$ is given by $u_1=u_3=0$. ∎

The proposition below expresses the fact that Z_φ and $\mathbb{R}^n$ are transverse in the sense of Łojasiewicz's regular separation.

Proposition 3. *There exists a constant C_{34}, depending only on φ, such that*

$$d(x,Z_\varphi)\ge C_{34}d(x,X_\varphi) \quad \textit{for any } x\in\mathbb{R}^n.$$

Proof. Let z be a point of Z_φ such that $|x-z|=d(x,Z_\varphi)$. Two cases have to be discussed, according to the value of $\lambda=|z|$.

First case: $\lambda\le |x|/2$. Then $d(x,Z_\varphi)\ge |x|/2$ and the inequality of Proposition 3 is trivial since, clearly, $d(x,X_\varphi)\le |x|$.

Second case: $\lambda>|x|/2$. Then $\lambda^{-1}x$ belongs to some fixed compact subset E of $\mathbb{R}^n$. We have $d(x,Z_\varphi)=\lambda|\lambda^{-1}x-\lambda^{-1}z|\ge \lambda d(\lambda^{-1}x,\mathbb{S}\cap Z_\varphi)$ since $\lambda^{-1}z$ belongs clearly to $\mathbb{S}\cap Z_\varphi$. Taking Lemmas 5 and 6 into account, we get $d(\lambda^{-1}x,\mathbb{S}\cap Z_\varphi)\approx |\varphi(\lambda^{-1}x)|+|\lambda^{-1}x|^2-1\approx d(\lambda^{-1}x,\mathbb{S}\cap X_\varphi)$ when $\lambda^{-1}x$ belongs to $E\cap\mathcal{W}$. When $\lambda^{-1}x$ belongs to $E\setminus\mathcal{W}$, it is easy to see that $d(\lambda^{-1}x,\mathbb{S}\cap Z_\varphi)\approx 1\approx d(\lambda^{-1}x,\mathbb{S}\cap X_\varphi)$. In particular, one always has

$$d(\lambda^{-1}x,\mathbb{S}\cap Z_\varphi)\gtrsim d(\lambda^{-1}x,\mathbb{S}\cap X_\varphi).$$

Let y be a point of $\mathbb{S}\cap X_\varphi$ such that $|\lambda^{-1}x-y|=d(\lambda^{-1}x,\mathbb{S}\cap X_\varphi)$. Gathering the preceding estimates and noting that λy belongs to X_φ, we thus obtain $d(x,Z_\varphi)\gtrsim \lambda|\lambda^{-1}x-y|=|x-\lambda y|\ge d(x,X_\varphi)$. ∎

We are now ready to conclude.

Proof of Theorem 2. By the Łojasiewicz inequality for analytic functions [5], we know that there exist positive constants C_{35} and ϱ such that

$$|\varphi(z)|\ge C_{35}d(z,Z_\varphi)^\varrho \tag{37}$$

for any z belonging to a neighborhood of 0 in $\mathbb{C}^n$. With the help of (37) and Proposition 3, the Cauchy formula on the polydisk $\{z \in \mathbb{C}^n : |z_i - x_i| \leq (1/(2\sqrt{n}))d(x, Z_\varphi),\ i = 1, \ldots, n\}$ yields

$$\left| D^J \left(\frac{1}{\varphi(x)} \right) \right| \leq C_{36}^{j+1} j! d(x, X_\varphi)^{-(j+\varrho)} \tag{38}$$

for any x sufficiently close to 0 in $\mathbb{R}^n \setminus X_\varphi$ and any multi-index J of length j. Now let a denote a point in $\mathcal{V} \cap X_\varphi$, to be specified later. If we take Proposition 2 into account, inequality (20) of Lemma 4 becomes

$$|D^K(f - \varphi T_a^l g)(x)| \leq C_{37} C_{38}^l k! M_l |x - a|^{l+1-k}, \tag{39}$$

for any integer $l \geq 0$ and any multi-index K of length $k \leq l + 1$, with constants C_{37} and C_{38} not depending on a (one can take explicitly $C_{37} = C_{10}C_{19}$ and $C_{38} = C_{11}C_{20}$). For the same reason, inequality (21) of Lemma 4 gives

$$|D^I T_a^l g(x)| \leq C_{37} C_{38}^l i! M_l \tag{40}$$

for any multi-index I. By (38), (39) and the Leibniz formula, we easily obtain

$$|D^I(g - T_a^l g)(x)| \leq C_{39} C_{40}^l i! M_l \mathcal{S} \quad \text{for } i \leq l + 1, \tag{41}$$

with some suitable constants C_{39}, C_{40}, and with

$$\mathcal{S} = \sup_{j+k=i} (|x - a|^{l+1-k} d(x, X_\varphi)^{-j-\varrho}).$$

Choose $l = i + [\varrho]$ and pick the point a in such a way that $|x - a| = d(x, X_\varphi)$ (clearly, a belongs to $\mathcal{V} \cap X_\varphi$ when x stays in a sufficiently small neighborhood of 0). It is now readily seen that we have $\mathcal{S} \leq 1$. Besides this, applying $[\varrho]$ times property (3) of the sequence M, we obtain $M_{i+[\varrho]} \leq C_{41}^{i+1} M_i$ for some C_{41} depending only on M and on the constant $[\varrho]$. Thus, adding (40) and (41) yields finally

$$|D^I g(x)| \leq C_{42} C_{43}^i i! M_i$$

for any multi-index I, hence the desired result. ■

Remark. The basic idea in the proof of Proposition 2 and Theorem 2 above is to estimate the derivatives of $g = f/\varphi$ by writing $g = (1/\varphi)(f - \varphi P) + P$, where P has good bounds and the rate of vanishing of $f - \varphi P$ at some point a cancels the explosion of $1/\varphi$. Here, P is a Taylor polynomial of g at a, whose degree has to change with the order of the derivative we want to estimate. This scheme of proof can be technically simplified when M is strongly regular. Indeed, in this particular case, we benefit from the C_M version of Borel's theorem. Thus, there is a C_M function $\widetilde{g}$ such that $g - \widetilde{g}$, and consequently $f - \varphi\widetilde{g}$, is flat at a. In this situation, Lemma 4 is no more needed, and the other proofs are slightly simpler. Details are left to the reader. Notice that similar arguments are used in the proof of Theorem 2.1 in [13].

Inspired by techniques of [12], the following example ends the article by showing that it is not possible to remove the assumption of isolated critical point in the statement of Theorem 2.

Example. For $\alpha > 0$, denote by ψ_α the function defined on $\mathbb{R}$ by $\psi_\alpha(t) = \exp(-t^{-1/\alpha})$ for $t > 0$ and $\psi_\alpha(t) = 0$ for $t \leq 0$. It is well known that this function has C_M regularity with $M_l = l!^\alpha$. Now pick two real numbers α and β with $0 < \beta < \alpha$ and put, for any $x = (x_1, x_2, x_3)$ in $\mathbb{R}^3$,

$$\varphi(x) = x_1^2 x_3 + x_2^3 \quad \text{and} \quad g(x) = \frac{\psi_\alpha(x_2)\psi_\beta(x_3)}{x_1^2 x_3 + x_2^3}.$$

Clearly, the critical locus of the homogeneous polynomial φ in $\mathbb{R}^3$ is the axis $\{0\} \times \{0\} \times \mathbb{R}$; the zero set Z_φ "draws a cusp" on the sphere $\mathbb{S}$. In view of the definition of ψ_α and ψ_β, the function g is well defined at each point of $\mathbb{R}^3$; it is identically zero outside the open set $\Omega = \{x : x_2 > 0,\ x_3 > 0\}$. Moreover, it is easy to see, by the flatness of ψ_α and ψ_β, that g is C^∞. Finally, we have $(\varphi g)(x) = \psi_\alpha(x_2)\psi_\beta(x_3)$, which shows that φg belongs to $C_M(\mathbb{R}^3, 0)$ with $M_l = l!^\alpha$. On the other hand, for $x \in \Omega$ and $|x_1| < (x_3 x_2^{-3})^{-1/2}$, we have

$$g(x) = \frac{1}{x_2^3} \cdot \frac{\psi_\alpha(x_2)\psi_\beta(x_3)}{1 + x_1^2 x_3 x_2^{-3}} = \sum_{j=0}^{\infty} (-1)^j \psi_\alpha(x_2)\psi_\beta(x_3) \frac{x_3^j}{x_2^{3j+3}}\, x_1^{2j}.$$

In particular, we deduce, for any integer $j \geq 0$, any $x_2 > 0$ and $x_3 > 0$, the equality

$$\frac{\partial^{2j} g}{\partial x_1^{2j}}(0, x_2, x_3) = (-1)^j (2j)! \psi_\alpha(x_2)\psi_\beta(x_3) \frac{x_3^j}{x_2^{3j+3}}.$$

One thus has

$$\frac{\partial^{2j} g}{\partial x_1^{2j}}(0, j^{-\alpha}, j^{-\beta}) = (-1)^j (2j)! e^{-2j} j^{(3\alpha-\beta)j+3\alpha}.$$

Using the Stirling formula, it is then easy to obtain

$$\left| \frac{\partial^{2j} g}{\partial x_1^{2j}}(0, j^{-\alpha}, j^{-\beta}) \right| \geq c^{j+1} (2j)!^{1+d\alpha} = c^{j+1} (2j)! M_{2j}^d,$$

with

$$d = \frac{3}{2} - \frac{\beta}{2\alpha},$$

and with some suitable constant $c > 0$. Hence g does not belong to any class strictly smaller than $C_{M^d}(\mathbb{R}^3, 0)$. Since we have $d > 1$, this shows that property $(\mathcal{P})$ fails. Moreover M is strongly regular. Therefore, by virtue of the converse part of Proposition 1, the ideal $I_{\varphi,M}$ is not closed.

References

[1] M. Baran and W. Pleśniak, *Polynomial inequalities on algebraic sets*, Studia Math. 141 (2000), 209–219.
[2] J. Chaumat et A.-M. Chollet, *Caractérisation des anneaux noethériens de séries formelles à croissance contrôlée. Application à la synthèse spectrale*, Publ. Mat. Barcelona 41 (1997), 545–561.
[3] —, —, *Sur la division et la composition dans des classes ultradifférentiables*, Studia Math. 136 (1999), 49–70.
[4] M. Gevrey, *Sur la nature analytique des solutions des équations aux dérivées partielles*, Ann. Sci. École Norm. Sup. 35 (1918), 129–190.
[5] B. Malgrange, *Ideals of Differentiable Functions*, Oxford Univ. Press, 1966.
[6] —, *Sur le théorème de Maillet*, Asymptotic Anal. 2 (1989), 1–4.
[7] A. Mouze, *Une nouvelle preuve du théorème de division dans les anneaux de séries formelles à croissance contrôlée*, doctoral thesis, appendix, Univ. of Lille, 2000.
[8] H.-J. Petzsche, *On E. Borel's theorem*, Math. Ann. 282 (1988), 299–313.
[9] J.-P. Ramis, *Dévissage Gevrey*, Astérisque 59–60 (1978), 173–204.
[10] C. Roumieu, *Ultradistributions définies sur $\mathbb{R}^n$ et sur certaines classes de variétés différentiables*, J. Anal. Math. 10 (1962–1963), 153–192.
[11] J. Siciak, *Extremal plurisubharmonic functions in $\mathbb{C}^N$*, Ann. Polon. Math. 39 (1981), 175–211.
[12] V. Thilliez, *On closed ideals in smooth classes*, Math. Nachr. 227 (2001), 143–157.
[13] —, *A sharp division estimate for ultradifferentiable germs*, Pacific J. Math., to appear.
[14] J.-C. Tougeron, *Sur les ensembles semi-analytiques avec conditions Gevrey au bord*, Ann. Sci. École Norm. Sup. 27 (1994), 173–208.
[15] C. Wagschal, *Le problème de Goursat non linéaire*, J. Math. Pures Appl. 58 (1979), 309–337.

CNRS–UMR 8524, Bâtiment M2
Université des Sciences et Technologies de Lille
59655 Villeneuve d'Ascq Cedex, France
E-mail: Vincent.Thilliez@univ-lille1.fr

Received April 18, 2001
Revised version September 10, 2001 (4721)