Rank and the Drazin inverse in Banach algebras

by

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Abstract. Let A be an arbitrary, unital and semisimple Banach algebra with nonzero socle. We investigate the relationship between the spectral rank (defined by B. Aupetit and H. Mouton) and the Drazin index for elements belonging to the socle of A. In particular, we show that the results for the finite-dimensional case can be extended to the (infinite-dimensional) socle of A.

1. Introduction. Throughout this paper we shall assume that A is a semisimple, complex and unital Banach algebra with the unit denoted by 1 and the group of invertible elements by A^{-1}. We denote by $\exp(A)$ the set of all exponentials of A, that is, $\exp(A) = \{e^x : x \in A\}$. For $a \in A$ we denote the spectrum of a by $\sigma(a, A)$, with the convention that we write $\sigma(a)$ if the algebra under discussion is clear from the context. The nonzero spectrum of $a \in A$ is denoted $\sigma'(a) = \sigma(a) \setminus \{0\}$. An element $a \in A$ belongs to the *socle* of A, denoted $\mathrm{Soc}(A)$, if there exist finitely many minimal left ideals or, equivalently, minimal right ideals such that a belongs to their sum. The set $\mathrm{Soc}(A)$ is a two-sided ideal of A. We assume $\mathrm{Soc}(A) \neq \{0\}$.

Following [3] we define the *rank* of an element $a \in A$ by

$$\mathrm{rank}(a) = \sup_{x \in A} \#\sigma'(xa) \leq \infty$$

where $\#$ denotes the number of elements in a set. By the semisimplicity of A we know that $\mathrm{rank}(a) = 0$ if and only if $a = 0$. Notice further that for $a \in A$ we have $\mathrm{rank}(a) \leq \mathrm{rank}(1)$ with equality whenever $a \in A^{-1}$. It can be shown ([3, Corollary 2.9]) that $\mathrm{Soc}(A) = \{a \in A : \mathrm{rank}(a) < \infty\}$, the set of *finite rank* elements of A. Also, if $a \in \mathrm{Soc}(A)$ then ([3, Theorem 2.2]) the set

$$E(a) = \{x \in A : \#\sigma'(xa) = \mathrm{rank}(a)\}$$

is a dense open subset of A. Amongst the elements of $\mathrm{Soc}(A)$ Aupetit and Mouton distinguish the *maximal finite rank* elements as those $a \in \mathrm{Soc}(A)$

with $\text{rank}(a) = \#\sigma'(a)$. Since the sets $E(a)$ are dense in A the set of maximal finite rank elements is dense in $\text{Soc}(A)$.

If A is finite-dimensional then $A = \text{Soc}(A)$, and conversely, if $A = \text{Soc}(A)$ then A is finite-dimensional. In general however, $\text{Soc}(A)$ is not necessarily closed in A and hence not necessarily finite-dimensional. A particularly good reason for studying this notion of (spectral) rank is that it coincides with the standard rank in the case $A = B(X)$, the Banach algebra of bounded linear operators on a Banach space X ([3, p. 118]); for some pathologies occurring on a previous rank definition used by several other authors see [3, p. 124]. The results obtained in this paper support the evidence pointing to the spectral rank as a suitable notion of "rank" in the case of general semisimple Banach algebras. In particular, our main result, Theorem 2.7, extends the relationship between rank and Drazin index (defined hereafter), which is known to hold for complex matrices ($M_n(\mathbb{C})$), to the socle of a semisimple Banach algebra.

An element $a \in A$ is said to be *Drazin invertible* in A if there exist $b \in A$ and $k \in \mathbb{Z}^+$ such that

(i) $a^k b a = a^k$,
(ii) $bab = b$,
(iii) $ba = ab$.

The unique element ([11, Lemma 1]) b satisfying (i)–(iii) is called the *Drazin inverse* of a. As is customary we denote the Drazin inverse of a by a^D and we call the least $k \in \mathbb{Z}^+$ such that (i) holds the *Drazin index* of a. If for $a \neq 0$ we define $a^0 = 1$ then it is clear that A^{-1} is precisely the subset of Drazin invertible elements with Drazin index zero. The subset of Drazin invertible elements with Drazin index equal to zero or one is called the *group invertibles*, where the term "group" refers to the fact that if a is group invertible with *group inverse* a^D, then $\{a, a^D\}$ generates an abelian group with identity aa^D. Notice that if a is Drazin invertible with Drazin inverse a^D then a^D is always group invertible (with group inverse $a^2 a^D$), even though a might not be group invertible. We denote the set of Drazin invertible elements by $\mathcal{D}(A)$ and the set of group invertibles by $\mathcal{G}(A)$. Although we restrict ourselves in this paper to the sets $\mathcal{D}(A)$ and $\mathcal{G}(A)$ we mention that the Drazin inverse belongs to the broad and intensively studied subject of generalized inverses, which is not confined only to the Banach algebra setting. For more on this topic (and some applications) see [6], [9].

In [5] it was shown that $\text{Soc}(A) \subseteq \mathcal{D}(A)$ and that $\text{Soc}(A)$ is the largest ideal contained in $\mathcal{D}(A)$. In fact, it was shown there that if J is a left or right multiplicative ideal (i.e. J only needs to absorb products in A from the left or the right) such that $J \subseteq \mathcal{D}(A)$ then $J \subseteq \text{Soc}(A)$. Since every maximal finite rank element can be written as a linear combination of orthogonal

rank 1 idempotents ([3, Theorem 2.8]) it follows that the maximal finite rank elements are all group invertible in $\text{Soc}(A)$. Hence $\text{Soc}(A) \cap \mathcal{G}(A)$ is dense in $\text{Soc}(A)$. In general (even for finite-dimensional cases) it is not true that every element of $\text{Soc}(A)$ is group invertible; it was shown in [5] that $\text{Soc}(A) \subseteq \mathcal{G}(A)$ if and only if every element of $\text{Soc}(A)$ belongs to the center of A. Using generalized inverses, we show here (Corollary 2.2 (ii)⇔(iii)) that the property "every element of $\text{Soc}(A)$ is central in A" is equivalent to an apparently much weaker condition on $\text{Soc}(A)$.

In order to prove Theorem 2.7 we need a structural characterization of Aupetit and Mouton's spectral definition of rank for elements belonging to $\text{Soc}(A)$. We start with some terminology and notation.

An element of an algebra A is *indecomposable* if it cannot be written as a sum $a = b + c$ where $b, c \in A$ are nonzero elements satisfying $bAc = 0$.

We denote by $M_{r,n}$, with $r \leq n \leq 2r$, the algebra of all $n \times n$ complex matrices $[a_{i,j}]$ satisfying $a_{i,j} = 0$ whenever $i > r$ or $j \leq n - r$.

We can now state the required result which we shall refer to as the *rank structure decomposition* (see [4, p. 289]) of an element belonging to $\text{Soc}(A)$.

THEOREM 1.1 (M. Brešar, P. Šemrl). *Let A be a semisimple unital complex Banach algebra and let $n \in \mathbb{N}$. Then $a \in A$ has $\text{rank}(a) = n$ if and only if there exist $a_1, \ldots, a_k \in A$ such that*

  (i) $a = a_1 + \cdots + a_k$,
  (ii) *each a_i is indecomposable,*
  (iii) $a_i A a_j = 0$ *whenever $i \neq j$,*
  (iv) $a_i A a_i \simeq M_{r_i, n_i}$ *for some $r_i, n_i \in \mathbb{N}$, $r_i \leq n_i \leq 2r_i$,*
  (v) $n = r_1 + \cdots + r_k$.

Moreover, $a_1, \ldots, a_k$ are unique nonzero elements in $\text{Soc}(A)$ satisfying (i)–(iii).

As a first application of Theorem 1.1 we give an alternative proof of [3, Theorem 2.12] which is perhaps a bit easier and in fact proves a little more.

THEOREM 1.2 (B. Aupetit, H. du T. Mouton). *For $a \in A$,*

$$\text{rank}(a) \leq \dim(aAa) \leq (\text{rank}(a))^2.$$

Moreover, if $a \in \text{Soc}(A)$ then $\text{rank}(a) = \dim(aAa)$ if and only if $aAa \simeq M_{1,n_1} \oplus \cdots \oplus M_{1,n_k}$ where $n_i \in \{1, 2\}$ and $k = \text{rank}(a)$.

Proof. As in the first part of the proof of [3, Theorem 2.12] it suffices to consider $a \in \text{Soc}(A)$. So let $a \in \text{Soc}(A)$ with rank structure decomposition $a = a_1 + \cdots + a_k$ and $\text{rank}(a) = r_1 + \cdots + r_k$ as in 1.1. For each i we first prove

(1.2.1) $a_i A a_i \cap (a_1 A a_1 + \cdots + a_{i-1} A a_{i-1} + a_{i+1} A a_{i+1} + \cdots + a_k A a_k) = \{0\}.$

If for some set $\{x_1, \ldots, x_k\} \in A$ we have

$$a_i x_i a_i = \sum_{\substack{j=1 \\ j \neq i}}^{k} a_j x_j a_j$$

then let $z \in A$ be such that $\mathrm{rank}(a_i x_i a_i) = \#\sigma'(a_i x_i a_i z)$. Using $a_i A a_j = \{0\}$ we obtain

$$\#\sigma'((a_i x_i a_i z)^2) = \#\sigma'\Big(a_i x_i a_i z \Big(\sum_{\substack{j=1 \\ j \neq i}}^{k} a_j x_j a_j z\Big)\Big) = 0$$

and by the spectral mapping theorem it follows that $\#\sigma'(a_i x_i a_i z) = 0$. Thus $\mathrm{rank}(a_i x_i a_i) = 0$ and hence $a_i x_i a_i = 0$, which proves (1.2.1).

It follows further that

$$aAa = (a_1 + \cdots + a_k)A(a_1 + \cdots + a_k) \subseteq a_1 A a_1 + \cdots + a_k A a_k$$

where $\dim(a_i A a_i) = r_i^2$ for each i. Let $V_i = \{e_1^{(i)}, \ldots, e_{r_i^2}^{(i)}\}$ be a basis for $a_i A a_i$. Since $a_i A a_i$ is finite-dimensional one may easily prove that

$$B_i = \{x \in A : a_i x a_i = \alpha_1 e_1^{(i)} + \cdots + \alpha_{r_i^2} e_{r_i^2}^{(i)} \text{ with } \alpha_j \neq 0 \text{ for each } j\}$$

is a dense and open subset of A. It follows from Baire's theorem that $\dim(aAa) = \dim(a_1 A a_1 + \cdots + a_k A a_k)$ and hence that

(1.2.2) $$aAa = a_1 A a_1 + \cdots + a_k A a_k.$$

We thus have

$$\begin{aligned}
\mathrm{rank}(a) = r_1 + \cdots + r_k &\leq r_1^2 + \cdots + r_k^2 \\
&= \dim(a_1 A a_1) + \cdots + \dim(a_k A a_k) \\
&= \dim(a_1 A a_1 + \cdots + a_k A a_k) = \dim(aAa) \\
&\leq (r_1 + \cdots + r_k)^2 = (\mathrm{rank}(a))^2.
\end{aligned}$$

If $\mathrm{rank}(a) = \dim(aAa)$ then, following the above arguments, we deduce that $r_i = 1$ for each i. Thus $a_i A a_i = \mathbb{C} a_i \simeq M_{1,r}$ where $r = 2$ if a_i is nilpotent and $r = 1$ if a_i is not nilpotent. The result follows by observing that (1.2.2) is actually a direct sum.

On the other hand, if $aAa \simeq M_{1,n_1} \oplus \cdots \oplus M_{1,n_k}$ where $n_i \in \{1, 2\}$ and $k = \mathrm{rank}(a)$ then it follows directly that $\dim(aAa) = k = \mathrm{rank}(a)$. ∎

The following lemma will be instrumental in some of our results and is a special case of [7, Theorem 6.4]. Note that, since every element of $\mathrm{Soc}(A)$ is algebraic, we can replace quasinilpotent in [7, Theorem 6.4] by nilpotent.

LEMMA 1.3 (core-nilpotent decomposition). *Let $a \in \mathrm{Soc}(A)$. Then a has a unique decomposition of the form $a = c_a + r_a$ where c_a is a group invertible element in $\mathrm{Soc}(A)$, r_a is a nilpotent element in $\mathrm{Soc}(A)$ and $c_a r_a = r_a c_a = 0$.*

REMARK. One may actually observe that, by the uniqueness of the core-nilpotent decomposition, $c_a = aa^D a$ and $r_a = a(1 - aa^D)$. The element c_a is called the *core* of a and r_a the *nilpotent part* of a. The idempotent $1 - aa^D$ is called the *spectral projection of a corresponding to 0*; the terminology being appropriate in view of the holomorphic functional calculus (see 1.6).

It is also appropriate, at this stage, to recall two facts that will be used throughout the remainder of this paper without specific reference:

(1.4)   If A is a semisimple Banach algebra with identity and $p \in A$ is an idempotent, then pAp is a semisimple Banach algebra with identity p. Moreover, $\sigma'(pxp, A) = \sigma'(pxp, pAp)$ for every $x \in A$.

(1.5)   If A is a Banach algebra and $a_1, \ldots, a_n \in A$ with $a_i a_j = a_j a_i = 0$ for $i \neq j$, then $\sigma'(a_1 + \cdots + a_n) = \bigcup_{i=1}^{n} \sigma'(a_i)$.

The first part of (1.4) and the containment $\sigma'(pxp, pAp) \subseteq \sigma'(pxp, A)$ are well known ([1, Lemma 2.5]). For the reverse containment notice that if $\lambda \neq 0$ and pbp is the inverse to $\lambda p - pap$ in pAp then $pbp + \frac{1}{\lambda}(1 - p)$ is the inverse to $\lambda - pap$ in A. For (1.5), use induction on the argument in the proof of [3, Lemma 2.15]. Concerning (1.4), recall that if pAp is finite-dimensional then the structure of pAp is given by the Wedderburn–Artin theorem.

A particularly useful representation of the Drazin inverse is given by the holomorphic functional calculus: if $a \in \mathrm{Soc}(A)$ then 0 is an isolated point of $\sigma(a)$ or possibly, in the finite-dimensional case, $0 \notin \sigma(a)$. If the latter instance occurs then the Drazin inverse is just the usual inverse a^{-1} and the representation follows easily. If 0 is isolated in $\sigma(a)$ then let U_0 be an open ball with center 0 and U_1 be an open set containing $\sigma(a) \setminus \{0\}$ such that U_0 and U_1 are separated in $\mathbb{C}$. Let Γ_0 be a circle in U_0 surrounding 0 and let Γ_1 be a smooth contour in U_1 surrounding $\sigma(a) \setminus \{0\}$. By the holomorphic functional calculus, the Drazin inverse of a is given by

(1.6)
$$a^D = \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} g(\lambda)(\lambda - a)^{-1} \, d\lambda$$

where

$$g(\lambda) = \begin{cases} 1/\lambda, & \lambda \in U_1, \\ 0, & \lambda \in U_0. \end{cases}$$

2. Rank and the Drazin inverse. For $a \in A$ we define the *commutant* of a as the set $\mathrm{comm}(a) = \{x \in A : xa = ax\}$. If B is a subset of A and $a \in \mathrm{Soc}(A)$ then a is said to *assume its rank on B* if there is $b \in B$ such that $\mathrm{rank}(a) = \#\sigma'(ba)$. Recall further that $x \in A$ is said to be *central* in A if x commutes with every element of A. As our first basic result, relating rank to group invertibility, we have

PROPOSITION 2.1. *Let $a \in \mathrm{Soc}(A)$. If a assumes its rank on $\mathrm{comm}(a)$ then $a \in \mathcal{G}(A)$.*

Proof. If $\mathrm{rank}(a) = 0$ then $a = 0$ and obviously the result holds. So let $a \in \mathrm{Soc}(A)$ with $\mathrm{rank}(a) \neq 0$ and let $u \in \mathrm{comm}(a)$ be such that $\mathrm{rank}(a) = \#\sigma'(ua)$. Now, since a and u commute, we cannot have $\sigma(a) = \{0\}$, and if $0 \notin \sigma(a)$ (which may happen in the finite-dimensional case) then the result follows trivially. Hence, using $2 \leq \#\sigma(a) < \infty$, it follows that

$$l = \inf\{|\lambda - \alpha| : \lambda, \alpha \in \sigma(a), \lambda \neq \alpha\} > 0.$$

Denote by B the Banach algebra generated by $\{1, a, u\}$. Choose $\alpha \in \mathbb{C}$ with $|\alpha| > \max\{\|u\|, l^{-1} \operatorname{diam} \sigma(ua)\}$. For every complex homomorphism Φ on B we have $\Phi(ua) \neq 0 \Rightarrow \Phi((\alpha - u)a) \neq 0$, thus establishing a correspondence $\sigma'(ua) \mapsto \sigma'((\alpha - u)a)$. Also, since $|\alpha| > l^{-1} \operatorname{diam} \sigma(ua)$, the correspondence is one-one. So there is $v \in \mathrm{comm}(a) \cap A^{-1}$ such that $\mathrm{rank}(a) = \#\sigma'(av)$. Moreover, it is easily seen that av is of maximal rank in $\mathrm{Soc}(A)$. By the diagonalisation theorem ([3, Theorem 2.8]) it follows that av is group invertible. If b is the group inverse for av then, in view of 1.6, b commutes with both a and v. It follows by the invertibility of v that vb is the group inverse of a. ∎

By definition, the set of maximal finite rank elements is contained in the set $R = \{a \in \mathrm{Soc}(A) : a \text{ assumes its rank on } \mathrm{comm}(a)\}$ and hence R is also dense in $\mathrm{Soc}(A)$. However, it is easy to see that the containment is strict in general. On the other hand, consideration of matrix algebras shows that Proposition 2.1 does not characterize elements of $\mathrm{Soc}(A)$ belonging to $\mathcal{G}(A)$. Our next result is an extension of [5, Theorem 11].

COROLLARY 2.2. *The following conditions on $\mathrm{Soc}(A)$ are equivalent:*

(i) $\mathrm{Soc}(A) \subseteq \mathcal{G}(A)$,

(ii) *every element of $\mathrm{Soc}(A)$ is central in A,*

(iii) *every $a \in \mathrm{Soc}(A)$ assumes its rank on $\mathrm{comm}(a)$,*

(iv) *0 is the only nilpotent element in $\mathrm{Soc}(A)$,*

(v) *the distance between any two distinct idempotents in $\mathrm{Soc}(A)$ is at least one,*

(vi) *the set of rank 1 idempotents is orthogonal.*

Proof. The equivalence of (i) and (ii) was shown in [5]. Obviously (ii) implies (iii), and (iii) implies (i) by Proposition 2.1. If 0 is the only nilpotent element in $\mathrm{Soc}(A)$ then, by the core-nilpotent decomposition, $\mathrm{Soc}(A) \subseteq \mathcal{G}(A)$. Conversely, if $a \in \mathrm{Soc}(A)$ is group invertible and $a^k = 0$ for some $k \in \mathbb{N}$ then, since a and a^D commute, we have $\sigma(aa^D) = \{0\}$, which implies $aa^D = 0$ because aa^D is an idempotent. Thus $aa^D a = a = 0$.

Suppose (ii) holds. If p and q are distinct idempotents in $\mathrm{Soc}(A)$ then it follows that $(p-q)^3 = p-q$ and hence that $\|p-q\| \geq 1$. Thus (ii) implies (v). On the other hand, if (v) holds then, since $\mathrm{Soc}(A)$ is a two-sided ideal, it follows that, for an idempotent $p \in \mathrm{Soc}(A)$ and $x \in A$ arbitrary, $e^{\lambda x} p e^{-\lambda x} \in \mathrm{Soc}(A)$ for each $\lambda \in \mathbb{C}$. By continuity of the map $\lambda \mapsto e^{\lambda x} p e^{-\lambda x} - p$ there exists an open ball $B_0 = B(0, \varepsilon)$, $\varepsilon > 0$, in $\mathbb{C}$ such that $\|e^{\lambda x} p e^{-\lambda x} - p\| < 1$ for all $\lambda \in B_0$. So, using the assumption in (v), $e^{\lambda x} p e^{-\lambda x} - p = 0$ for all $\lambda \in B_0$. Since the function $\lambda \mapsto e^{\lambda x} p e^{-\lambda x} - p$ is also analytic on $\mathbb{C}$ it follows by the scarcity theorem for the rank ([3, Theorem 2.3]) that $\mathrm{rank}(e^{\lambda x} p e^{-\lambda x} - p) = 0$ for all $\lambda \in \mathbb{C}$. So, for $\lambda = 1$, we have $e^x p = p e^x$, that is, p commutes with every element of $\exp(A)$. Since $\exp(A) + \exp(A) = A$ it follows that p is central in A. The implication (v)$\Rightarrow$(ii) now follows from the density of the maximal finite rank elements in $\mathrm{Soc}(A)$.

If (ii) holds and p, q are nonzero rank 1 idempotents then $pq \neq 0$ implies $pq = p = q$ using the minimality of p and q together with Theorem 1.2. Thus (ii) implies (vi). Finally, if (vi) holds then let p be a rank 1 idempotent and let $a \in A$ be arbitrary. If we choose $\alpha, \beta \in \mathbb{C}$ such that α, β belong to the resolvent of a with $\alpha \neq \beta$ then $(\alpha - a)p(\alpha - a)^{-1}$ and $(\beta - a)p(\beta - a)^{-1}$ are also rank 1 idempotents. If neither of them equals p then, by assumption, their respective products with p are zero. But this yields $\alpha p = pap = \beta p$, which is a contradiction. Hence $p = (\alpha - a)p(\alpha - a)^{-1}$ for some α in the resolvent of a, from which one obtains $pa = ap$. Again, by the density of the maximal finite rank elements, (ii) follows. ∎

As opposed to Corollary 2.2 we now identify those elements of $\mathrm{Soc}(A)$ belonging to $\mathcal{G}(A)$ which, together with Lemmas 2.5 and 2.6, lead to a connection between the Drazin index and the rank, thus generalizing the case $A = M_n(\mathbb{C})$ ([9, p. 126]) to the socle of an arbitrary semisimple Banach algebra.

THEOREM 2.3. *If $a \in \mathrm{Soc}(A)$ with Drazin inverse a^D then $a \in \mathcal{G}(A)$ if and only if $\mathrm{rank}(a) = \mathrm{rank}(a^D)$.*

Proof. $\Rightarrow$ If a is group invertible with group inverse a^D then

$$\mathrm{rank}(a) = \mathrm{rank}(a^2 a^D) \leq \mathrm{rank}(a^D) = \mathrm{rank}(a(a^D)^2) \leq \mathrm{rank}(a).$$

Thus $\mathrm{rank}(a) = \mathrm{rank}(a^D)$.

$\Leftarrow$ Suppose $\mathrm{rank}(a) = \mathrm{rank}(a^D)$. First note that since a^D is group invertible with group inverse $aa^D a$, they are both invertible in the semisimple algebra $aa^D A aa^D$ (which has aa^D as identity). Hence

$$\mathrm{rank}(a) = \mathrm{rank}(a^D) = \mathrm{rank}(aa^D a) = \mathrm{rank}(aa^D).$$

Choose $x \in E(aa^D a)$ and $y \in E((1 - aa^D)a)$ such that

$$\sigma(aa^D ax) \cap \sigma((1 - aa^D)ay) = \{0\}.$$

Since a and a^D commute we have

$$
\begin{aligned}
\operatorname{rank}(a) &\geq \operatorname{rank}(aa^D axaa^D + (1 - aa^D)ay(1 - aa^D)) \\
&\geq \#\sigma'(aa^D axaa^D + (1 - aa^D)ay(1 - aa^D)) \\
&= \#[\sigma'(aa^D axaa^D) \cup \sigma'((1 - aa^D)ay(1 - aa^D))] \\
&= \#\sigma'(aa^D axaa^D) + \#\sigma'((1 - aa^D)ay(1 - aa^D)) \\
&= \operatorname{rank}(aa^D a) + \operatorname{rank}((1 - aa^D)a) \\
&= \operatorname{rank}(a) + \operatorname{rank}((1 - aa^D)a).
\end{aligned}
$$

Thus, since the rank is nonnegative, we have $\operatorname{rank}((1 - aa^D)a) = 0$. It follows that $(1 - aa^D)a = 0$ and hence a is group invertible. ∎

THEOREM 2.4. *An element $a \in \operatorname{Soc}(A)$ belongs to $\mathcal{G}(A)$ if and only if* $\operatorname{rank}(a) = \operatorname{rank}(a^k)$ *for each $k \in \mathbb{N}$.*

Proof. $\Rightarrow$ If a is group invertible with group inverse b then, obviously, for each $k \in \mathbb{N}$, a^k is group invertible with group inverse b^k. But all these elements are invertible in the semisimple algebra $abAab$ and hence all of them have rank equal to $\operatorname{rank}(ab)$.

$\Leftarrow$ Suppose $\operatorname{rank}(a) = \operatorname{rank}(a^k)$ for each $k \in \mathbb{N}$ and suppose a has Drazin index $k_1 \in \mathbb{N}$. If b is the Drazin inverse of a then a^{k_1} is group invertible with group inverse b^{k_1}. But clearly, since b is group invertible, we have $\operatorname{rank}(b) = \operatorname{rank}(b^{k_1})$ by the first part of the proof. So, from $\operatorname{rank}(a^{k_1}) = \operatorname{rank}(b^{k_1})$, it follows that $\operatorname{rank}(a) = \operatorname{rank}(b)$. By Theorem 2.3, a is group invertible. ∎

LEMMA 2.5. *The Drazin index of $a \in \operatorname{Soc}(A)$ is the least nonnegative integer k such that a^k is group invertible.*

Proof. If the Drazin index of a is k then it is clear that a^k is group invertible. Now suppose a^j is also group invertible for some $j \in \mathbb{N}$ with $j < k$. Denote by b the Drazin inverse of a and by c the group inverse of a^j. By the core-nilpotent decomposition we have $a^j = (aba)^j + a^j(1 - ab)$ and also $a^j = a^j ca^j + 0$. It follows from the uniqueness of the decomposition that $a^j(1 - ab) = 0$. But this contradicts the definition of the Drazin index, that is, that k is the least nonnegative integer such that $a^k ba = a^k$. ∎

LEMMA 2.6. *For $a \in \operatorname{Soc}(A)$ the rank of a is additive on the rank structure decomposition of a.*

Proof. Let $a \in \operatorname{Soc}(A)$ with rank structure decomposition $a = \sum_{i=1}^{m} a_i$. Then clearly $\operatorname{rank}(a) = \operatorname{rank}(\sum_{i=1}^{m} \lambda_i a_i)$ for each choice $\lambda_1, \ldots, \lambda_m$ of non-zero scalars. Choose $x \in E(a) \cap E(a_1) \cap \cdots \cap E(a_m)$ such that $\operatorname{rank}(a) = \#\sigma'(ax) = \#\sigma'(\sum_{i=1}^{m} a_i x)$. Obviously we may choose $\lambda_1, \ldots, \lambda_m$ such that

$\sigma(\lambda_i a_i x) \cap \sigma(\lambda_j a_j x) = \{0\}$ for all $i \neq j$. It follows that

$$\text{rank}(a) \geq \#\sigma'\Big(\sum_{i=1}^{m} \lambda_i a_i x\Big) = \#\Big[\bigcup_{i=1}^{m} \sigma'(\lambda_i a_i x)\Big] = \sum_{i=1}^{m} \#\sigma'(\lambda_i a_i x)$$

$$= \sum_{i=1}^{m} \#\sigma'(a_i x) = \sum_{i=1}^{m} \text{rank}(a_i) \geq \text{rank}(a)$$

where the final inequality follows from the subadditivity of the rank ([3, Theorem 2.14]). Thus $\text{rank}(a) = \sum_{i=1}^{m} \text{rank}(a_i)$. ■

THEOREM 2.7. *An element $a \in \text{Soc}(A)$ has Drazin index k if and only if k is the least integer satisfying*

$$\text{rank}(a^k) = \text{rank}(a^{k+1}).$$

In particular, a is group invertible if and only if $\text{rank}(a) = \text{rank}(a^2)$.

Proof. $\Leftarrow$ Let $a \in \text{Soc}(A)$ and suppose k is the least nonnegative integer such that $\text{rank}(a^k) = \text{rank}(a^{k+1})$. If $a = \sum_{i=1}^{m} a_i$ is the rank structure decomposition of a then $a^k = \sum_{i=1}^{m} a_i^k$ and $a^{k+1} = \sum_{i=1}^{m} a_i^{k+1}$ are the rank structure decompositions of a^k and a^{k+1} respectively (see [4, Proposition 2.4]). By Lemma 2.6 it follows that

$$\text{rank}(a^k) = \sum_{i=1}^{m} \text{rank}(a_i^k) = \text{rank}(a^{k+1}) = \sum_{i=1}^{m} \text{rank}(a_i^{k+1}).$$

Now since $\text{rank}(a_i^k) \geq \text{rank}(a_i^{k+1})$ for each i we actually have $\text{rank}(a_i^k) = \text{rank}(a_i^{k+1})$ for each i. It follows from Theorem 1.1 that

$$a_i^k A a_i^k \simeq M_{r_i, n_i} \quad \text{and} \quad a_i^{k+1} A a_i^{k+1} \simeq M_{r_i, n_i'}$$

where $n_i' \leq n_i$. Note that r_i is the same for a_i^k and a_i^{k+1} since $\text{rank}(a_i^k) = \text{rank}(a_i^{k+1})$. Hence the dimension of $a_i^{k+1} A a_i^{k+1}$ does not diminish, that is,

$$\dim(a_i^{k+1} A a_i^{k+1}) = \dim(a_i^k A a_i^k).$$

Since $a_i^{k+1} A a_i^{k+1}$ is a vector subspace of $a_i^k A a_i^k$ we have $a_i^{k+1} A a_i^{k+1} = a_i^k A a_i^k$. Inductively it follows that $a_i^{k+r} A a_i^{k+r} = a_i^k A a_i^k$ for each $r \in \mathbb{N}$. Thus by Theorem 1.1, $\text{rank}(a_i^k) = \text{rank}(a_i^{k+r})$ for each $r \in \mathbb{N}$. This being true for all i, we have $\text{rank}(a^k) = \text{rank}(a^{k+r})$ for each $r \in \mathbb{N}$. In particular $\text{rank}(a^k) = \text{rank}((a^k)^n)$ for all $n \in \mathbb{N}$. So, using Theorem 2.4, we infer that a^k is group invertible. Also, if $j < k$ then, by assumption, we have

$$\text{rank}(a^j) > \text{rank}(a^{j+1}) \geq \cdots \geq \text{rank}((a^j)^2)$$

and hence a^j cannot be group invertible. So if k is the least integer such that $\text{rank}(a^k) = \text{rank}(a^{k+1})$ then k is the least integer such that a^k is group invertible. By Lemma 2.5, k is the Drazin index of a.

$\Rightarrow$ If a has Drazin index k then by Lemma 2.5, k is the least integer such that a^k is group invertible. Hence, as in the first part of the proof, we cannot have $\mathrm{rank}(a^k) > \mathrm{rank}(a^{k+1})$. Thus $\mathrm{rank}(a^k) = \mathrm{rank}(a^{k+1})$. ∎

COROLLARY 2.8. *If ab and ba belong to $\mathrm{Soc}(A)$ and $ab \in \mathcal{G}(A)$ then $ba \in \mathcal{G}(A)$ if and only if $\mathrm{rank}(ab) = \mathrm{rank}(ba)$.*

Proof. $\Leftarrow$ If ab is group invertible then by Theorem 2.4, $\mathrm{rank}((ab)^k) = \mathrm{rank}(ab)$ for each $k \in \mathbb{N}$. So if $\mathrm{rank}(ab) = \mathrm{rank}(ba)$ then, for $k > 1$,

$$\mathrm{rank}(ba) = \mathrm{rank}(ab) = \mathrm{rank}((ab)^k) = \mathrm{rank}(\underbrace{abab\dots ab}_{k\text{ times}})$$

$$\leq \mathrm{rank}(\underbrace{baba\dots ba}_{k-1\text{ times}}) = \mathrm{rank}((ba)^{k-1}) \leq \mathrm{rank}(ba).$$

Since $k > 1$ was arbitrary the result follows by Theorem 2.4.

$\Rightarrow$ If ab and ba are both group invertible then, by Theorem 2.7,

$$\mathrm{rank}(ab) = \mathrm{rank}(abab) \leq \mathrm{rank}(ba) = \mathrm{rank}(baba) \leq \mathrm{rank}(ab).$$

Hence $\mathrm{rank}(ab) = \mathrm{rank}(ba)$. ∎

In [5] it was shown that $1 - ab$ is Drazin invertible if and only if $1 - ba$ is Drazin invertible, with preservation of the index. This is not true for ab and ba: Consider a Banach algebra A with $a, b \in A$ satisfying $ab = 1$ but $ba \neq 1$. Then ab is Drazin invertible with Drazin index 0 but, since ba is not invertible, ba cannot have Drazin index equal to 0. However, using the fact that $ab = 1$, it is easy to see that ba is an idempotent and thus group invertible. So ba is also Drazin invertible but with Drazin index 1. Generally speaking this is the worst that could happen: it is not hard to prove, using the definition of the Drazin inverse, that if ab is Drazin invertible then ba is always Drazin invertible, with the respective indices differing by at most one.

Much of the existing theory of Drazin invertibility for matrix algebras (see [6, Chapter 7]) is derived through the use of matrix block forms. Now many more of these results may be generalized to the Banach algebra case (not only to socle elements) using the functional calculus representation (1.6). We only give one more example (thereby generalizing [6, Corollary 7.6.2(ii)]), suggesting how one may proceed.

THEOREM 2.9. *Let $a \in A$ be Drazin invertible. Then a has Drazin index k if and only if k is the least integer such that*

$$\lim_{\lambda \to 0}(a - \lambda)^{-1}a^k$$

exists.

Proof. $\Rightarrow$ Suppose a is Drazin invertible with Drazin index k. Let U_0 and U_1 be separated open sets in $\mathbb{C}$ such that $0 \in U_0$ and $\sigma'(a) \subseteq U_1$. Let (λ_n)

be a sequence in $\mathbb{C}$ such that $(\lambda_n) \subseteq U_0$ and $\lambda_n \to 0$. For each n let $V_n \subseteq U_0$ be an open ball with center 0 such that $\lambda_n \notin V_n$. Let Γ be a smooth contour in U_1 surrounding $\sigma'(a)$ and let Γ_0 be a small circle in U_0 surrounding 0. Finally, for each n, let Γ_n be a small circle in V_n surrounding 0. Now, for each n, $a - \lambda_n$ is invertible and

$$(a - \lambda_n)^{-1} = \frac{1}{2\pi i} \int_{\Gamma \cup \Gamma_n} (\alpha - \lambda_n)^{-1}(\alpha - a)^{-1} \, d\alpha.$$

Also, using the fact that $a^{k+1}a^D = a^k$, we find that

$$a^k = \frac{1}{2\pi i} \int_{\Gamma \cup \Gamma_0} f(\alpha)(\alpha - a)^{-1} \, d\alpha = \frac{1}{2\pi i} \int_{\Gamma \cup \Gamma_n} f_n(\alpha)(\alpha - a)^{-1} \, d\alpha$$

for each n, where f is defined by

$$f(\alpha) = \begin{cases} \alpha^k, & \alpha \in U_1, \\ 0, & \alpha \in U_0, \end{cases}$$

and f_n is the restriction of f to $V_n \cup U_1$. It follows by the holomorphic functional calculus that, for each n,

$$(a - \lambda_n)^{-1}a^k = \frac{1}{2\pi i} \int_{\Gamma \cup \Gamma_n} \frac{f_n(\alpha)}{\alpha - \lambda_n} (\alpha - a)^{-1} \, d\alpha = \frac{1}{2\pi i} \int_{\Gamma \cup \Gamma_0} g_n(\alpha)(\alpha - a)^{-1} \, d\alpha$$

where g_n is defined by

$$g_n(\alpha) = \begin{cases} \dfrac{\alpha^k}{\alpha - \lambda_n}, & \alpha \in U_1, \\ 0, & \alpha \in U_0. \end{cases}$$

If we define g by

$$g(\alpha) = \begin{cases} \alpha^{k-1}, & \alpha \in U_1, \\ 0, & \alpha \in U_0, \end{cases}$$

then (g_n) is a sequence of holomorphic functions on $U_0 \cup U_1$, converging uniformly to g on compact subsets of $U_0 \cup U_1$. We hence have

$$\lim_n (a - \lambda_n)^{-1}a^k = \lim_n \frac{1}{2\pi i} \int_{\Gamma \cup \Gamma_0} g_n(\alpha)(\alpha - a)^{-1} \, d\alpha$$

$$= \frac{1}{2\pi i} \int_{\Gamma \cup \Gamma_0} g(\alpha)(\alpha - a)^{-1} \, d\alpha = a^k a^D.$$

Thus, if the Drazin index of a is k, then k is an integer such that the limit $\lim_{\lambda \to 0}(a - \lambda)^{-1}a^k$ exists. Suppose now that $\lim_{\lambda \to 0}(a - \lambda)^{-1}a^l = b$ exists for some $l < k$. Then it follows easily that $a^l = ab = ba$ and also that

$$\lim_{\lambda \to 0} (a - \lambda)^{-1}a^l a^{k-l} = ba^{k-l} = a^k a^D.$$

Hence $a^l a^{k-l-1} = baa^{k-l-1} = ba^{k-l} = a^k a^D$, from which we obtain $a^{k-1} = a^k a^D$. But since $a^k a^D$ is group invertible we have a contradiction, because k is the least integer such that a^k is group invertible.

$\Leftarrow$ Suppose k is the least integer such that $\lim_{\lambda \to 0}(a - \lambda)^{-1} a^k$ exists, but that a has Drazin index $l < k$. By the first part of the proof it follows that $\lim_{\lambda \to 0}(a - \lambda)^{-1} a^l = a^l a^D$, which contradicts our assumption. Thus, under the above hypothesis, a must have Drazin index k. ∎

3. Rank and continuity. Continuity properties of the Drazin inverse were studied by several authors ([6], [8], [10], [11]), and in particular Koliha and Rakočević obtained an operator-theoretic generalization ([8, Theorem 5.1]) of a well known criterion ([6, Theorem 10.7.1]) for continuity of the Drazin inverse for matrices. Of course, their result implies Campbell and Meyer's theorem. We show here (Corollary 3.4) that with Aupetit and Mouton's spectral rank characterization, Campbell and Meyer's result can be extended to the socle of an arbitrary semisimple Banach algebra. We shall need the following lemma and then first prove the special case for group invertibility:

LEMMA 3.1. *Let A be a semisimple Banach algebra. If $a_n, a \in \mathrm{Soc}(A)$ are maximal rank with $a_n \to a$ and there exists $n_0 \in \mathbb{N}$ such that $\mathrm{rank}(a_n) = \mathrm{rank}(a)$ for $n \geq n_0$ then $a_n^D \to a^D$.*

Proof. For $n \geq n_0$ we have $\mathrm{rank}(a_n) = \#\sigma'(a_n) = \#\sigma'(a) = \mathrm{rank}(a)$. By continuity of the spectrum on $\mathrm{Soc}(A)$ we have $\inf_n d(0, \sigma'(a_n)) > 0$. It follows from [8, Theorem 2.4] that $a_n^D \to a^D$. ∎

THEOREM 3.2. *Let a_n and a be group invertible elements belonging to $\mathrm{Soc}(A)$ with $a_n \to a$. Then $a_n^D \to a^D$ if and only if there exists $n_0 \in \mathbb{N}$ such that $\mathrm{rank}(a_n) = \mathrm{rank}(a)$ for all $n \geq n_0$.*

Proof. $\Rightarrow$ If $a_n^D \to a^D$ then $a_n a_n^D \to aa^D$ and hence by [11, Lemma 12] for some $n_0 \in \mathbb{N}$ the elements $a_n a_n^D$ and aa^D are similar for $n \geq n_0$, that is, there are $x_n \in A^{-1}$ such that $x_n a_n a_n^D x_n^{-1} = aa^D$. By the rank definition $\mathrm{rank}(a_n a_n^D) = \mathrm{rank}(aa^D)$ for $n \geq n_0$. A simple argument, using $a_n = a_n a_n^D a_n$ and $a = aa^D a$, implies $\mathrm{rank}(a_n) = \mathrm{rank}(a)$ for $n \geq n_0$.

$\Leftarrow$ Set the idempotents $a_n a_n^D = q_n$ and $aa^D = q$ and consider the finite-dimensional, semisimple and unital Banach algebras $B_n = q_n A q_n$ and $B = qAq$. Using Baire's theorem we can find $x \in A$ such that $\mathrm{rank}(a_n) = \#\sigma'(a_n x) = \#\sigma'(ax) = \mathrm{rank}(a)$. Now, for $n \geq n_0$ we have

$$\mathrm{rank}(a_n) = \#\sigma'(a_n(q_n x q_n)) = \#\sigma'(a(qxq)) = \mathrm{rank}(a).$$

By the above equation and the fact that a_n and a are invertible in the algebras B_n and B respectively, the elements $q_n x q_n$ and qxq are also invertible in B_n and B (for all $n \geq n_0$). Moreover, the inverses of $q_n x q_n$ and qxq in

B_n and B are just the Drazin (group) inverses $(q_n x q_n)^D$ and $(qxq)^D$. We now show that $q_n x q_n \to qxq$ and that $(q_n x q_n)^D \to (qxq)^D$:

Since $a_n \to a$ we have $a_n x a_n = a_n(q_n x q_n)a_n \to a(qxq)a = axa$. Since $a_n(q_n x q_n)a_n$ and $a(qxq)a$ are maximal rank ($n \geq n_0$) it follows from Lemma 3.1 that $[a_n(q_n x q_n)a_n]^D \to [a(qxq)a]^D$. By the invertibility of a_n and $q_n x q_n$ in B_n and of a and qxq in B we can apply the inverse formula with D, that is, $a_n^D(q_n x q_n)^D a_n^D \to a^D(qxq)^D a^D$. Now, multiplying by a_n and a on both sides yields $a_n a_n^D(q_n x q_n)^D a_n^D a_n \to aa^D(qxq)^D a^D a$, which implies $(q_n x q_n)^D \to (qxq)^D$ because $a_n a_n^D$ and aa^D are the identities in B_n and B respectively. From Lemma 3.1 together with $[(q_n x q_n)^D]^D = q_n x q_n$ and $[(qxq)^D]^D = qxq$ it follows that $q_n x q_n \to qxq$. To complete the proof notice that $(q_n x q_n)(q_n x q_n)^D = a_n a_n^D$ and $qxq(qxq)^D = aa^D$, so that $a_n a_n^D \to aa^D$. The result now follows from [10, Theorem 4.1]. ∎

COROLLARY 3.3. *Let a_n, $a \in \mathrm{Soc}(A)$ be such that $a_n \to a$ and suppose the Drazin indices of the a_n's are bounded. Denote by c_{a_n} and c_a the core elements of a_n and a respectively. Then $a_n^D \to a^D$ if and only if there exists $n_0 \in \mathbb{N}$ such that $\mathrm{rank}(c_{a_n}) = \mathrm{rank}(c_a)$ for all $n \geq n_0$.*

Proof. $\Rightarrow$ Since $\mathrm{rank}(c_a = aa^D a) = \mathrm{rank}(aa^D)$ for socle elements, the result follows by consideration of the first part of the proof of Theorem 3.2.

$\Leftarrow$ Let k_n, k denote the Drazin indices of a_n, a. Since k_n, k are bounded there exists $m \in \mathbb{N}$ such that a_n^m, a^m are group invertible. So, as $\mathrm{rank}(c_{a_n}) = \mathrm{rank}(c_a)$ for all $n \geq n_0$, we have $\mathrm{rank}(a_n^m) = \mathrm{rank}(a^m)$ for $n \geq n_0$. From Theorem 3.2 we obtain $(a_n^m)^D \to (a^m)^D$ and since $a_n^m \to a^m$ it follows that $a_n a_n^D \to aa^D$. Thus [10, Theorem 4.1] implies $a_n^D \to a^D$. ∎

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