Plus-operators in Krein spaces and dichotomous behavior of irreversible dynamical systems with discrete time

by

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Abstract. We study dichotomous behavior of solutions to a non-autonomous linear difference equation in a Hilbert space. The evolution operator of this equation is not continuously invertible and the corresponding unstable subspace is of infinite dimension in general. We formulate a condition ensuring the dichotomy in terms of a sequence of indefinite metrics in the Hilbert space. We also construct an example of a difference equation in which dichotomous behavior of solutions is not compatible with the signature of the indefinite metric.

1. Introduction. In [Kh-Z2] we considered a linear fractional transformation of the ball $\mathcal{K}^-$ of all angular operators corresponding to the set of all maximal non-positive subspaces of a Krein space, i.e., of a Hilbert space $\mathcal{H}$ with an indefinite metric, for which both positive and negative components can be infinite-dimensional in general ([Kh2], [Kh3], [Az-I], [Kr]). This transformation is generated by a continuous linear operator U in $\mathcal{H}$ (so-called bistrict plus-operator: see for example [Kh1]–[Kh3], [Az-I] and [Kr]). We did not assume U to be continuously invertible. The weak compactness of the image and preimage of $\mathcal{K}^-$ under the associated linear fractional transformation F_U has been established ([Kh-Z2, Theorem 2.1]).

In the present paper we consider much more general operator linear fractional relations and we obtain some additional properties of the image and preimage of the ball $\mathcal{K}^-$ (see Theorem 2.1, Proposition 2.1 and Example 3.1). We apply the above results to the study of dichotomous behavior of solutions to a non-autonomous linear difference equation in a Hilbert space $\mathcal{H}$,

$$(1.1) \qquad y_{n+1} = A_n y_n \quad (y_n \in \mathcal{H},\, n = 0, 1, 2, \ldots),$$

where A_n are bounded linear operators acting in $\mathcal{H}$. This equation describes a non-autonomous dynamical system, in which the integer n plays the role of discrete time. We do not assume that the operators A_n are continuously invertible. This means that the corresponding dynamical system is irreversible in general. The *dichotomous behavior* means that solutions with initial values belonging to some subspace of the phase space $\mathcal{H}$ (the *stable subspace*) stabilize to zero at infinity, but all the other solutions grow indefinitely. For differential equations in Hilbert and Banach spaces this property was studied in [Mz], [Ms-Sh], [Kh-Z1], [Kh4], [Kh-Z2] and [Kh-Sh].

We formulate a condition ensuring dichotomous behavior in terms of the increment of a sequence $\{[\cdot,\cdot]_n\}_{n\in\mathbb{N}}$ of indefinite metrics with respect to equation (1.1) (see Definition 3.1 and Theorem 3.1) and apply the method of embedded bicones used in [Z], [Kh-Z1], [Kh4] and [Kh-Z2]. We obtain estimates for the velocity of the dichotomy. We pay a special attention to the case where the stable subspace is a maximal non-positive subspace associated with the indefinite metric $[\cdot,\cdot]_n$ for a large enough n (see Theorem 3.1(iv)). In this case we say that the dichotomy is *compatible with the signature of the indefinite metric.* Close questions were considered and a similar approach was applied in [B-G1]–[B-G3]. In [B-G1] and [B-G3] the dichotomy of solutions of (1.1) was studied with the help of indefinite metrics in the case of a finite-dimensional Hilbert space $\mathcal{H}$. In particular, in [B-G3] the case of invertible operators A_n was considered (a reversible system) and the method used was close to the method of embedded bicones (see the proof of Lemma 4.4 of [B-G3]). In [B-G2] the case of an infinite-dimensional Hilbert space was studied and the so-called inertia theorems for (1.1) were obtained (see Remark 3.1 of the present paper).

If the operators A_n in (1.1) are not continuously invertible, application of the method of embedded bicones faces certain topological difficulties. To overcome these, we use Theorem 2.1 of [Kh-Z2] mentioned above. We also construct an example of a dichotomous dynamical system in which the dichotomy is not compatible with the signature of the indefinite metric (Example 3.2).

Notice that another approach to the dichotomy of the dynamical system (1.1) (see [B-G1], [B-G4], [B-G5] ([1]), [A-M-Z], [A-S] and [B-P]) is to study the connection of the dichotomy with the Fredholm properties of the operator

$$(L\mathbf{y})_n = y_{n+1} - A_n y_n$$

(so-called "weighted shift"), acting in the Hilbert space $L_2(\mathbb{Z}_+, \mathcal{H})$ of sequences $\mathbf{y} = \{y_n\}_{n\in\mathbb{Z}_+}$ ($y_n \in \mathcal{H}$). In particular, it turns out [B-G4] that the exponential dichotomy of a reversible dynamical system of the form (1.1)

([1]) In [B-G1]–[B-G5] a more general version of equation (1.1) was considered.

in a finite-dimensional Hilbert space $\mathcal{H}$ is equivalent to the continuous invertibility of the operator L acting in $L_2(\mathbb{Z}, \mathcal{H})$ (after a suitable extension of $\{A_n\}_{n \in \mathbb{Z}_+}$ to the negative time axis).

Now we describe the contents of the paper.

At the beginning of Section 2 we give the needed background from the theory of spaces with an indefinite metric. In particular, the concepts of plus-operator, strict plus-operator and bistrict plus-operator are defined. Moreover, we introduce a couple of new notions: a linear fractional relation F_U (LFR for brevity), defined by the block-matrix of a plus-operator U, the dual notion of an LFR G_T defined by a minus-operator T, and some operator sets $\mathcal{G}_U^{\pm}$. These notions have a conceptual character and Theorem 2.1 is an evidence of this. It asserts that in the case of an invertible bistrict plus-operator U the image $\mathrm{Im}(G_T)\,(T = U^{-1})$ is exactly the set $\mathcal{G}_U^-$ (equation (2.6)). And vice versa: if (2.6) holds with $T = U^{-1}$, then U is a bistrict plus-operator (consequently, T is a bistrict minus-operator, and G_T is a usual linear fractional transformation, LFT for brevity). This fact defines the borders of the method used in [Kh-Sh], where the case of a reversible continuous time dynamical system was considered, that is, the evolution operator $U(t)$ was assumed to be invertible. In [Kh-Sh] the compactness of $\mathrm{Im}(G_T)\,(T = U^{-1})$ in the weak operator topology (WOT) was established by using (2.6) and the fact that the image of an LFT is compact in WOT. This compactness is a key point of the proof of the dichotomy of solutions $x(t) = U(t)x_0$ to the evolution equation in [Kh-Sh]. In view of Theorem 2.1, we now see that in the general case of a plus-operator $U(t)$ (not necessarily bistrict as in [Kh-Z2] or invertible as in [Kh-Sh]) one has to develop a general theory of LFRs of types (2.3)–(2.4) (see also Remark 2.1). In [Kh4, Section 2] a number of results on geometrical and topological structure of images and domains of LFRs (called there generalized linear fractional transformations) were obtained. Our Theorem 2.1 is also a result of such type.

The final Proposition 2.1 of Section 2 asserts that the spectrum $\sigma(U)$ of a strict plus-operator U such that $\mathcal{G}_U^- = \emptyset$ surrounds the origin of the complex plane $\mathbb{C}$. This result is closely related to a very interesting problem concerning existence of a continuous time dynamical system with the dichotomy of solutions which is not compatible with the signature of an indefinite metric (see details in Remark 3.4).

In Section 3 we prove the above mentioned Theorem 3.1 about dichotomy of solutions of irreversible dynamical systems with discrete time. Furthermore, we construct examples of dynamical systems with dichotomy of solutions incompatible with an indefinite metric (see Examples 3.1 and 3.2).

Let us introduce some notation:

- $\mathbb{C}$ is the set of all complex numbers;

- $\mathbb{R}_+ = [0, \infty)$;
- $\mathbb{N}$ is the set of all natural numbers;
- $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$;
- $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$;
- $\mathrm{cl}(\mathcal{S})$ is the closure of a subset $\mathcal{S}$ of a Banach space $\mathcal{B}$;
- $\mathcal{H}$ is a complex or real Hilbert space with inner product (x, y) and norm $\|x\| = (x, x)^{1/2}$; we will also use $\mathcal{H}$ with subscripts, but they will be omitted in the notation of norms and inner products;
- $\mathrm{span}(\mathcal{S})$ is the minimal closed subspace of a Hilbert space $\mathcal{H}$ containing a subset $\mathcal{S}$ of $\mathcal{H}$;
- $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all bounded linear operators acting from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$; if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, we will write $\mathcal{L}(\mathcal{H})$;
- $\mathcal{SL}(\mathcal{H})$ is the set of all selfadjoint operators from $\mathcal{L}(\mathcal{H})$;
- $\sigma(A)$ is the spectrum of a linear operator A acting in $\mathcal{H}$;
- $(\mathcal{H}, V)$ is a *Krein space*, i.e. $\mathcal{H}$ is a Hilbert space with an indefinite metric $[x, y] = (Vx, y)$, where $V \in \mathcal{SL}(\mathcal{H})$ is continuously invertible;
- $\mathcal{H}^+$, $\mathcal{H}^-$ (possibly with subscripts) are the invariant subspaces of V, corresponding to the positive and negative parts of its spectrum, respectively; $d_V^+ = \dim(\mathcal{H}^+)$, $d_V^- = \dim(\mathcal{H}^-)$ (these numbers can be infinite in general);
- $\mathcal{P}^+$, $\mathcal{P}^-$ (possibly with subscripts) are respectively the non-negative and non-positive bicones generated by the indefinite metric $[x, y]$: $\mathcal{P}^+ = \{x \in \mathcal{H} : [x, x] \geq 0\}$, $\mathcal{P}^- = \{x \in \mathcal{H} : [x, x] \leq 0\}$.

2. Plus-operators and linear fractional relations. Let $(\mathcal{H}, V)$ be a Krein space, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Denote by P^+, P^- the orthogonal projections on $\mathcal{H}^+$, $\mathcal{H}^-$ respectively. It is well known that one can reduce the regular indefinite square form $[x, x] = (Vx, x)$ to the form

$$[x, x] = \|P^+ x\|^2 - \|P^- x\|^2,$$

by passing to an equivalent norm in $\mathcal{H}$ [Az-I]. In the following we will assume such passage to be realized. Recall that a subspace $L \subseteq \mathcal{H}$ is called *non-positive* (respectively, *non-negative*) if it is contained in $\mathcal{P}^-$ (respectively, in $\mathcal{P}^+$). A non-positive (respectively, non-negative) subspace $L \subseteq \mathcal{H}$ is called *maximal non-positive* (respectively, *maximal non-negative*) if it is not a proper subset of any non-positive (respectively, non-negative) subspace.

Consider two Krein spaces $(\mathcal{H}_i, V_i)$ $(i = 1, 2)$ with indefinite metrics of the form defined above, i.e. $V_i = P_i^+ - P_i^-$, where P_i^+, P_i^- are the orthogonal projections in $\mathcal{H}_i$ on the subspaces $\mathcal{H}_i^+, \mathcal{H}_i^-$ respectively. This means that we have orthogonal splittings

$$(2.1) \qquad \mathcal{H}_i = \mathcal{H}_i^+ \oplus \mathcal{H}_i^-, \quad i = 1, 2.$$

Assume that the signatures of the quadratic forms $[x, x]_i = (V_i x, x)$ are the same, i.e. $d_{V_1}^+ = d_{V_2}^+$ and $d_{V_1}^- = d_{V_2}^-$. Let $\mathcal{P}_i^+$, $\mathcal{P}_i^-$ ($i = 1, 2$) be the non-negative and non-positive bicones respectively generated by the above indefinite metrics.

Recall that a bounded linear operator U acting from $(\mathcal{H}_1, V_1)$ to $(\mathcal{H}_2, V_2)$ is called a *plus-operator* if $x \in \mathcal{P}_1^+$ implies $Ux \in \mathcal{P}_2^+$, and it is called a *strict plus-operator* if

$$[Ux, Ux]_2 \geq \mu(U)[x, x]_1 \quad \forall x \in \mathcal{H}_1,$$

where $\mu(U) > 0$. If together with U the adjoint operator $U^\star$ is also a strict plus-operator, then U is called a *bistrict plus-operator*. The definitions of a minus-operator (strict and bistrict) are analogous (see [Az-I]).

Consider the block matrix representation of the operator U, corresponding to the orthogonal decompositions (2.1):

$$(2.2) \qquad\qquad U = \{U_{i,j}\}_{i,j=1}^2.$$

Let $\mathcal{M}_i^+$ (resp., $\mathcal{M}_i^-$) be the set of all maximal non-negative (resp., maximal non-positive) subspaces of $(\mathcal{H}_i, V_i)$, $i = 1, 2$. We denote by $\mathcal{K}_i^+$ (resp., $\mathcal{K}_i^-$) the set of all angular operators of subspaces $L_i^+ \in \mathcal{M}_i^+$ (resp., $L_i^- \in \mathcal{M}_i^-$), $i = 1, 2$. Recall that these are the linear operators $K_i^+ : \mathcal{H}_i^+ \to \mathcal{H}_i^-$ (resp., $K_i^- : \mathcal{H}_i^- \to \mathcal{H}_i^+$) for which $L_i^+ = \mathrm{graph}(K_i^+)$ ($L_i^- = \mathrm{graph}(K_i^-)$). Let us also recall [Az-I] that any non-negative (resp., non-positive) subspace is contained in some maximal non-negative (resp., maximal non-positive) subspace. In other words, for any $E_i^+ \subset \mathcal{P}_i^+$ (resp., $E_i^- \subset \mathcal{P}_i^-$) there exists $K_i^+ \in \mathcal{K}_i^+$ (resp., $K_i^- \in \mathcal{K}_i^-$) such that

$$E_i^+ \subset L_i^+ = (P_i^+ + K_i^+)\mathcal{H}_i^+ \quad (\text{resp., } E_i^- \subset L_i^- = (P_i^- + K_i^-)\mathcal{H}_i^-)$$

(see details in [Az-I], [Kh1]). Define

$$\mathcal{G}_U^- = \{K_1^- \in \mathcal{K}_1^- : U(P_1^- + K_1^-)\mathcal{H}_1^- \subset \mathcal{P}_2^-\},$$
$$\mathcal{G}_U^+ = \{K_2^+ \in \mathcal{K}_2^+ : (P_2^+ + K_2^+)\mathcal{H}_2^+ \supseteq U(P_1^+ + K_1^+)\mathcal{H}_1^+ \text{ for some } K_1^+ \in \mathcal{K}_1^+\}.$$

For a plus-operator U define a linear fractional relation (LFR for brevity) $F_U : \mathcal{K}_1^+ \to \mathcal{K}_2^+$ by

$$(2.3) \qquad F_U(K_1^+) = \{K_2^+ \in \mathcal{K}_2^+ : U_{2,1} + U_{2,2}K_1^+ = K_2^+(U_{1,1} + U_{1,2}K_1^+)\}.$$

In the particular case of a bistrict plus-operator U, the LFR F_U becomes a usual single-valued linear fractional mapping (LFM for brevity) of the form

$$F_U(K_1^+) = (U_{2,1} + U_{2,2}K_1^+)(U_{1,1} + U_{1,2}K_1^+)^{-1}.$$

Analogously, for a minus-operator $T : \mathcal{H}_2 \to \mathcal{H}_1$ define an LFR $G_T : \mathcal{K}_2^- \to \mathcal{K}_1^-$ as follows:

$$(2.4) \qquad G_T(K_2^-) = \{K_1^- \in \mathcal{K}_1^- : T_{1,1}K_2^- + T_{1,2} = K_1^-(T_{2,1}K_2^- + T_{2,2})\}.$$

Let us notice that $K_1^- \in G_T(K_2^-)$ is equivalent to $TL_2^- \subseteq L_1^-$, where $L_i^- = (P_i^- + K_i^-)\mathcal{H}_i^- \in \mathcal{M}_i^-$. In the particular case when $TL_2^- \in \mathcal{M}_i^-$ we have the equality $TL_2^- = L_1^-$. Hence in this case $G_T(K_2^-)$ is a single point of $\mathcal{K}_1^-$ and $K_1^- = G_T(K_2^-)$. In the general case $G_T(K_2^-)$ is the set of all the operators $\widetilde{K}_1^- \in \mathcal{K}_1^-$ such that the restriction $\widetilde{K}_1^-|_{P_1^- TL_2^-}$ coincides with $P_1^+(P_1^-|_{TL_2^-})^{-1}$ (see [Az-I]).

Consider some basic properties of LFRs. First note that it follows immediately from the definition that

$$(2.5) \qquad \mathcal{G}_{cU}^+ = \mathcal{G}_U^+, \qquad \mathcal{G}_{cU}^- = \mathcal{G}_U^-, \qquad \text{for } 0 \neq c \in \mathbb{C}.$$

Evidently, $\mathrm{Im}(F_U) = \mathcal{G}_U^+$. The following assertion establishes the relation between $\mathcal{G}_U^-$ and $\mathrm{Im}(G_{U^{-1}})$.

THEOREM 2.1. *Let U be an invertible plus-operator. Then $T = U^{-1}$ is a minus-operator. Furthermore,*

$$(2.6) \qquad \mathcal{G}_U^- = \mathrm{Im}(G_T)$$

if and only if U is a bistrict plus-operator.

Proof. Since U is an invertible plus-operator, it follows that $T = U^{-1}$ is a minus-operator. As a consequence of invertibility of U we see [Az-I] that U and U^{-1} are respectively a strict plus-operator and a strict minus-operator. Now suppose in addition U is bistrict, that is, both U and $U^\star$ are strict plus-operators. Then ([Kh1], [Az-I]) T is a bistrict minus-operator and

$$(2.7) \qquad TL_2^- \in \mathcal{M}_1^- \qquad \forall L_2^- \in \mathcal{M}_2^-.$$

Now let $K_1^- \in \mathrm{Im}(G_T)$ and $L_1^- = (P_1^- + K_1^-)\mathcal{H}_1^- \in \mathcal{M}_1^-$. Then by the definition of $\mathrm{Im}(G_T)$ there exists $K_2^- \in \mathcal{K}_2^-$ such that $TL_2^- \subseteq L_1^-$, where $L_2^- = (P_2^- + K_2^-)\mathcal{H}_2^- \in \mathcal{M}_2^-$. Hence by (2.7), $TL_2^- = L_1^-$, and consequently, $UL_1^- = L_2^-$, that is, $K_1^- \in \mathcal{G}_U^-$.

Now let $K_1^- \in \mathcal{G}_U^-$ and as above $L_1^- = (P_1^- + K_1^-)\mathcal{H}_1^-$. Then by (2.7), $L_2^- = UL_1^- \in \mathcal{M}_2^-$, that is, there exists $K_2^- \in \mathcal{K}_2^-$ such that $L_2^- = (P_2^- + K_2^-)\mathcal{H}_2^-$. As a result we have $TL_2^- = L_1^-$, hence $K_1^- \in \mathrm{Im}(G_T)$. So, we have proved the equality (2.6).

Now assume that $\mathcal{G}_U^- = \mathrm{Im}(G_T)$. Our aim is to prove that U is a bistrict plus-operator. As mentioned above, U is strict. Take any $K_1^- \in \mathcal{G}_U^-$ and set $L_1^- = (P_1^- + K_1^-)\mathcal{H}_1^-$. By the definition of $\mathcal{G}_U^-$, $\widetilde{L}_2^- = UL_1^-$ is a non-positive subspace, so it belongs to some maximal non-positive subspace L_2^- with the angular operator $K_2^- : L_2^- = (P_2^- + K_2^-)\mathcal{H}_2^-$ and $UL_1^- \subseteq L_2^-$. At the same time $K_1^- \in \mathrm{Im}(G_T)$ and $T = U^{-1}$. Hence $L_1^- \supseteq U^{-1}L_3^-$, where $L_3^- = (P_2^- + K_3^-)\mathcal{H}_2^-$ for some $K_3^- \in \mathcal{K}_2^-$. As a result we obtain

$$L_3^- \subseteq UL_1^- \subseteq L_2^-,$$

where both L_2^- and L_3^- are maximal non-positive subspaces (that is, $L_2^-, L_3^- \in \mathcal{M}_2^-$). Consequently,

$$UL_1^- = L_2^- = L_3^-, \quad U^{-1}L_2^- = U^{-1}L_3^- = L_1^-.$$

This means that the strict minus-operator U^{-1} takes the maximal non-positive subspace L_2^- onto the maximal non-positive subspace L_1^-. Therefore [Az-I], U^{-1} is a bistrict minus-operator and consequently U is a bistrict plus-operator. ∎

REMARK 2.1. In [Kh-Sh] a reversible dynamical system was considered with an invertible bistrict evolution plus-operator U, and dichotomy of solutions, compatible with the signature of the indefinite metric, was proved. The method used in [Kh-Sh] is based on replacing $\mathcal{G}_U^-$ by $\mathrm{Im}(G_{U^{-1}})$. Theorem 2.1 shows that even in the case of an invertible strict but not bistrict plus-operator U the method of [Kh-Sh] cannot be used to prove the compatible dichotomy of solutions.

In [Kh-Z2] we proved the following

THEOREM 2.2. *Let U be a bounded bistrict plus-operator acting from $(\mathcal{H}_1, V_1)$ to $(\mathcal{H}_2, V_2)$. If $d_{V_1}^- \neq 0$ then the set $\mathcal{G}_U^-$ is non-empty and compact with respect to the weak operator topology in $\mathcal{L}(\mathcal{H}_1^-, \mathcal{H}_1^+)$.*

REMARK 2.2. In [Kh4] the compactness of $\mathcal{G}_U^-$ with respect to the weak operator topology was also proved in the general case of an arbitrary bounded plus-operator U (not bistrict and even not strict). But even in the case of a strict (but not bistrict) plus-operator U the set $\mathcal{G}_U^-$ can be empty in the non-trivial situation $d_{V_1}^- \neq 0$. The corresponding example was constructed in [Kh-Z2, Example 2.1] (see also Example 3.1 of the present paper).

Let us introduce the following

DEFINITION 2.1. A plus-operator U is called *J-expansive* if

$$[Ux, Ux]_2 \geq [x, x]_1 \quad \forall x \in \mathcal{H}_1,$$

and *J-biexpansive* if both U and $U^\star$ are *J*-expansive.

Let U act in a Krein space $(\mathcal{H}, V)$, that is, $U : \mathcal{H} \to \mathcal{H}$. Then we have

PROPOSITION 2.1. *If U is a strict plus-operator in $\mathcal{H}$ and*

$$(2.8) \qquad\qquad \mathcal{G}_U^- = \emptyset,$$

then there exists a circle C_r with center zero and radius $r > 0$ such that $C_r \subseteq \sigma(U)$.

Proof. Define $\widetilde{U} = (\mu(U))^{-1/2}U$. Then $\widetilde{U}$ is a *J*-expansive operator: $[\widetilde{U}x, \widetilde{U}x] \geq [x, x]$ for all $x \in \mathcal{H}$. Assume that there is a complex number

λ with $|\lambda| = 1$ such that λ is a regular point of $\widetilde{U}$. Without loss of generality consider $\lambda = 1$. Then $\widetilde{U}$ (see [Az-I, proof of Theorem II.4.31] $(^2)$) is a J-biexpansive operator, and consequently, U is a bistrict plus-operator. Hence by [Kh-Z2] we conclude that $\mathcal{G}_U^- \neq \emptyset$. ∎

REMARK 2.3. Notice that in the case $d_V^- = 0$ the only maximal nonpositive subspace is $\mathcal{H}^- = \{0\}$. Each linear operator U maps it into itself, so (2.8) never holds. On the other hand, any operator U such that $\|Ux\| \geq d\|x\|$ with $d > 0$ is a strict plus-operator. That is, in this case $\sigma(U)$ may surround the origin for some strict plus-operators (for example, for the bilateral shift) and not surround it for others (for example, for the identity operator $U = I$).

3. Dichotomy of solutions to the evolution equation. We now turn to the linear difference equation of the form (1.1) in a Hilbert space $\mathcal{H}$. Recall that we have assumed in the Introduction that all the operators A_n in this equation belong to the class $\mathcal{L}(\mathcal{H})$. Consider the evolution operator $U(n, m)$ $(n, m \in \mathbb{Z}_+, \; n \geq m)$ of equation (1.1). Recall that it associates with each $y_0 \in \mathcal{H}$ the solution y_n of (1.1) satisfying the initial condition $y_n|_{n=m} = y_0$, that is,

$$(3.1) \qquad U(n, m)y_0 = A_{n-1}A_{n-2}\cdots A_{m+1}A_m y_0 \quad \text{if } n > m,$$

and $U(n, n) = I$. We shall write briefly $U(n) = U(n, 0)$. Along with (1.1) consider the following sequence of difference equations in $\mathcal{H}$:

$$(3.2) \qquad z_{n+1} = A_{N-n}^\star z_n,$$

where $N \in \mathbb{Z}_+$, $n = 0, 1, \ldots, N$. For each fixed $N \in \mathbb{Z}_+$ denote by $\widetilde{U}_N(n, m)$ the evolution operator of this equation. In view of (3.1), these operators are linked to the evolution operator of (1.1) by

$$\widetilde{U}_{n+m-1}(n, m) = U^\star(n, m),$$

hence for $m = 0$,

$$(3.3) \qquad U^\star(n) = \widetilde{U}_{n-1}(n).$$

Consider a sequence of indefinite metrics in $\mathcal{H}$ of the form

$$(3.4) \qquad [x, y]_n = (V_n x, y) \quad (n \in \mathbb{Z}_+),$$

where each operator V_n belongs to $\mathcal{SL}(\mathcal{H})$, is continuously invertible and satisfies the following conditions:

(A) *The numbers $d_{V_n}^+$, $d_{V_n}^-$ do not depend on n.*
(B) $\sup_{n \in \mathbb{Z}_+} \|V_n\| = 1.$

Let us introduce the following

$(^2)$ The idea to use this theorem was suggested to us by T. Ya. Azizov.

DEFINITION 3.1. We call the sequence of quadratic forms

$$[U(n+1,n)y_0, U(n+1,n)y_0]_{n+1} - [y_0, y_0]_n \qquad (n \in \mathbb{Z}_+)$$

the *increment of the sequence of the quadratic forms* $[y_0, y_0]_n$ *with respect to the difference equation* (1.1) and denote it by $\Delta_{(1.1)}([y_0, y_0]_n)$.

From (3.4), (1.1) we obtain

$$(3.5) \qquad \Delta_{(1.1)}([y_0, y_0]_n) = [A_n y_0, A_n y_0]_{n+1} - [y_0, y_0]_n$$
$$= ((A_n^\star V_{n+1} A_n - V_n) y_0, y_0).$$

In the analogous manner we obtain

$$(3.6) \qquad \Delta_{(3.2)}([y_0, y_0]_n) = [A_{N-n}^\star y_0, A_{N-n}^\star y_0]_{n+1} - [y_0, y_0]_n$$
$$= ((A_{N-n} V_{n+1} A_{N-n}^\star - V_n) y_0, y_0).$$

We shall assume the following condition to be satisfied for the metrics (3.4) and equation (1.1):

(C) *There exists a non-increasing sequence* $\{\varepsilon_n\}_{n \in \mathbb{Z}_+}$ *of positive numbers such that*

$$\Delta_{(1.1)}([y_0, y_0]_n) \geq \varepsilon_n \|y_0\|^2 \qquad \forall n \in \mathbb{Z}_+, \, y_0 \in \mathcal{H},$$

that is,

$$A_n^\star V_{n+1} A_n - V_n \geq \varepsilon_n I \qquad \forall n \in \mathbb{Z}_+.$$

As a consequence of (C) we obtain

$$(3.7) \quad [U(n+1)y_0, U(n+1)y_0]_{n+1} - [U(n_0)y_0, U(n_0)y_0]_{n_0}$$
$$\geq \sum_{k=n_0}^{n} \varepsilon_k \|U(k)y_0\|^2 \qquad \forall y_0 \in \mathcal{H}, \, n \geq n_0.$$

Consider the following bicones, connected with the metrics (3.4) and the evolution operator of (1.1):

$$(3.8) \qquad C_n^- = \{y_0 \in \mathcal{H} : [U(n)y_0, U(n)y_0]_n \leq 0\} \qquad (n \in \mathbb{Z}_+).$$

Using this definition and (3.7) it is easy to show that

$$(3.9) \qquad\qquad\qquad C_{n+1}^- \subseteq C_n^-.$$

We set

$$(3.10) \qquad\qquad\qquad C_\infty^- = \bigcap_{n \in \mathbb{Z}_+} C_n^-.$$

LEMMA 3.1. *Assume that, besides conditions* (A)–(C), *the condition*

$$(3.11) \qquad \Delta_{(3.2)}([y_0, y_0]_k) \geq 0 \qquad \forall k \in \{0, 1, \ldots, N\}, \, y_0 \in \mathcal{H}$$

is satisfied, that is,

$$A_{N-k} V_{k+1} A_{N-k}^\star \geq V_k \qquad \forall k \in \{0, 1, \ldots, N\}, \, N \in \mathbb{N}.$$

Then the set C_∞^-, defined by (3.10), *contains a maximal non-positive subspace L_∞^- of the Krein space* $(\mathcal{H}, V_0)$.

Proof. In view of (C), (3.11) with $N = n - 1$ and (3.3), we have

$$[U(n)y, U(n)y]_n \geq [y, y]_0, \quad [U^\star(n)y, U^\star(n)y]_n \geq [y, y]_0 \quad \forall y \in \mathcal{H}, \, n \in \mathbb{N}.$$

This means that each $U(n)$ is a J-biexpansive operator. Let M_n^- be the set of maximal subspaces of the bicone C_n^-, defined by (3.8), and $\mathcal{M}_n^-$ be the set of angular operators K^- in the Krein space $(\mathcal{H}, V_0)$, corresponding to the subspaces $L^- \in M_n^-$. By Theorem 2.2, each $\mathcal{M}_n^-$ is compact with respect to the weak operator topology in $\mathcal{L}(\mathcal{H}_0^-, \mathcal{H}_0^+)$, where $\mathcal{H}_0^-$ and $\mathcal{H}_0^+$ are the components of the canonical decomposition of the Krein space $(\mathcal{H}, V_0)$. At the same time, (3.9) implies that $\mathcal{M}_{n+1}^- \subseteq \mathcal{M}_n^-$. Thus $\bigcap_{n \in \mathbb{N}} \mathcal{M}_n^-$ is non-empty. Hence a subspace $L_\infty^- \in \bigcap_{n \in \mathbb{N}} M_n^-$ is the desired maximal non-positive subspace of the Krein space $(\mathcal{H}, V_0)$ contained in C_∞^-. ∎

Let $\{\varepsilon_n\}_{n \in \mathbb{Z}_+}$ be as in condition (C) and $L_{2,\varepsilon}(\mathbb{Z}_+, \mathcal{H})$ be the Hilbert space of sequences $\overline{y} = \{y_n\}_{n \in \mathbb{Z}_+}$ of vectors in $\mathcal{H}$ satisfying the condition

$$\sum_{n=1}^\infty \varepsilon_n \|y_n\|^2 < \infty$$

with the inner product

$$(\overline{y}, \overline{z})_\varepsilon = \sum_{n=1}^\infty \varepsilon_n (y_n, z_n).$$

Denote by $\mathcal{N}$ the set of solutions of (1.1) which belong to $L_{2,\varepsilon}(\mathbb{Z}_+, \mathcal{H})$, i.e.,

(3.12) $\quad \mathcal{N} = \{\overline{y} = \{y_n\}_{n \in \mathbb{Z}_+} \in L_{2,\varepsilon}(\mathbb{Z}_+, \mathcal{H}) \,|\, y_{n+1} = A_n y_n \; \forall n \in \mathbb{Z}_+\}.$

Let $\mathcal{N}_0$ be the "slice" of $\mathcal{N}$ at time $n = 0$, i.e.

(3.13) $\qquad\qquad \mathcal{N}_0 = \{y_0 \,|\, (y_0, y_1, \ldots, y_n, \ldots) \in \mathcal{N}\}.$

We now turn to the main result of the present paper.

THEOREM 3.1. *Assume that besides conditions* (A)–(C), *the condition*

(3.14) $$\sum_{n=0}^\infty \varepsilon_n = +\infty$$

is satisfied, where $\{\varepsilon_n\}_{n \in \mathbb{Z}_+}$ is as in (C). *Then:*

(i) *The set C_∞^- is a closed subspace of $\mathcal{H}$, and*

(3.15) $\qquad\qquad\qquad\qquad C_\infty^- = \mathcal{N}_0;$

(ii) *For any $y_0 \in C_\infty^-$ the solution $y_n = U(n)y_0$ of* (1.1) *satisfies*

(3.16) $$\sum_{k=n+1}^\infty \varepsilon_k \|y_k\|^2 \leq I(y_0) \prod_{k=0}^n (1 + \varepsilon_k)^{-1},$$

where

$$I(y_0) = \sum_{k=1}^{\infty} \varepsilon_k \|y_k\|^2; \tag{3.17}$$

(iii) *For any* $y_0 \notin C_{\infty}^{-}$ *there exists* $C(y_0) > 0$ *such that the solution* $y_n = U(n)y_0$ *of* (1.1) *satisfies*

$$\|y_n\|^2 \geq C(y_0) \prod_{k=0}^{n} (1 + \varepsilon_k); \tag{3.18}$$

(iv) *If, in addition, condition* (3.11) *is satisfied, then* C_{∞}^{-} *is a maximal non-positive subspace of the Krein space* $(\mathcal{H}, V_0)$.

Proof. First of all, let us prove (iii). Assume that $y_0 \notin C_{\infty}^{-}$. Then, in view of (3.8)–(3.10), there exists $n_0 \in \mathbb{Z}_+$ such that

$$[U(n_0)y_0, U(n_0)y_0]_{n_0} > 0. \tag{3.19}$$

Using condition (B), we obtain from (3.7) the following inequality:

$$\|U(n+1)y_0\|^2 \geq [U(n_0)y_0, U(n_0)y_0]_{n_0} \tag{3.20}$$
$$+ \sum_{k=n_0}^{n} \varepsilon_k \|U(k)y_0\|^2 \quad \forall n \geq n_0,$$

which can be rewritten in the form

$$Y_{n+1} - Y_n \geq \varepsilon_{n+1} Y_n \quad (n \geq n_0), \tag{3.21}$$

where

$$Y_n = [U(n_0)y_0, U(n_0)y_0]_{n_0} + \sum_{k=n_0}^{n} \varepsilon_k \|U(k)y_0\|^2. \tag{3.22}$$

Inequalities (3.19) and (3.21) imply that

$$Y_n \geq [U(n_0)y_0, U(n_0)y_0]_{n_0} \prod_{k=n_0}^{n} (1 + \varepsilon_{k+1}) \quad (n \geq n_0).$$

Taking into account (3.19), (3.20) and (3.22), we obtain from the last inequality the desired estimate (3.18) with some positive constant $C(y_0)$.

To prove (i), observe that (3.18) and (3.14) imply that, if $y_0 \notin C_{\infty}^{-}$, the solution $y_n = U(n)y_0$ $(n \in \mathbb{Z}_+)$ of (1.1) does not belong to $L_{2,\varepsilon}(\mathbb{Z}_+)$. This means that

$$\mathcal{N}_0 \subseteq C_{\infty}^{-}. \tag{3.23}$$

Conversely, assume that $y_0 \in C_{\infty}^{-}$. Then, by (3.8) and (3.9),

$$[U(n)y_0, U(n)y_0]_n \leq 0 \quad \forall n \in \mathbb{Z}_+.$$

This fact and (3.7) imply that

$$(3.24) \qquad \sum_{k=n_0}^{n} \varepsilon_k \|U(k)y_0\|^2 \leq [U(n_0)y_0, U(n_0)y_0]_{n_0} \quad \forall n \geq n_0,$$

hence the solution $y_n = U(n)y_0$ $(n \in \mathbb{Z}_+)$ of (1.1) belongs to $L_{2,\varepsilon}(\mathbb{Z}_+)$, i.e., $y_0 \in \mathcal{N}_0$. So, we have proved (3.15). Since C_∞^- is closed in $\mathcal{H}$ as the intersection of the closed sets C_n^-, and the set $\mathcal{N}_0$ is linear (by (3.12), (3.13)), we have proved (i).

To prove (ii), assume, as above, that $y_0 \in C_\infty^-$. Let $n \to \infty$ in (3.24) and then substitute n for n_0 to obtain, in view of (B),

$$\sum_{k=n+1}^{\infty} \varepsilon_k \|U(k)y_0\|^2 \leq \|U(n)y_0\|^2 \quad \forall n \in \mathbb{Z}_+.$$

Define

$$(3.25) \qquad Z_n = \sum_{k=n+1}^{\infty} \varepsilon_k \|U(k)y_0\|^2.$$

Then the last inequality can be rewritten in the form

$$Z_{n+1} \leq (1 + \varepsilon_{n+1})^{-1} Z_n \quad \forall n \in \mathbb{Z}_+.$$

This estimate and (3.25) imply the desired estimate (3.16).

Assertion (iv) follows from the obvious inclusion $C_\infty^- \subseteq C_0^-$ and Lemma 3.1. ∎

REMARK 3.1. In [B-G2] an irreversible dynamical system similar to (1.1) was considered under condition (C) with $\varepsilon_n = $ const. But the results obtained there differ from those of Theorem 3.1. In particular, in [B-G2] the so-called *inertia theorem* was proved (Theorem 2.5). It asserts that the operator $\zeta I - T$ is left-invertible for any $\zeta \in \mathbb{T}$, where T is the operator defined in $L_2(\mathbb{Z}_+, \mathcal{H})$ by the block matrix

$$(3.26) \qquad \qquad \{\delta_{i,j+1} A_j\}_{i,j \in \mathbb{Z}_+},$$

and, furthermore, a formula for the dimension of $\ker(\zeta I - T)^\star$ is valid in terms of the operators A_n and V_n (see formula (2.6) of [B-G2]). On the other hand, Theorem 3.1 of the present paper gives information about the asymptotic behavior of sequences from the subspace $\ker(I - T)$, which coincides with the subspace $\mathcal{N}$ defined by (3.12). Theorem 3.1(iv) also gives information about the dimension of the latter subspace. Furthermore, Theorem 3.1 states that all those solutions of (1.1) which do not belong to $\mathcal{N}$ grow to infinity with a certain rate.

The following example shows that condition (3.11) cannot be eliminated in assertion (iv) of the previous theorem.

EXAMPLE 3.1. Let

(3.27)
$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where

$$\mathcal{H}^+ = \mathrm{cl}(\mathrm{span}(\{e_i\}_{i=1}^\infty)),$$
$$(e_i, e_j) = \delta_{i,j}, \; i, j = 1, 2, \ldots, \quad \mathcal{H}^- = \mathrm{span}(\{e_-\}), \; (e_-, e_-) = 1.$$

Define an indefinite metric in $\mathcal{H}$ by $[x, y] = (Vx, y)$, where $V = P_+ - P_-$ and P_+, P_- are the orthogonal projections onto $\mathcal{H}^+$ and $\mathcal{H}^-$, respectively. Consider the operator $U = U(\lambda)$ acting in $\mathcal{H}$ and having the following block matrix representation with respect to the splitting (3.27):

(3.28)
$$U = (U_{i,j})_{i,j=1}^2,$$

where

(3.29)
$$U_{1,1}e_k = e_{k+1}, \; k \in \mathbb{N}, \quad U_{1,2}e_- = e_1,$$
$$U_{2,1} = 0, \quad U_{2,2} = \lambda I, \quad \lambda \in \mathbb{C}, \; |\lambda| \le \sqrt{2}.$$

Then $U = U(\lambda)$ is a bounded left-invertible operator, and $[Ux, Ux] - [x, x] = |\beta|^2(2 - |\lambda|^2) \ge 0$ for any $x = \sum_{k=1}^\infty \alpha_k e_k + \beta e_- \in \mathcal{H}$, that is, U is a J-expansion. Let us calculate $\mathcal{G}_{U(\lambda)}^-$. A simple criterion for $K^- \in \mathcal{G}_{U(\lambda)}^-$ is the following (see [Kh4, proof of Theorem 2.1]):

(3.30)
$$(K^-)^\star(U_{1,1}^\star U_{1,1} - U_{2,1}^\star U_{2,1})K^- + (K^-)^\star(U_{1,1}^\star U_{1,2} - U_{2,1}^\star U_{2,2})$$
$$+ (U_{1,2}^\star U_{1,1} - U_{2,2}^\star U_{2,1})K^- + (U_{1,2}^\star U_{1,2} - U_{2,2}^\star U_{2,2}) \le 0.$$

In our case

$$U_{2,1} = U_{1,1}^\star U_{1,2} = 0, \quad U_{1,1}^\star U_{1,1} = U_{1,2}^\star U_{1,2} = I, \quad U_{2,2}^\star U_{2,2} = |\lambda|^2 I,$$

and (3.30) becomes

$$\|K^-\|^2 + 1 - |\lambda|^2 \le 0.$$

This shows that

(3.31)
$$\mathcal{G}_{U(\lambda)}^- = \emptyset \quad \text{for } |\lambda| < 1,$$

(3.32)
$$\mathcal{G}_{U(\lambda)}^- \ne \emptyset \quad \text{for } 1 \le |\lambda| \le \sqrt{2}.$$

Consider the operator $A = cU$, $|c| > 1$, and the autonomous difference equation

(3.33)
$$y_{n+1} = Ay_n.$$

Take any $y_0 = \sum_{k=1}^\infty \alpha_k e_k + \beta e_- \in \mathcal{H}$. Then

$$[Ay_0, Ay_0] - [y_0, y_0] = (|c|^2 - 1)\sum_{k=1}^\infty |\alpha_k|^2 + |\beta|^2(1 + |c|^2(1 - |\lambda|^2)) \ge d\|y_0\|^2,$$

where $d = \min\{|c|^2 - 1, 1 + |c|^2(1 - |\lambda|^2)\}$. Evidently, $d > 0$ for $|\lambda| \le 1$ and any c with $|c| > 1$, and also for $1 < |\lambda| \le \sqrt{2}$ and $1 < |c| < 1/\sqrt{|\lambda|^2 - 1}$, i.e.,

in these two cases condition (C) is satisfied for equation (3.33) with $\varepsilon_n = d$. It is clear that also conditions (A), (B) are satisfied for $[x,y]_n = [x,y]$. On the other hand, in view of (2.5) and (3.31), $C_n^- = \{0\}$ for each $n \in \mathbb{N}$, hence $C_\infty^- = \{0\}$ for all $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and

$$(3.34) \qquad \dim(C_\infty^-) = 1 \quad \forall \lambda \in \mathbb{C},\, 1 \leq |\lambda| \leq \sqrt{2}.$$

Thus for $|\lambda| < 1$ Theorem 3.1 implies that for any $0 \neq y_0 \in \mathcal{H}$ the solution $y_n = U(n)y_0$ of (3.33) tends to infinity exponentially as $n \to \infty$. This means that the stable subspace of (3.33) is $\{0\}$, which is evidently not a maximal subspace in C_0^-. So, for $|\lambda| < 1$ assertion (iv) of Theorem 3.1 is not valid.

But if $1 \leq |\lambda| \leq \sqrt{2}$ we have the following picture in view of (3.32): by Theorem 3.1(ii), (iii), for any y_0 in the one-dimensional subspace C_∞^- the solution $y_n = U(n)y_0$ of (3.33) tends to zero exponentially as $n \to \infty$, and for any $0 \neq y_0 \in \mathcal{H} \setminus C_\infty^-$ it tends to infinity exponentially as $n \to \infty$. Observe that, in view of (3.34), in this case assertion (iv) of Theorem 3.1 is valid.

Let us show that condition (3.11) holds for neither $|\lambda| < 1$ nor $1 \leq |\lambda| \leq \sqrt{2}$. Indeed, for $y_0 \in \mathcal{H}^+$,

$$[U^\star y_0, U^\star y_0] - [y_0, y_0] = -2.$$

REMARK 3.2. A simple version (with $\lambda = 0$) of Example 3.1 has been considered in [B-G2] as an example of equation (1.1) which satisfies condition (C), but $\zeta I - T$ is not right-invertible for any $\zeta \in \mathbb{T}$, where T is defined in $L_2(\mathbb{Z}_+, \mathcal{H})$ by the block matrix (3.26).

REMARK 3.3. The case $1 \leq |\lambda| \leq \sqrt{2}$ of Example 3.1 shows that condition (3.11) is not necessary for assertion (iv) of Theorem 3.1 to be valid.

REMARK 3.4. It would be interesting to construct an example analogous to the case $|\lambda| < 1$ of Example 3.1 for an evolution equation with continuous time:

$$(3.35) \qquad \frac{dx}{dt} = A(t)x.$$

Observe that there is little chance to construct such an example for the autonomous equation ($A(t) = A = \text{const}$). If we have a strict plus-operator A with $\mathcal{G}_A^- = \emptyset$ and try to extend the discrete semigroup $\{A^n\}_{n \in \mathbb{Z}_+}$ to a semigroup with continuous time $t \in [0, \infty)$, we see that the operator A^t does not exist in the sense of Dunford for $t \notin \mathbb{Z}_+$ by Proposition 2.1. But it would be interesting to construct such an example for a non-autonomous dynamical system of the type (3.35).

EXAMPLE 3.2. In a Hilbert space $\mathcal{H}$ consider the operator A defined in Example 3.1 with $\lambda = 0$, that is, $A = cU(0)$ ($c > 1$), where $U(\lambda)$ is the operator defined by the block matrix representation (3.28), (3.29). Consider

the Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathbb{C}$ and denote by $(\cdot, \cdot)_{\widetilde{\mathcal{H}}}$ the inner product in it. Define an indefinite inner product in $\widetilde{\mathcal{H}}$ by $[x, y]_{\widetilde{\mathcal{H}}} = (\widetilde{V}x, y)_{\widetilde{\mathcal{H}}}$, where $\widetilde{V} = V \oplus (-I)$ and V is the operator from Example 3.1, which defines the indefinite metric in $\mathcal{H}$. Also consider the operator $\widetilde{A} = A \oplus \theta I$ with $\theta \in (0, 1)$ and the corresponding autonomous difference equation in $\widetilde{\mathcal{H}}$:

$$(3.36) \qquad\qquad y_{n+1} = \widetilde{A}y_n.$$

It is easy to check that for some $\varepsilon > 0$,

$$\Delta_{(3.36)}([y_0, y_0]_{\widetilde{\mathcal{H}}}) \geq \varepsilon(y_0, y_0)_{\widetilde{\mathcal{H}}} \quad \forall n \in \mathbb{Z}_+,\ y_0 \in \widetilde{\mathcal{H}}.$$

Furthermore, $d_{\widetilde{V}}^- = 2$, because $d_V^- = 1$ in $\mathcal{H}$. On the other hand, in view of the result of Example 3.1, equation (3.36) exhibits the exponential dichotomy with a one-dimensional stable subspace. So, in the constructed example the dichotomy is not compatible with the signature of the indefinite metric.

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