

On positive solutions of nonlinear elliptic equations involving concave and critical nonlinearities

by

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Abstract. We study the existence of nonnegative solutions of elliptic equations involving concave and critical Sobolev nonlinearities. Applying various variational principles we obtain the existence of at least two nonnegative solutions.

1. Introduction. The main purpose of this paper is to investigate the existence of solutions of the following nonlinear elliptic problem:

$$(1_\varepsilon) \quad \begin{cases} -\Delta u + u = \varepsilon h(x)u^q + u^{2^*-1} & \text{in } \mathbb{R}^N, \\ u > 0, \quad h \in H^1(\mathbb{R}^N), \end{cases}$$

where $\varepsilon > 0$ is a parameter, $0 < q < 1$ and $2^* = 2N/(N-2)$, $N \geq 3$, is the critical Sobolev exponent. We assume that h is a nonnegative and $\neq 0$ function in $L^r(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, where $r = 2^*/(2^* - q - 1)$.

It is well known that equation (1_ε) with $h \equiv 0$, that is,

$$(1) \quad -\Delta u + u = u^{2^*-1} \quad \text{in } \mathbb{R}^N,$$

does not have a positive solution. This is a consequence of the Pokhozhaev identity. By contrast, the equation

$$(2) \quad -\Delta u = u^{2^*-1} \quad \text{in } \mathbb{R}^N$$

has a family of positive solutions $\varepsilon^{-(N-2)/2}U((x-y)/\varepsilon)$, where the function U , called an *instanton*, is given by (see [17])

$$U(x) = \left[\frac{N(N-2)}{N(N-2) + |x|^2} \right]^{(N-2)/2}.$$

We also have $\int_{\mathbb{R}^N} |\nabla U|^2 dx = \int_{\mathbb{R}^N} U^{2^*} dx = S^{N/2}$. The constant S is the best

2000 *Mathematics Subject Classification*: 35J20, 35J60, 35J70.

Key words and phrases: elliptic equations, critical Sobolev exponent, mountain-pass principle, local minimization, concentration-compactness principle.

Research of J. Chabrowski supported by Small ARC Grant.

Research of P. Drábek supported by the Grant Agency of the Czech Republic, Grant no. 201/00/0376.

Sobolev constant given by

$$S = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^{2^*} dx = 1 \right\},$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ denotes the Sobolev space obtained as the completion of the space $C_0^1(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

However, the equation

$$(3) \quad -\Delta u + a(x)u = u^{2^*-1} \quad \text{in } \mathbb{R}^N$$

with nonnegative and nonconstant coefficient $a(x)$ may have a positive solution. In fact, Benci–Cerami [6] proved the existence of a positive solution of (3) provided $\|a\|_{N/2} \leq S(2^{N/2} - 1)$. This result has been extended in [9], where the multiplicity of solutions was expressed in terms of the category of the set $a^{-1}(0)$. Further results, under some integrability assumptions on the coefficient $a(x)$, can be found in [1], [13] and [14]. In Section 3 we show that problem (1_ε) , which is a small concave perturbation of (1), admits a positive solution. This solution is obtained as a local minimizer of a variational functional for (1_ε) . In Section 4, we consider an equation of the form

$$(1_{\varepsilon,s}) \quad -\Delta u + \varepsilon^s u = \varepsilon h(x)u^q + u^{2^*-1} \quad \text{in } \mathbb{R}^N,$$

where $s > 1$. Theorem 4.1 of Section 4 gives the existence of two positive solutions: the first is a mountain-pass type solution and the second is obtained through local minimization of the variational functional for $(1_{\varepsilon,s})$. In Section 5 we establish the existence of infinitely many solutions of equations (1_ε) and $(1_{\varepsilon,s})$ for $\varepsilon > 0$ small. The results of Section 3 show that at least one of the solutions of the equation in (1_ε) is positive. By Theorem 4.1 of Section 4 at least two solutions of $(1_{\varepsilon,s})$ are positive under an additional assumption that $s < 2/(1 - q)$. In Section 6, we extend the local minimization to some nonlinear problems involving the p -Laplacian. Theorems 7.1 and 8.1 of Sections 7 and 8, respectively, complement the results obtained in [16]. In the case of the p -Laplacian the exponent q satisfies the inequality $q < p$ and the corresponding nonlinearity is not necessarily concave.

Throughout our paper we use standard notation and terminology. In a given Banach space X , we denote by “ \rightharpoonup ” weak convergence and by “ \rightarrow ” strong convergence. For $u \in \mathbb{R}$, we let $u^+ = \max(0, u)$ and $u^- = \max(0, -u)$.

Let $F \in C^1(X, \mathbb{R})$. A sequence $\{u_m\}$ is said to be a *Palais–Smale sequence* for F at level c ($(\text{PS})_c$ sequence for short) if $F(u_m) \rightarrow c$ and $F'(u_m) \rightarrow 0$ in X^* as $m \rightarrow \infty$. We say that F satisfies the *Palais–Smale condition at level c* ($(\text{PS})_c$ condition for short) if any $(\text{PS})_c$ sequence is relatively compact in X . By $H^1(\mathbb{R}^N)$ we denote the usual Sobolev space

equipped with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx.$$

2. Palais–Smale condition. For $u \in H^1(\mathbb{R}^N)$ we define the energy functional associated with (1_ε) :

$$J_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx - \frac{\varepsilon}{q+1} \int_{\mathbb{R}^N} h(x)|u|^{q+1} dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx.$$

An elementary analysis of the real-valued function

$$a \mapsto \frac{a^{2^*}}{N} - \varepsilon \left(\frac{1}{q+1} - \frac{1}{2} \right) \|h\|_r a^{q+1}, \quad a \geq 0,$$

shows that it has a global minimum greater than or equal to $-C^* \varepsilon^r$ with some $C^* = C^*(N, q, \|h\|_r) > 0$. Thus we have

$$(4) \quad \frac{a^{2^*}}{N} - \varepsilon \left(\frac{1}{q+1} - \frac{1}{2} \right) \|h\|_r a^{q+1} \geq -C^* \varepsilon^r$$

for every $a \geq 0$.

PROPOSITION 2.1. *The functional J_ε satisfies the $(PS)_c$ condition for*

$$(5) \quad c < \frac{S^{N/2}}{N} - \varepsilon^r C^*.$$

Proof. If $\{u_m\}$ is a $(PS)_c$ sequence with c satisfying (5), then $\{u_m\}$ is bounded in $H^1(\mathbb{R}^N)$. Indeed, there exists an integer m_0 such that

$$\begin{aligned} c + 1 + o(\|u_m\|) &\geq J_\varepsilon(u_m) - \frac{1}{2} \langle J'_\varepsilon(u_m), u_m \rangle \\ &= \frac{q-1}{2(q+1)} \varepsilon \int_{\mathbb{R}^N} h|u_m|^{q+1} dx + \frac{1}{N} \int_{\mathbb{R}^N} |u_m|^{2^*} dx \end{aligned}$$

for all $m \geq m_0$. By the Hölder and Sobolev inequalities, we deduce

$$\int_{\mathbb{R}^N} |u_m|^{2^*} dx \leq C_1 + C_2 \|u_m\|^{q+1} + o(\|u_m\|)$$

for some constants $C_1, C_2 > 0$ and all $m \geq m_0$. On the other hand,

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_m|^2 + u_m^2) dx &= J_\varepsilon(u_m) + \frac{\varepsilon}{q+1} \int_{\mathbb{R}^N} h|u_m|^{q+1} dx + \frac{1}{2^*} \int_{\mathbb{R}^N} |u_m|^{2^*} dx \\ &\leq c + 1 + \frac{\varepsilon}{q+1} \int_{\mathbb{R}^N} h|u_m|^{q+1} dx + \frac{1}{2^*} \int_{\mathbb{R}^N} |u_m|^{2^*} dx \end{aligned}$$

for $m \geq m_0$. Combining the last two estimates we easily derive the boundedness of $\{u_m\}$ in $H^1(\mathbb{R}^N)$. Therefore, we may assume that $u_m \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, $u_m \rightarrow u$ in $L^p(\Omega)$ for each $1 \leq p < 2^*$ and each bounded domain

$\Omega \subset \mathbb{R}^N$. Also, we assume that $u_m \geq 0$ on \mathbb{R}^N . Otherwise we replace J_ε by a functional

$$J_\varepsilon^+(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx - \frac{\varepsilon}{q+1} \int_{\mathbb{R}^N} h(u^+)^{q+1} dx - \frac{1}{2^*} \int_{\mathbb{R}^N} (u^+)^{2^*} dx$$

and observe that every bounded $(\text{PS})_c$ sequence $\{u_n\}$ for J_ε^+ has the property that $u_n^- \rightarrow 0$ in $H^1(\mathbb{R}^N)$. We set

$$\begin{aligned} \mu_\infty &= \lim_{R \rightarrow \infty} \limsup_{m \rightarrow \infty} \int_{|x| \geq R} (|\nabla u_m|^2 + u_m^2) dx, \\ \nu_\infty &= \lim_{R \rightarrow \infty} \limsup_{m \rightarrow \infty} \int_{|x| \geq R} |u_m|^{2^*} dx. \end{aligned}$$

Then by the concentration-compactness principle (see [15] and [8]) we have

$$\begin{aligned} |\nabla u_m|^2 + u_m^2 &\rightharpoonup d\mu = |\nabla u|^2 + u^2 + \sum_{j \in J} \mu_j \delta_{x_j} + \mu_\infty \delta_\infty, \\ |u_m|^{2^*} &\rightharpoonup d\nu = |u|^{2^*} + \sum_{j \in J} \nu_j \delta_{x_j} + \nu_\infty \delta_\infty, \end{aligned}$$

where J is at most a countable set, and $\nu_j, \mu_j > 0$ are constants satisfying

$$(6) \quad S\nu_j^{(N-2)/N} \leq \mu_j$$

for $j \in J \cup \{\infty\}$. Using a family of test functions concentrating at x_j and a family of test functions concentrating at ∞ , we check as in [2] and [8] that

$$(7) \quad \nu_j = \mu_j$$

for all $j \in J \cup \{\infty\}$. We now observe that if $\nu_j \neq 0$ for some $j \in J \cup \{\infty\}$, then by (6) and (7) we have

$$(8) \quad \nu_j \geq S^{N/2}.$$

We show that $\nu_j = 0$ for every $j \in J \cup \{\infty\}$. Arguing by contradiction, assume that $\nu_j > 0$ for some $j \in J \cup \{\infty\}$. Then we have

$$J_\varepsilon(u_m) - \frac{1}{2} \langle J'_\varepsilon(u_m), u_m \rangle = \frac{1}{N} \int_{\mathbb{R}^N} |u_m|^{2^*} dx + \frac{(q-1)\varepsilon}{2(q+1)} \int_{\mathbb{R}^N} h(x) |u_m|^{q+1} dx.$$

Letting $m \rightarrow \infty$ and using (8) we deduce that

$$\begin{aligned} (9) \quad c &= \lim_{m \rightarrow \infty} \left(J_\varepsilon(u_m) - \frac{1}{2} \langle J'_\varepsilon(u_m), u_m \rangle \right) \\ &\geq \frac{1}{N} \int_{\mathbb{R}^N} |u|^{2^*} dx + \frac{S^{N/2}}{N} + \frac{(q-1)\varepsilon}{2(q+1)} \int_{\mathbb{R}^N} h(x) |u|^{q+1}. \end{aligned}$$

If $u \equiv 0$, we deduce from (9) that $c \geq S^{N/2}/N$, which is impossible. If $u \not\equiv 0$, we then derive from (9) and (4) that

$$c \geq \frac{S^{N/2}}{N} - C^* \varepsilon^r,$$

which is impossible. Since $\nu_j = 0$ for every $j \in J \cup \{\infty\}$, we see that $u_m \rightarrow u$ in $L^{2^*}(\mathbb{R}^N)$. To show that $u_m \rightarrow u$ in $H^1(\mathbb{R}^N)$ we write

$$\begin{aligned} o(1) &= \langle J'_\varepsilon(u_n) - J'_\varepsilon(u_m), u_n - u_m \rangle \\ &= \int_{\mathbb{R}^N} (|\nabla(u_n - u_m)|^2 + (u_n - u_m)^2) dx \\ &\quad - \varepsilon \int_{\mathbb{R}^N} h(|u_n|^{q-1}u_n - |u_m|^{q-1}u_m)(u_n - u_m) dx \\ &\quad - \int_{\mathbb{R}^N} (|u_n|^{2^*-2}u_n - |u_m|^{2^*-2}u_m)(u_n - u_m) dx. \end{aligned}$$

Since $u_n \rightarrow u$ in $L^{2^*}(\mathbb{R}^N)$, it is easy to check that the last two integrals converge to 0 as $n, m \rightarrow \infty$. Hence $\{u_n\}$ satisfies the Cauchy condition in $H^1(\mathbb{R}^N)$ and the convergence of $\{u_n\}$ in $H^1(\mathbb{R}^N)$ follows. ■

3. Local minimum. We are now in a position to establish the existence of a local minimum for J_ε .

THEOREM 3.1. *There exists an $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$ problem (1_ε) has a solution u_ε which is a local minimum of J_ε .*

Proof. Using the Sobolev embedding theorem and the Hölder inequality we obtain the following estimate from below for the functional J_ε :

$$\begin{aligned} J_\varepsilon(u) &\geq \frac{1}{2} \|u\|^2 - \frac{\varepsilon}{q+1} \|h\|_r \|u\|^{q+1} - \frac{S^{-2^*/2}}{2^*} \|u\|^{2^*} \\ &= \|u\|^2 \left(\frac{1}{2} - \frac{\varepsilon}{q+1} \|h\|_r \|u\|^{q-1} - \frac{S^{-2^*/2}}{2^*} \|u\|^{2^*-1} \right). \end{aligned}$$

Let $\|u\| = \varrho$ and choose $\varrho > 0$ so that

$$\frac{1}{2} - \frac{S^{-2^*/2}}{2^*} \varrho^{2^*-1} > \varrho_0 > 0,$$

where ϱ_0 is a constant. It then follows from the last two estimates that there exist constants $\varepsilon_0 > 0$ and $c_1 > 0$ such that

$$J_\varepsilon(u) \geq c_1 \varrho^2$$

for all $\|u\| = \varrho$ and $0 < \varepsilon \leq \varepsilon_0$. If $\varphi > 0$, $\varphi \in H^1(\mathbb{R}^N)$, then

$$J_\varepsilon(t\varphi) = \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla\varphi|^2 + \varphi^2) dx - \frac{t^{q+1}\varepsilon}{q+1} \int_{\mathbb{R}^N} h(x)\varphi^{q+1} dx - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^N} \varphi^{2^*} dx < 0$$

for $t > 0$ sufficiently small. Therefore, we have

$$M = \inf_{\|u\| \leq \varrho} J_\varepsilon(u) < 0.$$

We can also assume that ε_0 is chosen so that $S^{N/2}/N - \varepsilon^r C^* \geq 0$ for $0 < \varepsilon \leq \varepsilon_0$. By the Ekeland variational principle [11] there exists a minimizing sequence $\{u_m\} \subset B(0, \varrho)$ such that $J_\varepsilon(u_m) \rightarrow M$ and $J'_\varepsilon(u_m) \rightarrow 0$ in $H^{-1}(\mathbb{R}^N)$. Since the functional J_ε satisfies the $(PS)_M$ condition it is clear from the above construction that the functional J_ε achieves a minimum u_ε at an interior point of $B(0, \varrho)$. Since $J_\varepsilon(u_\varepsilon) = J_\varepsilon(|u_\varepsilon|)$ we may assume that $u_\varepsilon \geq 0$ and by the maximum principle we have $u_\varepsilon > 0$ on \mathbb{R}^N . This completes the proof. ■

REMARK 3.2. It follows from the Hölder and Sobolev inequalities that $\|u_\varepsilon\| \leq r(\varepsilon)$ with $r(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Indeed, since u_ε satisfies (1_ε) we deduce from the inequality

$$J_\varepsilon(u_\varepsilon) - \frac{1}{2} \langle J'_\varepsilon(u_\varepsilon), u_\varepsilon \rangle < 0$$

that

$$\int_{\mathbb{R}^N} u_\varepsilon^{2^*} dx \leq N\varepsilon \frac{1-q}{2(q+1)} \int_{\mathbb{R}^N} h u_\varepsilon^{q+1} dx \leq N\varepsilon \frac{1-q}{2(q+1)} \|h\|_r \left(\int_{\mathbb{R}^N} u_\varepsilon^{2^*} dx \right)^{(q+1)/2^*}$$

and hence

$$\left(\int_{\mathbb{R}^N} u_\varepsilon^{2^*} dx \right)^{(2^*-q-1)/2^*} \leq N\varepsilon \frac{1-q}{2(q+1)} \|h\|_r.$$

Combining this with the inequality $J_\varepsilon(u_\varepsilon) < 0$ we derive our assertion.

REMARK 3.3. Theorem 3.1 remains true if h changes sign, that is, $h^+ \not\equiv 0$ and $h^- \not\equiv 0$.

This follows from the fact that positivity of h was only used to show that $\inf_{\|u\| \leq \varrho} J_\varepsilon(u) < 0$ for small $\varrho > 0$. This also can be shown by choosing $\varphi \in H^1(\mathbb{R}^N)$, $\varphi \geq 0$, $\not\equiv 0$ and with $\text{supp } \varphi \subset \{x : h(x) > 0\}$. This obviously implies that $J_\varepsilon(t\varphi) < 0$ for small $t > 0$.

In Proposition 3.4 below, we show that problem (1_ε) has no solution for ε large.

PROPOSITION 3.4. *There exists $\varepsilon^* > 0$ such that problem (1_ε) has no solution for $\varepsilon \geq \varepsilon^*$.*

Proof. We follow the argument from [12]. Suppose that problem (1_ε) has a solution u_ε for every $\varepsilon > 0$. Let $\varepsilon_k \rightarrow \infty$ and set $u_k = u_{\varepsilon_k}$. Since $h \geq 0$ and $\not\equiv 0$ on \mathbb{R}^N , we may assume without loss of generality that $h(x) > 0$ on $\overline{B(0, R)}$ for some $R > 0$. We denote by $\lambda_1(R) > 0$ the first eigenvalue of the

problem

$$(10) \quad -\Delta u + u = \lambda u, \quad u \in W_0^{1,2}(B(0, R)).$$

Given $\delta > 0$ we choose ε_k sufficiently large so that

$$\varepsilon_k h(x) u^q + u^{2^*} \geq (\lambda_1(R) + \delta) u$$

for all $u \geq 0$ and $x \in \overline{B(0, R)}$. Then u_k is a supersolution of the problem

$$(11) \quad -\Delta u + u = (\lambda_1(R) + \delta) u, \quad u \in W_0^{1,2}(B(0, R)).$$

If $\varphi > 0$ is an eigenfunction corresponding to $\lambda_1(R)$, then for each $t > 0$, $t\varphi$ is a subsolution of (11). We now choose $t > 0$ sufficiently small so that $t\varphi \leq u_k$ on $\overline{B(0, R)}$. Hence we can find a solution v of (11) such that $t\varphi \leq v \leq u_k$ on $B(0, R)$. However, this is impossible for $\delta > 0$ sufficiently small, since the first eigenvalue of (10) is isolated. ■

4. Mountain-pass solution. Inspection of the proofs of Proposition 2.1 and Theorem 3.1 shows that the method of local minimization can be extended to problem $(1_{\varepsilon,s})$. A variational functional for problem $(1_{\varepsilon,s})$ has the form

$$J_{\varepsilon,s}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \varepsilon^s u^2) dx - \frac{\varepsilon}{q+1} \int_{\mathbb{R}^N} h(x) |u|^{q+1} dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx.$$

Using the mountain-pass principle [3] and local minimization we show that problem $(1_{\varepsilon,s})$ has two distinct positive solutions.

THEOREM 4.1. *Suppose that $N \geq 5$ and $1 < s < 2/(1 - q)$. Then there exists $\bar{\varepsilon} > 0$ such that for every $0 < \varepsilon \leq \bar{\varepsilon}$ problem $(1_{\varepsilon,s})$ admits two distinct positive solutions.*

Proof. First, we check that the functional $J_{\varepsilon,s}$ has mountain-pass geometry. We set

$$\|u\|_\varepsilon^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + \varepsilon^s u^2) dx$$

for $\varepsilon > 0$. This is a norm equivalent to $\|\cdot\|$. First, we notice that if $\|v\| = \varrho$, then $\|v^\varepsilon\|_\varepsilon = \varrho \varepsilon^{s/2}$, where $v^\varepsilon(x) = \varepsilon^{Ns/4} v(\varepsilon^{s/2} x)$. We now consider a sphere $\|u\|_\varepsilon = \varrho \varepsilon^{s/2}$ in $H^1(\mathbb{R}^N)$, where $\varrho > 0$ is fixed. For $\|u\|_\varepsilon = \varrho \varepsilon^{s/2}$ we have the following estimate:

$$\begin{aligned} J_{\varepsilon,s}(u) &\geq \varepsilon^s \varrho^2 / 2 - C_1 \varepsilon^{1+(q+1)s/2} \varrho^{q+1} - C_2 \varepsilon^{s2^*/2} \varrho^{2^*} \\ &= \varepsilon^s (\varrho^2 / 2 - C_1 \varepsilon^{1+(q-1)s/2} \varrho^{q+1} - C_2 \varepsilon^{s(2^*-2)/2} \varrho^{2^*}). \end{aligned}$$

From this and $s < 2/(1 - q)$ we deduce that there exists $\varepsilon_1 > 0$ such that to every $0 < \varepsilon \leq \varepsilon_1$ there corresponds $\varrho_\varepsilon > 0$ such that $J_{\varepsilon,s}(u) \geq \varrho_\varepsilon$ for $\|u\|_\varepsilon = \varrho \varepsilon^{s/2}$. It is clear that $J_{\varepsilon,s}(tU) < 0$ for $t > 0$ sufficiently large, where U is the instanton defined in Section 1. We point out here that $J_{\varepsilon,s}(U)$ is

well defined since $N \geq 5$. Therefore, we can define the mountain-pass level

$$c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J_{\varepsilon,s}(u),$$

where

$$\Gamma = \{\gamma \in C([0, 1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) < 0\}.$$

We now show that

$$(12) \quad c_\varepsilon < \frac{S^{N/2}}{N} - C^* \varepsilon^r$$

for $\varepsilon > 0$ sufficiently small. First, we choose $t^* > 0$ and $\varepsilon^* > 0$ small enough so that

$$(13) \quad J_{\varepsilon,s}(tU) \leq \frac{S^{N/2}}{N} - C^* \varepsilon^r$$

for $0 \leq t \leq t^*$ and $0 < \varepsilon \leq \varepsilon^*$. Here ε^* is chosen so that

$$\frac{S^{N/2}}{N} - C^* \varepsilon^r \geq c_0 > 0$$

for all $0 < \varepsilon \leq \varepsilon^*$. To estimate $J_{\varepsilon,s}(tU)$ for $t^* \leq t$ we observe that the function

$$t \mapsto \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla U|^2 + \varepsilon^s U^2) dx - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^N} U^{2^*} dx, \quad t \geq 0,$$

achieves its maximum on $[0, \infty)$ at a point $t_\varepsilon > 0$. If $\varepsilon = 0$, then $t_0 = 1$ and we also have $1 \leq t_\varepsilon \leq t_1$ for $0 \leq \varepsilon \leq 1$. We then have, for $t^* \leq t$,

$$\begin{aligned} J_{\varepsilon,s}(tU) &\leq \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} (|\nabla U|^2 + \varepsilon^s U^2) dx - \frac{t_\varepsilon^{2^*}}{2^*} \int_{\mathbb{R}^N} U^{2^*} dx \\ &\quad - \frac{\varepsilon(t^*)^{q+1}}{q+1} \int_{\mathbb{R}^N} h(x) U^{q+1} dx. \end{aligned}$$

Since

$$\frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} |\nabla U|^2 dx - \frac{t_\varepsilon^{2^*}}{2^*} \int_{\mathbb{R}^N} U^{2^*} dx \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla U|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} U^{2^*} dx = \frac{S^{N/2}}{N},$$

we deduce from the previous inequality that

$$(14) \quad J_{\varepsilon,s}(tU) \leq \frac{S^{N/2}}{N} + O(\varepsilon^s) - \frac{\varepsilon(t^*)^{q+1}}{q+1} \int_{\mathbb{R}^N} h(x) U^{q+1} dx$$

for $t^* \leq t$ and $0 < \varepsilon \leq \varepsilon^*$. Taking ε^* smaller if necessary, we can assume

$$(15) \quad O(\varepsilon^s) - \frac{\varepsilon(t^*)^{q+1}}{q+1} \int_{\mathbb{R}^N} h(x) U^{q+1} dx \leq -C^* \varepsilon^r$$

for all $0 < \varepsilon \leq \varepsilon^*$. Combining (13)–(15) we obtain (12). The argument used in the proof of Proposition 2.1 shows that the functional $J_{\varepsilon,s}$ satisfies

the $(PS)_c$ condition with c satisfying (5). This completes the proof of the existence of a mountain-pass solution. The above estimate of the functional $J_{\varepsilon,s}$ on the sphere $\|u\|_\varepsilon = \varrho\varepsilon^{s/2}$ shows that the second solution can be obtained by local minimization as in Theorem 3.1. ■

In the next result we examine the behaviour, as $\varepsilon \rightarrow 0$, of the solutions from Theorem 4.1.

PROPOSITION 4.2. *Suppose that the assumptions of Theorem 4.1 hold.*

(i) *If $\{u_\varepsilon\}$, $0 < \varepsilon \leq \bar{\varepsilon}$, are mountain-pass solutions of problem $(1_{\varepsilon,s})$, then $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-s} \|u_\varepsilon\|^2 = \infty$.*

(ii) *If $\{u_\varepsilon\}$, $0 < \varepsilon \leq \bar{\varepsilon}$, are solutions of problem $(1_{\varepsilon,s})$ which are local minimizers, then $\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\| = 0$.*

Proof. It follows from the proof of Theorem 4.1 that for every $\varrho > 0$ there exists an $\varepsilon_0 > 0$ such that a mountain-pass solution u_ε satisfies

$$J_{\varepsilon,s}(u_\varepsilon) \geq \frac{\varepsilon^s \varrho^2}{4} \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.$$

This implies that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_\varepsilon|^2 + \varepsilon^s u_\varepsilon^2) dx &\geq \frac{\varepsilon^s \varrho^2}{4} + \frac{\varepsilon}{2^\star} \int_{\mathbb{R}^N} h u_\varepsilon^{q+1} dx + \frac{1}{2^\star} \int_{\mathbb{R}^N} u_\varepsilon^{2^\star} dx \\ &= \frac{\varepsilon^s \varrho^2}{4} + \frac{1}{2^\star} \int_{\mathbb{R}^N} (|\nabla u_\varepsilon|^2 + \varepsilon^s u_\varepsilon^2) dx. \end{aligned}$$

From this we deduce that

$$\frac{1}{N} \int_{\mathbb{R}^N} (|\nabla u_\varepsilon|^2 + \varepsilon^s u_\varepsilon^2) dx \geq \frac{\varepsilon^s \varrho^2}{4}$$

and the result follows.

(ii) Following the argument of Remark 3.2 we see that

$$J_{\varepsilon,s}(u_\varepsilon) - \frac{1}{2} \langle J'_{\varepsilon,s}(u_\varepsilon), u_\varepsilon \rangle < 0$$

implies that

$$\int_{\mathbb{R}^N} u_\varepsilon^{2^\star} dx \leq N \frac{1-q}{2(q+1)} \varepsilon \|h\|_r \|u_\varepsilon\|_\varepsilon^{q+1}.$$

We then have

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_\varepsilon|^2 + \varepsilon^s u_\varepsilon^2) dx &= J_{\varepsilon,s}(u_\varepsilon) + \frac{\varepsilon}{q+1} \int_{\mathbb{R}^N} h u_\varepsilon^{q+1} dx + \frac{1}{2^\star} \int_{\mathbb{R}^N} u_\varepsilon^{2^\star} dx \\ &\leq \frac{\varepsilon}{q+1} \|h\|_r \|u_\varepsilon\|_\varepsilon^{q+1} + \frac{1}{2^\star} \int_{\mathbb{R}^N} u_\varepsilon^{2^\star} dx \leq \varepsilon C_1 \|u_\varepsilon\|_\varepsilon^{q+1} \end{aligned}$$

for some constant $C_1 > 0$ independent of ε . Since

$$\varepsilon^{s/2} \|u\| \leq \|u\|_\varepsilon \leq \|u\|$$

for every $u \in H^1(\mathbb{R}^N)$, we deduce from the last estimate for u_ε that

$$\|u_\varepsilon\| \leq C\varepsilon^{1/(1-q)-s/2}$$

for some constant $C > 0$ independent of ε . This estimate completes the proof of assertion (ii). ■

REMARK 4.3. Theorem 4.1 continues to hold if h changes sign and satisfies $\int_{\mathbb{R}^N} h(x)U^{q+1} dx > 0$.

This assumption allows one to show that the mountain-pass level c_ε for the functional $J_{\varepsilon,s}$ satisfies (12).

5. Existence of infinitely many solutions. Since the right-hand side of the equation in (1_ε) involves concave and convex nonlinearities we can establish the existence of infinitely many solutions. Our approach is based on the Bartsch–Willem fountain theorem [4].

Let $\{e_k\}$, $k = 1, 2, \dots$, be an orthonormal basis for $H^1(\mathbb{R}^N)$. We set

$$X(j) = \text{span}(e_1, \dots, e_j), \quad X_k = \bigoplus_{j \geq k} X(j), \quad X^k = \bigoplus_{j \leq k} X(j)$$

for each $k \in \mathbb{N}$.

THEOREM 5.1 (Bartsch–Willem [5]). *Let $F : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ be a C^1 even functional satisfying the following conditions:*

- (A₁) *There exists an integer k_0 such that for every $k \geq k_0$ there exists $R_k > 0$ such that $F(u) \geq 0$ for every $u \in X_k$ with $\|u\| = R_k$.*
- (A₂) *$b_k = \inf_{B_k} F(u) \rightarrow 0$ as $k \rightarrow \infty$, where $B_k = \{u \in X_k : \|u\| \leq R_k\}$.*
- (A₃) *For every $k \geq 1$ there exist $r_k \in (0, R_k)$ and $d_k < 0$ such that $F(u) \leq d_k$ for every $u \in X^k$ with $\|u\| = r_k$.*
- (A₄) *Every sequence $u_n \in X^n$ with $F(u_n) < 0$ and $F'|_{X^n} \rightarrow 0$ as $n \rightarrow \infty$ has a subsequence which converges to a critical point of F .*

Then for each $k \geq k_0$, F has a critical value $c_k \in [b_k, d_k]$.

THEOREM 5.2. *There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ the equation in (1_ε) admits infinitely many solutions.*

Proof. It suffices to check that the functional J_ε satisfies the assumptions of Theorem 5.1. For each $k \in \mathbb{N}$ we define

$$\lambda_k = \sup_{u \in X_k - \{0\}} \frac{(\int_{\mathbb{R}^N} h|u|^{q+1} dx)^{1/(q+1)}}{\|u\|}.$$

It is clear that $\{\lambda_k\}$ is a decreasing sequence. Since $u \mapsto \int_{\mathbb{R}^N} h(x)|u(x)|^{q+1} dx$ is a completely continuous functional on $H^1(\mathbb{R}^N)$, we can show as in [18] that $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$. We now proceed as in the proof of Theorem 3 in [5]. Let $u \in H^1(\mathbb{R}^N)$. Then we have

$$J_\varepsilon(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda_k^{q+1}\varepsilon}{q+1}\|u\|^{q+1} - \frac{S^{-2^*/2}}{2^*}\|u\|^{2^*}.$$

If $\|u\| < R$ with $R > 0$ small, then

$$\frac{S^{-2^*/2}}{2^*}\|u\|^{2^*} \leq \frac{1}{4}\|u\|^2.$$

We set $R_k = (4\varepsilon\lambda_k^{q+1}/(q+1))^{1/(1-q)}$. We see that R_k satisfies

$$\frac{R_k^2}{4} = \frac{\varepsilon}{q+1}\lambda_k^{q+1}R_k^{q+1}$$

and moreover $R_k \rightarrow 0$ as $k \rightarrow \infty$. Therefore we can find $k_0 \in \mathbb{N}$ such that $R_k \leq R$ for $k \geq k_0$. Consequently, if $u \in X_k$, $k \geq k_0$, satisfies $\|u\| = R_k$, we then have

$$J_\varepsilon(u) \geq \frac{1}{4}\|u\|^2 - \frac{\varepsilon\lambda_k^{q+1}}{q+1}\|u\|^{q+1} = 0.$$

This shows that (A_1) holds and since $R_k \rightarrow 0$, condition (A_2) is also satisfied. To check (A_3) we observe that on the finite-dimensional space X^k all norms are equivalent. Hence

$$J_\varepsilon(u) \leq \frac{1}{2}\|u\|^2 - A\|u\|^{q+1} - B\|u\|^{2^*}$$

for some constants $A, B > 0$. Since $q+1 < 2$, taking r_k sufficiently small, we can satisfy (A_3) . The Palais–Smale condition (A_4) follows from Proposition 2.1. We only need to select $\varepsilon_0 > 0$ so that $S^{N/2}/N - \varepsilon^r C^* > 0$ for $0 < \varepsilon \leq \varepsilon_0$. ■

A similar argument can be employed to show the existence of infinitely many solutions of equation $(1_{\varepsilon,s})$.

THEOREM 5.3. *Let $s > 1$. There exists $\bar{\varepsilon} > 0$ such that equation $(1_{\varepsilon,s})$ for $0 < \varepsilon \leq \bar{\varepsilon}$ has infinitely many solutions.*

6. The p -Laplacian. In this section we study the problem

$$(1_\lambda) \quad \begin{cases} -\Delta_p u + u^{p-1} = u^{p^*-1} + \lambda f(x, u) & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{on } \mathbb{R}^N, \end{cases}$$

where $\Delta_p = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian of u , $p^* = Np/(N-p)$, $N > p$, is the critical Sobolev exponent and $\lambda > 0$ is a positive parameter.

We assume that the function $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following conditions:

(f₁) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $f(x, 0) = 0$ for $x \in \mathbb{R}^N$,

(f₂) for every $R > 0$ there exist $\theta_R \in [p, p^*)$ and constants $a_R, b_R > 0$ such that

$$|f(x, s)| \leq a_R s^{\theta_R - 1} + b_R$$

for every $|x| \leq R$ and $s \geq 0$,

(f₃) there exist constants $r_1 \in (1, p^*)$ and $r_2 \in (p, p^*)$ such that

$$f(x, s) \leq b(x)s^{r_1 - 1} + cs^{r_2 - 1}$$

for every $x \in \mathbb{R}^N$ and $s \geq 0$, where $b \in L^{p^*/(p^* - r_1)}(\mathbb{R}^N)$ and $b(x) \geq 0$ on \mathbb{R}^N and $c > 0$ is a constant. Furthermore, we assume that there exist $q \in (1, p)$ and an open set $\Omega \subset \mathbb{R}^N$ such that the function $F(x, s) = \int_0^s f(x, t) dt$ satisfies

$$F(x, s) \geq as^q$$

for every $x \in \Omega$ and $s \geq 0$ and some positive constant $a > 0$.

An example of the nonlinearity f satisfying the above conditions is

$$f(x, s) = b(x)s^{q-1} - d(x)s^{l-1} + Cs^{r_2-1},$$

where $C > 0$ is a constant, $1 \leq l < p^*$, $b(x) \geq 0$ on \mathbb{R}^N and belongs to $L^{p^*/(p^* - q)}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $d(x)$ is a continuous and nonnegative function on \mathbb{R}^N . Moreover, it is assumed that $b(x) \geq a_0 > 0$ on $B(0, R)$ and $d(x) = 0$ on $B(0, R)$. In this example $q = r_1$ and $\Omega = B(0, R)$.

A similar problem

$$\begin{cases} -\Delta_p u = u^{p^* - 1} + \lambda f(x, u) & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{on } \mathbb{R}^N, \end{cases}$$

has been studied in [16]. Under some additional assumptions, guaranteeing the mountain-pass geometry of the variational functional for this problem, the authors established the existence of a nontrivial solution. First, we establish the existence of a solution for problem (1_λ) through local minimization. Under assumptions (f_1) , (f_2) and (f_3) a variational functional corresponding to (1_λ) is not well defined on $W^{1,p}(\mathbb{R}^N)$. Therefore, following the paper [16] we truncate the nonlinearity f . We also point out here that in [16] the constant q (see assumption (f_3)) satisfies $1 < q < p^*$ and some other restrictions. This assumption is replaced here by $1 < q < p$, since we construct a solution through local minimization (see Lemma 6.2 below).

Let $\varphi \in C^1(\mathbb{R}^N)$ be a function such that $\varphi(x) = 1$ for $|x| \leq 1$, $\varphi(x) = 0$ for $|x| \geq 2$ and $0 \leq \varphi(x) \leq 1$ on \mathbb{R}^N . We set $\varphi_n(x) = \varphi(x/n)$ and extend f by 0 for $s \leq 0$, that is, $f(x, s) = 0$ for $x \in \mathbb{R}^N$ and $s \leq 0$. We define a modified nonlinearity $f_n(x, s) = \varphi_n(x)f(x, s)$. For each $n \in \mathbb{N}$ we consider the following problem:

$$(1_{\lambda,n}) \quad \begin{cases} -\Delta_p u + u^{p-1} = u^{p^*-1} + \lambda f_n(x, u) & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{on } \mathbb{R}^N. \end{cases}$$

Let $F_n(x, s) = \int_0^s f_n(x, t) dt$. We associate with problem $(1_{\lambda,n})$ the variational functional

$$I_{\lambda,n}(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx - \frac{1}{p^*} \int_{\mathbb{R}^N} (u^+)^{p^*} dx - \lambda \int_{\mathbb{R}^N} F_n(x, u) dx$$

for $u \in W^{1,p}(\mathbb{R}^N)$, where $W^{1,p}(\mathbb{R}^N)$ is the usual Sobolev space equipped with the norm

$$\|u\|^p = \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx.$$

The functional $I_{\lambda,n}$ is well defined on $W^{1,p}(\mathbb{R}^N)$ and is of class C^1 . Its Fréchet derivative is given by

$$\begin{aligned} \langle I'_{\lambda,n}(u), v \rangle &= \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla v + |u|^{p-2} uv) dx - \int_{\mathbb{R}^N} (u^+)^{p^*-1} v dx \\ &\quad - \lambda \int_{\mathbb{R}^N} f_n(x, u) v dx \quad \text{for } v \in W^{1,p}(\mathbb{R}^N). \end{aligned}$$

LEMMA 6.1. (i) *Let $r_1 \leq p$. Then there exist $\bar{\lambda}, \varrho, \alpha > 0$ such that for every $0 < \lambda < \bar{\lambda}$,*

$$(16) \quad I_{\lambda,n}(u) \geq \alpha \quad \text{for } \|u\| = \varrho,$$

and each $n \in \mathbb{N}$.

(ii) *If $p < r_1 < p^*$, then for each $\lambda > 0$ there exist $\varrho, \alpha > 0$ such that (16) holds for each $n \in \mathbb{N}$.*

Proof. (i) It follows from (f_3) and the Sobolev inequality that

$$\begin{aligned} I_{\lambda,n}(u) &\geq \frac{1}{p} \|u\|^p - \frac{1}{p^*} \|u\|_{p^*}^{p^*} - \frac{\lambda}{r_1} \|b\|_{p^*/(p^*-r_1)} \|u\|_{p^*}^{r_1} - \frac{\lambda c}{r_2} \|u\|_{r_2}^{r_2} \\ &\geq \frac{1}{p} \|u\|^p - \frac{1}{p^* S^{p^*/p}} \|u\|^{p^*} - \frac{\lambda}{r_1 S^{r_1/p}} \|b\|_{p^*/(p^*-r_1)} \|u\|^{r_1} - \frac{\lambda c}{r_2 C_{r_2}^{r_2}} \|u\|^{r_2} \\ &= \|u\|^p \left(\frac{1}{p} - \frac{1}{p^* S^{p^*/p}} \|u\|^{p^*-p} \right) - \lambda \left(\frac{\|b\|_{p^*/(p^*-r_1)}}{r_1 S^{r_1/p}} \|u\|^{r_1} + \frac{c}{r_2 C_{r_2}^{r_2}} \|u\|^{r_2} \right), \end{aligned}$$

where C_{r_2} is the best Sobolev constant for the embedding of $W^{1,p}(\mathbb{R}^N)$ into $L^{r_2}(\mathbb{R}^N)$. First we choose constants $\varrho, \varrho_1 > 0$ so that

$$\frac{1}{p} - \frac{1}{p^* S^{p^*/p}} \varrho^{p^*-p} > \varrho_1.$$

Then we choose a constant $\bar{\lambda} > 0$ such that the inequality (16) holds for all $0 < \lambda \leq \bar{\lambda}$ and some constant $\alpha > 0$.

(ii) If $p < r_1 < p^*$, then the above estimate takes the form

$$I_{\lambda,n}(u) \geq \|u\|^p \left[\frac{1}{p} - \frac{1}{p^* S^{p^*/N}} \|u\|^{p^*-p} - \lambda(C_1 \|u\|^{r_1-p} + C_2 \|u\|^{r_2-p}) \right]$$

for some constants $C_1, C_2 > 0$ independent of λ and u . From this estimate we easily deduce assertion (ii). ■

LEMMA 6.2. *In both cases (i) and (ii) of Lemma 6.1 there exist constants $0 < \varrho_1 < \varrho_2$ such that*

$$(17) \quad -\varrho_2 \leq \inf_{\|u\| \leq \varrho} I_{\lambda,n}(u) \leq -\varrho_1$$

for all $n \in \mathbb{N}$ sufficiently large.

Proof. Let $w \in C_0^1(\mathbb{R}^N)$, $w \geq 0$, $w \not\equiv 0$, $\text{supp } w \subset \Omega$ and $\|w\| \leq \varrho$. Then for $t > 0$,

$$I_{\lambda,n}(tw) \leq \frac{t^p}{p} \int_{\mathbb{R}^N} (|\nabla w|^p + |w|^p) dx - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} |w|^{p^*} dx - at^q \int_{\mathbb{R}^N} |w|^q dx,$$

where n is so large that $\text{supp } w \subset B(0, n)$. Since $q < p$ we can choose $t > 0$ small enough so that $I_{\lambda,n}(tw) < 0$ and consequently the upper bound in (17) holds for n sufficiently large. Finally, the lower bound in (17) follows from the estimates in the proof of Lemma 6.1. ■

7. Existence result for $(1_{\lambda,n})$. For each $n \in \mathbb{N}$ we set

$$c_{\lambda,n} = \inf_{\|u\| \leq \varrho} I_{\lambda,n}(u).$$

According to Lemma 6.2 we have $-\varrho_2 \leq c_{\lambda,n} \leq -\varrho_1$. Therefore we may assume that up to a subsequence $-\varrho_2 \leq c_\lambda = \lim_{n \rightarrow \infty} c_{\lambda,n} \leq -\varrho_1$. For each $n \in \mathbb{N}$ there exists a sequence $\{u_j^n\} \subset B(0, \varrho)$ such that

$$I_{\lambda,n}(u_j^n) \rightarrow c_{\lambda,n} \quad \text{and} \quad I'_{\lambda,n}(u_j^n) \rightarrow 0 \quad \text{in } W^{-1,p'}(\mathbb{R}^N)$$

as $j \rightarrow \infty$, where $p' = p/(p-1)$. Given $\varepsilon > 0$ satisfying $c_\lambda + \varepsilon < 0$ we can find $n_0 \in \mathbb{N}$ such that $c_{\lambda,n} \in (c_\lambda - \varepsilon, c_\lambda + \varepsilon)$ for each $n \geq n_0$. Thus, for each $n \geq n_0$ we can select $u_n = u_{j_n}^n$ satisfying

$$c_\lambda - \varepsilon < I_{\lambda,n}(u_n) < c_\lambda + \varepsilon \quad \text{and} \quad \|I'_{\lambda,n}(u_n)\| \leq 1/n.$$

Since $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$ we may assume that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$.

We are now in a position to formulate the existence result for problem $(1_{\lambda,n})$. We need the following assumption:

(f₄) There exist constants $p < \tau < p^*$ and $1 < \mu < p^*$ such that

$$\frac{1}{\tau} f(x, s)s - F(x, s) \geq c_1(x)s^\mu$$

for all $x \in \mathbb{R}^N$ and $s \geq 0$, where c_1 is a function in $L^{p^*/(p^*-\mu)}(\mathbb{R}^N)$.

The nonlinearity f from the example in the paragraph following the assumption (f_3) satisfies (f_4) with $c_1(x) = (1/\tau - 1/q)b(x)$, $\mu = q$, provided $l < \tau < r_2$.

THEOREM 7.1. (i) *If $r_1 \leq p$, then there exists $\bar{\lambda} > 0$ such that problem (1_λ) has a nontrivial solution for each $0 < \lambda < \bar{\lambda}$.*

(ii) *If $p < r_1 < p^*$, then problem (1_λ) has a nontrivial solution for each $\lambda > 0$.*

Proof. It is sufficient to show that the weak limit of the sequence $\{u_n\}$ is not identically equal to 0. The proof of this fact is similar to that of Theorem 1.1 in [16]. Therefore we only sketch it in some details.

Step 1. Applying P. L. Lions' concentration-compactness principle [15] we may assume that up to a subsequence

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^N), 1 \leq s < p^*, \quad u_n(x) \rightarrow u(x) \quad \text{a.e. on } \mathbb{R}^N, \\ |u_n|^{p^*} &\rightharpoonup \nu = |u|^{p^*} + \sum_i \nu_i \delta_{x_i} + \nu_\infty \delta_\infty, \\ |\nabla u|^p &\rightharpoonup \mu \quad \text{and} \quad \sum_i \nu_i^{p/p^*} < \infty, \end{aligned}$$

where

$$\nu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| \geq R} |u_n|^{p^*} dx.$$

Step 2. In each bounded subset of \mathbb{R}^N there are only a finite number of the points x_j . To establish this claim we use a family of smooth functions concentrating at x_j and assumption (f_2) and show as in [16] that $\mu(\{x_i\}) \leq a_R \nu(\{x_i\}) = a_R \nu_i$. On the other hand we always have $\mu(\{x_i\}) \geq S \nu_i^{p/p^*}$. Therefore $\nu_i \geq (S/a_R)^{N/p}$. Since $\sum_i \nu_i^{p/p^*} < \infty$, we see that there are at most a finite number of x_j in $B(0, r)$ for every $r > 0$ and the claim easily follows.

Step 3. Using Step 2 one can show that $u_n \rightarrow u$ in $L^{p^*}(K)$ for each compact set $K \subset \mathbb{R}^N - \bigcup \{x_i\}$. Indeed, as an immediate consequence of Step 2 we obtain $\int_K |u_n|^{p^*} dx \rightarrow \int_K |u|^{p^*} dx$ as $n \rightarrow \infty$. By the uniform convexity of the space $L^{p^*}(K)$ we find that $u_n \rightarrow u$ in $L^{p^*}(K)$. From this, using the fact that $\int_{\mathbb{R}^N} |\nabla u|^p dx$ and $\int_{\mathbb{R}^N} |u|^p dx$ are convex functionals, we deduce that $\nabla u_n \rightarrow \nabla u$ in $(L^p(K))^N$ for each compact set $K \subset \mathbb{R}^N - \bigcup \{x_i\}$. This implies that $\nabla u_n \rightarrow \nabla u$ a.e. on \mathbb{R}^N . Applying Vitali's convergence theorem we show as in [16] that u is a solution of (1_λ) in the distributional sense. For details we refer to pages 9–10 of [16]. It remains to show that

$u \not\equiv 0$. Arguing indirectly assume that $u \equiv 0$. Thus, up to a subsequence

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|\nabla u_n|^p + |u_n|^p) dx = l > 0,$$

since otherwise $I_{\lambda,n}(u_n) \rightarrow 0$, which is impossible. By (f_4) we have

$$\begin{aligned} 0 &> \lim_{n \rightarrow \infty} \left[I_{\lambda,n}(u_n) - \frac{1}{\tau} \langle I'_{\lambda,n}(u_n), u_n \rangle \right] \\ &\geq \lim_{n \rightarrow \infty} \left[\left(\frac{1}{p} - \frac{1}{\tau} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^p + |u_n|^p) dx + \left(\frac{1}{\tau} - \frac{1}{p^*} \right) \int_{\mathbb{R}^N} (u_n^+)^{p^*} dx \right. \\ &\quad \left. + \int_{\mathbb{R}^N} c_1(x) |u_n|^\mu dx \right] \\ &\geq \left(\frac{1}{p} - \frac{1}{\tau} \right) l, \end{aligned}$$

which gives a contradiction. ■

8. Remark on the existence of a mountain-pass solution for problem (1_λ) . In the final section of this paper we indicate how to construct a mountain-pass solution for problem (1_λ) . This requires an additional assumption on the nonlinearity f :

(f_5) There exists an open set $\Omega_1 \subset \mathbb{R}^N$ such that

$$F(x, s) \geq bs^{q_1}$$

for every $x \in \Omega_1$, $s \geq 0$ and some constants $b > 0$ and $1 < q_1 < p^*$. Moreover, $p < q_1$ if $p^2 \leq N$, and $q_1 > p^* - p/(p-1)$ if $p^2 > N$.

This extra assumption in our model nonlinearity

$$f(x) = b(x)s^{q-1} - d(x)s^{l-1} + Cs^{r_2-1}$$

is satisfied provided $d(x) = 0$ for $x \in B(0, R)$ with $q_1 = r_2$ and r_2 satisfying the conditions for q_1 from (f_5) .

We recall that the best Sobolev constant S in $W^{1,p}(\mathbb{R}^N)$ is defined by

$$(18) \quad S = \inf_{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) - \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{p/p^*}}.$$

It is well known [10] that the infimum in (18) is achieved by a family of functions

$$W_\varepsilon(x) = \frac{c_N \varepsilon^{(N-p)/p^2}}{(\varepsilon + |x|^{p/(p-1)})^{(N-p)/p}}, \quad \varepsilon > 0,$$

for $x \in \mathbb{R}^N$, where

$$c_N = \left\{ N \left[\frac{N-p}{p-1} \right]^{p-1} \right\}^{(N-p)/p^2}.$$

We also have $\int_{\mathbb{R}^N} |\nabla W_\varepsilon|^p dx = \int_{\mathbb{R}^N} W_\varepsilon^{p^*} dx = S^{N/p}$.

THEOREM 8.1. *Suppose that assumptions $(f_1), \dots, (f_5)$ hold. Then in both cases (i) and (ii) of Theorem 7.1 problem (1_λ) admits a mountain-pass solution. (This means that in both cases (i) and (ii) problem (1_λ) has at least two distinct solutions.)*

Proof. We follow the ideas from [16] (see the proof of Theorem 1.1 there).

Step I. We use the estimate from the proof of Lemma 6.2 to obtain

$$I_{\lambda,n}(tw) \leq \frac{t^p}{p} \int_{\mathbb{R}^N} (|\nabla w|^p + |w|^p) dx - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} |w|^{p^*} dx - at^q \int_{\mathbb{R}^N} |w|^q dx.$$

However, this time for sufficiently large $t > 0$ we get $I_{\lambda,n}(tw) < 0$. Combining this with Lemma 6.1 we see that in both cases (i) and (ii), $I_{\lambda,n}$ satisfies (uniformly in n) the conditions of the geometry of the mountain-pass theorem.

Step II. Let $x_0 \in \Omega_1$ and choose $r > 0$ so that $B(x_0, 2r) \subset \Omega_1$. We assume that $B(x_0, 2r) \subset B(0, n)$ for all $n \geq n_0$ for some integer n_0 . We define a function $\varphi \in C_0^1(\mathbb{R}^N)$ such that $\varphi(x) = 1$ on $B(x_0, r)$, $\varphi(x) = 0$ on $\mathbb{R}^N - B(x_0, 2r)$ and $0 \leq \varphi(x) \leq 1$ on \mathbb{R}^N and set $v_\varepsilon = \varphi W_\varepsilon(\cdot - x_0) / \|\varphi W_\varepsilon(\cdot - x_0)\|_{p^*}$. We claim that there exist $\varepsilon, d_\lambda > 0$ such that

$$(19) \quad \max_{t \geq 0} I_{\lambda,n}(tv_\varepsilon) \leq d_\lambda < \frac{S^{N/p}}{N}$$

for $n \geq n_0$. Setting $Y_\varepsilon = \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^p + |v_\varepsilon|^p) dx$, we derive from assumption (f_4) that

$$\begin{aligned} I_{\lambda,n}(tv_\varepsilon) &= \frac{t^p Y_\varepsilon}{p} - \frac{t^{p^*}}{p^*} - \lambda \int_{\mathbb{R}^N} F_n(x, tv_\varepsilon) dx \\ &\leq \frac{t^p Y_\varepsilon}{p} - \frac{t^{p^*}}{p^*} - \lambda t^{q_1} b \int_{\mathbb{R}^N} |v_\varepsilon|^{q_1} dx. \end{aligned}$$

Let

$$J_\lambda(tv_\varepsilon) = \frac{t^p Y_\varepsilon}{p} - \frac{t^{p^*}}{p^*} - \lambda t^{q_1} b \int_{\mathbb{R}^N} |v_\varepsilon|^{q_1} dx.$$

It is sufficient to show that there exist $d_\lambda, \varepsilon > 0$ such that

$$\max_{t \geq 0} J_\lambda(tv_\varepsilon) \leq d_\lambda < \frac{S^{N/p}}{N}.$$

For each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\max_{t \geq 0} J_\lambda(tv_\varepsilon) = J_\lambda(t_\varepsilon v_\varepsilon),$$

with $0 < t_\varepsilon \leq Y_\varepsilon^{1/(p-1)}$. Consequently, we have

$$J_\lambda(t_\varepsilon v_\varepsilon) \leq \frac{1}{N} Y_\varepsilon^{N/p} - \lambda b_1 \int_{\mathbb{R}^N} |v_\varepsilon|^{q-1} dx$$

for some constant $b_1 > 0$. Suppose that $p^2 \leq N$. Using the explicit form of W_ε we easily derive the following estimate:

$$\begin{aligned} J_\lambda(t_\varepsilon v_\varepsilon) &\leq \frac{S^{N/p}}{N} + O(\varepsilon^{(N-p)/p}) + O(\varepsilon^{p-1}) - \lambda b_1 \int_{\mathbb{R}^N} |v_\varepsilon|^{q_1} dx \\ &= \frac{S^{N/p}}{p} + O(\varepsilon^{p-1}) - \lambda b_1 \int_{\mathbb{R}^N} |v_\varepsilon|^{q_1} dx. \end{aligned}$$

The term $O(\varepsilon^{p-1})$ is due to the integral $\int_{\mathbb{R}^N} |v_\varepsilon|^p dx$ appearing in Y_ε . Thus, there exists $M > 0$ such that

$$J_\lambda(t_\varepsilon v_\varepsilon) \leq \frac{S^{N/p}}{N} + \varepsilon^{p-1} \left(M - \frac{\lambda b_1}{\varepsilon^{p-1}} \int_{\mathbb{R}^N} |v_\varepsilon|^{q_1} dx \right).$$

Arguing as in the proof of Proposition 5.1 in [16] we derive from this the estimate

$$(20) \quad J_\lambda(t_\varepsilon v_\varepsilon) \leq \frac{S^{N/p}}{N} + \varepsilon^{p-1} (M - \lambda b_2 \varepsilon^{[(\frac{N-p}{p^2} - \frac{N-p}{p})q_1 + \frac{(p-1)N}{p} + 1 - p]})$$

for some constant $b_2 > 0$. Since

$$\left[\frac{N-p}{p^2} - \frac{N-p}{p} \right] q_1 + (p-1) \left(\frac{N}{p} - 1 \right) < 0$$

is equivalent to $p < q_1$, our claim follows from the estimate (20). In a similar manner we consider the case $p^2 > N$.

Step III. We now apply the mountain-pass theorem to the functionals $I_{\lambda,n}$ for every $n \geq n_0$ to obtain the mountain-pass levels $c_{\lambda,n}$ and Palais-Smale sequences $\{u_j^n\}$ in $W^{1,p}(\mathbb{R}^N)$ satisfying

$$I_{\lambda,n}(u_j^n) \rightarrow c_{\lambda,n} \quad \text{and} \quad I'_{\lambda,n}(u_j^n) \rightarrow 0 \quad \text{in } W^{-1,p'}(\mathbb{R}^N)$$

as $j \rightarrow \infty$. We then choose a subsequence $u_n = u_{j_n}^n$ satisfying

$$c_\lambda - \varepsilon \leq I_{\lambda,n}(u_n) \leq c_\lambda + \varepsilon, \quad \|I'_{\lambda,n}(u_n)\| \leq 1/n$$

for $n \geq n_0$ and some $\varepsilon > 0$, where up to a subsequence $\lim_{n \rightarrow \infty} c_{\lambda,n} = c_\lambda$. Arguing as in Lemma 6.1 of [16] we show that $u_n \rightharpoonup u \neq 0$ in $W^{1,p}(\mathbb{R}^N)$ and u is a solution of (1_λ) . ■

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