Factorization theorem for product Hardy spaces

by

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Abstract. We extend the well known factorization theorems on the unit disk to product Hardy spaces, which generalizes the previous results obtained by Coifman, Rochberg and Weiss. The basic tools are the boundedness of a certain bilinear form on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ and the characterization of $\mathrm{BMO}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ recently obtained by Ferguson, Lacey and Sadosky.

1. Introduction. The purpose of this note is to extend some well known factorization theorems to product Hardy spaces. We begin by recalling the known results. In the one-dimensional case, let F be holomorphic in $\{|z| < 1\}$ and satisfy

$$\sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})| \, d\theta \leq 1.$$

It is well known that $F = G_1 G_2$ with G_1 and G_2 holomorphic and

$$\|G_i\|^2_{H^2} = \sup_{0<r<1} \int_0^{2\pi} |G_i(re^{i\theta})|^2 \, d\theta \leq 1, \quad i = 1, 2.$$

Coifman, Rochberg and Weiss extended this theorem to Hardy spaces in several variables. To be precise, the Hardy space $H^1(\mathbb{R}^n)$ used in [3] is defined by

$$\|f\|_{H^1(\mathbb{R}^n)} = \|f\|_1 + \sum_{j=1}^n \|R_j(f)\|_1$$

where $R_j(f)$, $j = 1, \ldots, n$, are the Riesz transforms of f. In [3] it was proved that for every $f \in H^1(\mathbb{R}^n)$,

$$f = \sum_{i=1}^{\infty} \sum_{j=1}^{n} [g_i^j R_j(h_i^j) + h_i^j R_j(g_i^j)]$$

for some $\{g_i^j, h_i^j\} \subset L^2(\mathbb{R}^n)$, and moreover, there exist two constants c_1 and c_2 such that

$$c_1 \sum_{i=1}^{\infty} \sum_{j=1}^{n} \|g_i^j\|_2 \|h_i^j\|_2 \leq \|f\|_{H^1} \leq c_2 \sum_{i=1}^{\infty} \sum_{j=1}^{n} \|g_i^j\|_2 \|h_i^j\|_2.$$

The basic idea of the proof in [3] was the duality between H^1 and BMO and a new characterization of BMO in terms of the L^2 boundedness of the commutator of a singular integral operator and a multiplication operator. It is worth pointing out that $(\text{VMO})^* = H^1$ was used. See [3, (2.4), p. 614].

Uchiyama [7] extended the result of Coifman, Rochberg and Weiss to spaces of homogeneous type. The new idea used in [7] is the boundedness of a certain bilinear form and the atomic decomposition of H^p on spaces of homogeneous type.

More recently, Ferguson, Sadosky and Lacey (see [6] and [5]) gave a new characterization of BMO on the product space $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ in terms of the nested commutator determined by a function b and the one-dimensional Hilbert transforms H_j, $j = 1, 2$, in the jth variable. See [5] for the details.

In this note, we extend the Coifman, Rochberg and Weiss factorization theorem to $H^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$. Since $H^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ is not a simple generalization of the classical $H^1(\mathbb{R}^2)$, so far no characterization of the predual space of $H^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ is known, and the atomic decomposition of H^p used by Uchiyama does not work for product H^p spaces either. The basic idea in this note is to introduce a certain bilinear form on $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ in terms of the Hilbert transforms H_j, $j = 1, 2$, and to prove its boundedness on $H^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$. Finally, this result together with the characterization of $\text{BMO}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ obtained in [5] gives the factorization theorem for $H^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$.

We begin by recalling the definition of Hardy spaces in product domains (see [1], [2] and [4]). Let $\psi^{(i)} \in C_0^{\infty}(\mathbb{R})$, $i = 1, 2$, be supported in the unit interval, with $\int_{\mathbb{R}} \psi^{(i)}(x) \, dx = 1$. Set $\psi_{t_i}^{(i)}(x_i) = t_i^{-1} \psi^{(i)}(x_i/t_i)$, $i = 1, 2$, for $t > 0$. For $f \in L^1(\mathbb{R} \times \mathbb{R})$, let

$$f^*(x_1, x_2) = \sup_{t_1, t_2 > 0} \left| \iint f(y_1, y_2) \psi_{t_1}^{(1)}(x_1 - y_1) \psi_{t_2}^{(2)}(x_2 - y_2) \, dy_1 \, dy_2 \right|.$$

We say $f \in H^1(\mathbb{R} \times \mathbb{R})$ if $f^* \in L^1(\mathbb{R} \times \mathbb{R})$ and we define

$$\|f\|_{H^1(\mathbb{R} \times \mathbb{R})} = \|f^*\|_{L^1(\mathbb{R} \times \mathbb{R})}.$$

The main results of this note are the following theorems.

THEOREM 1. *If* $1 < q < \infty$, $g \in L^q(\mathbb{R} \times \mathbb{R})$ *and* $h \in L^{q'}(\mathbb{R} \times \mathbb{R})$, $1/q + 1/q' = 1$, *then*

$$\|D(g,h)\|_{H^1(\mathbb{R}\times\mathbb{R})} \le c\|g\|_{L^q(\mathbb{R}\times\mathbb{R})}\|h\|_{L^{q'}(\mathbb{R}\times\mathbb{R})},$$

where

$$D(g,h)(x_1, x_2)$$
$$= h(H_2(H_1(g))) + H_1(g)H_2(h) + H_2(g)H_1(h) + g(H_2(H_1(h))),$$

and c *is a positive constant depending only on* q.

THEOREM 2. *If* $f \in H^1(\mathbb{R} \times \mathbb{R})$, *then there exist some* $\{g_i\} \subset L^2(\mathbb{R} \times \mathbb{R})$ *and* $\{h_i\} \subset L^2(\mathbb{R} \times \mathbb{R})$ *and two positive constants* c_1 *and* c_2 *such that*

$$f = \sum_{i=1}^{\infty} D(g_i, h_i)$$

and

$$c_1 \sum_{i=1}^{\infty} \|g_i\|_2 \|h_i\|_2 \le \|f\|_{H^1} \le c_2 \sum_{i=1}^{\infty} \|g_i\|_2 \|h_i\|_2.$$

2. Proofs. To prove the main results of this note, we need other definitions and notation. For $f \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R})$, the *strong maximal function* is defined by

$$\mathcal{M}f(x_1, x_2) = \sup_{(x_1,x_2)\in R} \frac{1}{|R|} \iint\limits_R |f(y_1, y_2)| \, dy_1 \, dy_2,$$

where the supremum is taken over all rectangles R which contain (x_1, x_2). For $0 < r < \infty$, $0 < \varepsilon < 1$, we define

$$M_r^{(1)}f(x_1, x_2) = \sup_{t_1>0} \left(\frac{1}{t_1} \int\limits_{|y_1-x_1|<t_1} |f(y_1, x_2)|^r \, dy_1 \right)^{1/r},$$

$$I_\varepsilon^{(1)}f(x_1, x_2) = \text{p.v.} \int \frac{f(y_1, x_2)}{|x_1 - y_1|^{1-\varepsilon}} \, dy_1.$$

$M_r^{(2)}$ and $I_\varepsilon^{(2)}$ are defined in a similar way. When $r = 1$, we write $M^{(j)}$ for $M_1^{(j)}$, $j = 1, 2$. Finally, we define the maximal operators of the Hilbert transforms H_1 and H_2 in the first and second variable respectively by

$$H_1^* f(x_1, x_2) = \sup_{t_1>0} \left| \int\limits_{|y_1-x_1|>t_1} \frac{f(y_1, x_2)}{x_1 - y_1} \, dy_1 \right|,$$

$$H_2^* f(x_1, x_2) = \sup_{t_2>0} \left| \int\limits_{|y_2-x_2|>t_2} \frac{f(x_1, y_2)}{x_2 - y_2} \, dy_2 \right|.$$

Proof of Theorem 1. Fix $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ and let $t_1, t_2 > 0$. Then

$$(2.1) \quad \iint D(g,h)(y_1,y_2)\psi_{t_1}^{(1)}(x_1 - y_1)\psi_{t_2}^{(2)}(x_2 - y_2)\,dy_1\,dy_2$$

$$= \iint \frac{g(z_1,z_2)}{(y_1 - z_1)(y_2 - z_2)}\,[\psi_{t_1}^{(1)}(x_1 - z_1) - \psi_{t_1}^{(1)}(x_1 - y_1)]$$

$$\times [\psi_{t_2}^{(2)}(x_2 - z_2) - \psi_{t_2}^{(2)}(x_2 - y_2)]\,dz_1\,dz_2\,h(y_1,y_2)\,dy_1\,dy_2$$

$$= \iint \eta(y_1,y_2,z_1,z_2)\,dz_1\,dz_2\,h(y_1,y_2)\,dy_1\,dy_2,$$

where

$$\eta(y_1,y_2,z_1,z_2) = \frac{g(z_1,z_2)}{(y_1 - z_1)(y_2 - z_2)}[\psi_{t_1}^{(1)}(x_1 - z_1) - \psi_{t_1}^{(1)}(x_1 - y_1)]$$

$$\times [\psi_{t_2}^{(2)}(x_2 - z_2) - \psi_{t_2}^{(2)}(x_2 - y_2)].$$

CASE I: $|x_1 - y_1| > 4t_1$, $|x_2 - y_2| > 4t_2$. Define

$$\Omega_1 = \{(y_1, y_2) \in \mathbb{R} \times \mathbb{R} : |x_1 - y_1| > 4t_1,\ |x_2 - y_2| > 4t_2\}.$$

Then $\psi_{t_1}^{(1)}(x_1 - y_1) = \psi_{t_2}^{(2)}(x_2 - y_2) = 0$ when $(y_1, y_2) \in \Omega_1$, and

$$(2.2) \quad \iint \eta(y_1,y_2,z_1,z_2)\,dz_1\,dz_2$$

$$= \iint \left[\frac{1}{y_1 - z_1} - \frac{1}{y_1 - x_1}\right]\left[\frac{1}{y_2 - z_2} - \frac{1}{y_2 - x_2}\right]\psi_{t_1}^{(1)}(x_1 - z_1)\psi_{t_2}^{(2)}(x_2 - z_2)$$

$$\times g(z_1,z_2)\,dz_1\,dz_2$$

$$+ \frac{1}{y_1 - x_1}\iint \left[\frac{1}{y_2 - z_2} - \frac{1}{y_2 - x_2}\right]\psi_{t_1}^{(1)}(x_1 - z_1)\psi_{t_2}^{(2)}(x_2 - z_2)g(z_1,z_2)\,dz_1\,dz_2$$

$$+ \frac{1}{y_2 - x_2}\iint \left[\frac{1}{y_1 - z_1} - \frac{1}{y_1 - x_1}\right]\psi_{t_1}^{(1)}(x_1 - z_1)\psi_{t_2}^{(2)}(x_2 - z_2)g(z_1,z_2)\,dz_1\,dz_2$$

$$+ \frac{1}{(y_1 - x_1)(y_2 - x_2)}\iint \psi_{t_1}^{(1)}(x_1 - z_1)\psi_{t_2}^{(2)}(x_2 - z_2)g(z_1,z_2)\,dz_1\,dz_2$$

$$= \eta_1(y_1,y_2) + \eta_2(y_1,y_2) + \eta_3(y_1,y_2) + \eta_4(y_1,y_2).$$

If $\psi_{t_1}^{(1)}(x_1 - z_1) \neq 0$, then $|x_1 - z_1| < |x_1 - y_1|/2$. Similarly, if $\psi_{t_2}^{(2)}(x_2 - z_2) \neq 0$, then $|x_2 - z_2| < |x_2 - y_2|/2$. Hence

$$|\eta_1(y_1,y_2)| \le c\,\frac{t_1}{(t_1 + |y_1 - x_1|)^2}\frac{t_2}{(t_2 + |y_2 - x_2|)^2}\,\mathcal{M}g(x_1,x_2).$$

So,

$$(2.3) \quad \left|\iint_{\Omega_1} \eta_1(y_1,y_2)h(y_1,y_2)\,dy_1\,dy_2\right| \le c\mathcal{M}g(x_1,x_2)\mathcal{M}h(x_1,x_2).$$

For the other three terms $\eta_2(y_1, y_2), \eta_3(y_1, y_2)$ and $\eta_4(y_1, y_2)$, by integrating against $h(y_1, y_2)$ with respect to y_1 and y_2 directly, we obtain

(2.4) $\left| \iint\limits_{\Omega_1} \eta_2(y_1, y_2) h(y_1, y_2) \, dy_1 \, dy_2 \right|$

$$\leq c \mathcal{M} g(x_1, x_2) \int |H_1^* h(x_1, y_2)| \frac{t_2}{(t_2 + |y_2 - x_2|)^2} \, dy_2$$

$$\leq c \mathcal{M} g(x_1, x_2) M^{(2)}(H_1^* h)(x_1, x_2),$$

(2.5) $\left| \iint\limits_{\Omega_1} \eta_3(y_1, y_2) h(y_1, y_2) \, dy_1 \, dy_2 \right|$

$$\leq c \mathcal{M} g(x_1, x_2) \int |H_2^* h(y_1, x_2)| \frac{t_1}{(t_1 + |y_1 - x_1|)^2} \, dy_1$$

$$\leq c \mathcal{M} g(x_1, x_2) M^{(1)}(H_2^* h)(x_1, x_2),$$

and

(2.6) $\left| \iint\limits_{\Omega_1} \eta_4(y_1, y_2) h(y_1, y_2) \, dy_1 \, dy_2 \right| \leq c \mathcal{M} g(x_1, x_2) H_1^*(H_2^* h)(x_1, x_2).$

CASE II: $|x_1 - y_1| \leq 4t_1$, $|x_2 - y_2| \leq 4t_2$. Define

$$\Omega_2 = \{(y_1, y_2) \in \mathbb{R} \times \mathbb{R} : |x_1 - y_1| \leq 4t_1, \ |x_2 - y_2| \leq 4t_2\}$$

and let $\phi \in C^\infty(0, \infty)$ be a fixed nonnegative function such that $\phi(t) = 0$ on $(0, 1/2)$, $\phi(t) = 1$ on $(1, \infty)$ and $|d\phi/dt| < 3$. Set $\phi' = 1 - \phi$. Then

(2.7) $\iint \eta(y_1, y_2, z_1, z_2) \, dz_1 \, dz_2$

$$= \iint \frac{g(z_1, z_2)}{(y_1 - z_1)(y_2 - z_2)} [\psi_{t_1}^{(1)}(x_1 - z_1) - \psi_{t_1}^{(1)}(x_1 - y_1)]$$

$$\times [\psi_{t_2}^{(2)}(x_2 - z_2) - \psi_{t_2}^{(2)}(x_2 - y_2)][\phi(|x_1 - z_1|/16t_1) + \phi'(|x_1 - z_1|/16t_1)]$$

$$\times [\phi(|x_2 - z_2|/16t_2) + \phi'(|x_2 - z_2|/16t_2)] \, dz_1 \, dz_2$$

$$= \psi_{t_1}^{(1)}(x_1 - y_1) \psi_{t_2}^{(2)}(x_2 - y_2) \iint \left[\frac{1}{y_1 - z_1} - \frac{1}{x_1 - z_1} \right] \left[\frac{1}{y_2 - z_2} - \frac{1}{x_2 - z_2} \right]$$

$$\times \phi(|x_1 - z_1|/16t_1) \phi(|x_2 - z_2|/16t_2) g(z_1, z_2) \, dz_1 \, dz_2$$

$$+ \psi_{t_1}^{(1)}(x_1 - y_1) \psi_{t_2}^{(2)}(x_2 - y_2) \iint \frac{1}{x_1 - z_1} \left[\frac{1}{y_2 - z_2} - \frac{1}{x_2 - z_2} \right]$$

$$\times \phi(|x_1 - z_1|/16t_1) \phi(|x_2 - z_2|/16t_2) g(z_1, z_2) \, dz_1 \, dz_2$$

$$+ \psi_{t_1}^{(1)}(x_1 - y_1) \psi_{t_2}^{(2)}(x_2 - y_2) \iint \frac{1}{x_2 - z_2} \left[\frac{1}{y_1 - z_1} - \frac{1}{x_1 - z_1} \right]$$

$$\times \phi(|x_1 - z_1|/16t_1) \phi(|x_2 - z_2|/16t_2) g(z_1, z_2) \, dz_1 \, dz_2$$

$$+ \psi_{t_1}^{(1)}(x_1 - y_1) \psi_{t_2}^{(2)}(x_2 - y_2) \iint \frac{1}{(x_1 - z_1)(x_2 - z_2)}$$

$$\times \phi(|x_1 - z_1|/16t_1)\phi(|x_2 - z_2|/16t_2)g(z_1, z_2)\,dz_1\,dz_2$$

$$+\psi_{t_1}^{(1)}(x_1 - y_1)\iint \left[\frac{1}{y_1 - z_1} - \frac{1}{x_1 - z_1}\right]\frac{1}{y_2 - z_2}[\psi_{t_2}^{(2)}(x_2 - y_2) - \psi_{t_2}^{(2)}(x_2 - z_2)]$$

$$\times \phi(|x_1 - z_1|/16t_1)\phi'(|x_2 - z_2|/16t_2)g(z_1, z_2)\,dz_1\,dz_2$$

$$+\psi_{t_1}^{(1)}(x_1 - y_1)\iint \frac{1}{(x_1 - z_1)(y_2 - z_2)}[\psi_{t_2}^{(2)}(x_2 - y_2) - \psi_{t_2}^{(2)}(x_2 - z_2)]$$

$$\times \phi(|x_1 - z_1|/16t_1)\phi'(|x_2 - z_2|/16t_2)g(z_1, z_2)\,dz_1\,dz_2$$

$$+\psi_{t_2}^{(2)}(x_2 - y_2)\iint \left[\frac{1}{y_2 - z_2} - \frac{1}{x_2 - z_2}\right]\frac{1}{y_1 - z_1}[\psi_{t_1}^{(1)}(x_1 - y_1) - \psi_{t_1}^{(1)}(x_1 - z_1)]$$

$$\times \phi'(|x_1 - z_1|/16t_1)\phi(|x_2 - z_2|/16t_2)g(z_1, z_2)\,dz_1\,dz_2$$

$$+\psi_{t_2}^{(2)}(x_2 - y_2)\iint \frac{1}{(y_1 - z_1)(x_2 - z_2)}[\psi_{t_1}^{(1)}(x_1 - y_1) - \psi_{t_1}^{(1)}(x_1 - z_1)]$$

$$\times \phi'(|x_1 - z_1|/16t_1)\phi(|x_2 - z_2|/16t_2)g(z_1, z_2)\,dz_1\,dz_2$$

$$+\iint \frac{1}{(y_1 - z_1)(y_2 - z_2)}[\psi_{t_1}^{(1)}(x_1 - z_1) - \psi_{t_1}^{(1)}(x_1 - y_1)]g(z_1, z_2)$$

$$\times [\psi_{t_2}^{(2)}(x_2 - z_2) - \psi_{t_2}^{(2)}(x_2 - y_2)]\phi'(|x_1 - z_1|/16t_1)\phi'(|x_2 - z_2|/16t_2)]\,dz_1\,dz_2$$

$$= \sum_{j=5}^{13} \eta_j(y_1, y_2).$$

If $\phi(|x_1 - z_1|/16t_1) \neq 0$, then $|x_1 - z_1| \geq 8t_1$, hence $|x_1 - y_1| < |x_1 - z_1|/2$. Similarly, if $\phi(|x_2 - z_2|/16t_2) \neq 0$, then $|x_2 - z_2| \geq 8t_2$ and $|x_2 - y_2| < 2|x_2 - z_2|/2$. Hence,

$$|\eta_5(y_1, y_2)| \leq c|\psi_{t_1}^{(1)}(x_1 - y_1)\psi_{t_2}^{(2)}(x_2 - y_2)|\iint \frac{|y_1 - x_1|}{|x_1 - z_1|^2}\frac{|y_2 - x_2|}{|x_2 - z_2|^2}$$

$$\times \phi(|x_1 - z_1|/16t_1)\phi(|x_2 - z_2|/16t_2)|g(z_1, z_2)|\,dz_1\,dz_2$$

$$\leq c|\psi_{t_1}^{(1)}(x_1 - y_1)\psi_{t_2}^{(2)}(x_2 - y_2)|\mathcal{M}g(x_1, x_2).$$

To estimate the other terms, define a variant of the Hilbert transform in the first variable,

$$\widetilde{H}_1 f(x_1, x_2) = \lim_{t_1 \to 0}\int \frac{f(z_1, x_2)}{x_1 - z_1}\phi(|x_1 - z_1|/16t_1)\,dz_1,$$

and its maximal operator

$$\widetilde{H}_1^* f(x_1, x_2) = \sup_{t_1 > 0}\left|\int \frac{f(z_1, x_2)}{x_1 - z_1}\phi(|x_1 - z_1|/16t_1)\,dz_1\right|.$$

Similarly, we define a variant of the Hilbert transform in the second variable,

$$\widetilde{H}_2 f(x_1, x_2) = \lim_{t_2 \to 0}\int \frac{f(x_1, z_2)}{x_2 - z_2}\phi(|x_2 - z_2|/16t_2)\,dz_2,$$

and its maximal operator

$$\widetilde{H}_2^* f(x_1, x_2) = \sup_{t_2 > 0} \left| \int \frac{f(x_1, z_2)}{x_2 - z_2} \phi(|x_2 - z_2|/16t_2) \, dz_2 \right|.$$

By standard arguments, it is easy to prove that $\widetilde{H}_j$ and $\widetilde{H}_j^*$ are $L^p \to L^p$ bounded for all $1 < p < \infty$, $j = 1, 2$.

By means of $\widetilde{H}_1$ and $\widetilde{H}_1^*$, we obtain an estimate for $\eta_6(y_1, y_2)$:

$$|\eta_6(y_1, y_2)| \leq c |\psi_{t_1}^{(1)}(x_1 - y_1) \psi_{t_2}^{(2)}(x_2 - y_2)| M^{(2)}(\widetilde{H}_1^* g)(x_1, x_2).$$

The estimates for $\eta_7(y_1, y_2)$ and $\eta_8(y_1, y_2)$ are similar:

$$|\eta_7(y_1, y_2)| \leq c |\psi_{t_1}^{(1)}(x_1 - y_1) \psi_{t_2}^{(2)}(x_2 - y_2)| M^{(1)}(\widetilde{H}_2^* g)(x_1, x_2),$$

$$|\eta_8(y_1, y_2)| \leq c |\psi_{t_1}^{(1)}(x_1 - y_1) \psi_{t_2}^{(2)}(x_2 - y_2)| \widetilde{H}_1^*(\widetilde{H}_2^* g)(x_1, x_2).$$

To estimate the remaining terms, we choose $\varepsilon \in (0, 1)$ such that

$$|\psi_{t_1}^{(1)}(x_1 - z_1) - \psi_{t_1}^{(1)}(x_1 - y_1)| \leq c |y_1 - z_1|^\varepsilon / t_1^{1+\varepsilon},$$

$$|\psi_{t_2}^{(2)}(x_2 - z_2) - \psi_{t_2}^{(2)}(x_2 - y_2)| \leq c |y_2 - z_2|^\varepsilon / t_2^{1+\varepsilon}.$$

Then

$$|\eta_9(y_1, y_2)| \leq \frac{c}{t_2^{1+\varepsilon}} |\psi_{t_1}^{(1)}(x_1 - y_1)| I_\varepsilon^{(2)}(M^{(1)}(g)\phi'(|x_2 - \cdot|/16t_2))(x_1, y_2),$$

$$|\eta_{10}(y_1, y_2)| \leq \frac{c}{t_2^{1+\varepsilon}} |\psi_{t_1}^{(1)}(x_1 - y_1)| I_\varepsilon^{(2)}(\widetilde{H}_1^*(g)\phi'(|x_2 - \cdot|/16t_2))(x_1, y_2).$$

The estimates for the last three terms are similar:

$$|\eta_{11}(y_1, y_2)| \leq \frac{c}{t_1^{1+\varepsilon}} |\psi_{t_2}^{(2)}(x_2 - y_2)| I_\varepsilon^{(1)}(M^{(2)}(g)\phi'(|x_1 - \cdot|/16t_1))(y_1, x_2),$$

$$|\eta_{12}(y_1, y_2)| \leq \frac{c}{t_1^{1+\varepsilon}} |\psi_{t_2}^{(2)}(x_2 - y_2)| I_\varepsilon^{(1)}(\widetilde{H}_2^*(g)\phi'(|x_1 - \cdot|/16t_1))(y_1, x_2),$$

$$|\eta_{13}(y_1, y_2)| \leq \frac{c}{t_1^{1+\varepsilon} t_2^{1+\varepsilon}} I_\varepsilon^{(1)}(I_\varepsilon^{(2)}(|g|\phi'(|x_2 - \cdot|/16t_2))\phi'(|x_1 - \cdot|/16t_1))(y_1, y_2).$$

Integrating $\eta_j(y_1, y_2)$ against $h(y_1, y_2)$ over Ω_2 with respect to y_1 and y_2 together with the above estimates for $\eta_j(y_1, y_2)$, $j = 5, 6, \ldots, 13$, we obtain estimates for the first four terms:

(2.8) $\quad \left| \int_* \eta_5(y_1, y_2) h(y_1, y_2) \, dy_1 \, dy_2 \right| \leq c \mathcal{M}g(x_1, x_2) \mathcal{M}h(x_1, x_2),$

(2.9) $\quad \left| \int_* \eta_6(y_1, y_2) h(y_1, y_2) \, dy_1 \, dy_2 \right| \leq c M^{(2)}(\widetilde{H}_1^* g)(x_1, x_2) \mathcal{M}h(x_1, x_2),$

(2.10) $\quad \left| \int_* \eta_7(y_1, y_2) h(y_1, y_2) \, dy_1 \, dy_2 \right| \leq c M^{(1)}(\widetilde{H}_2^* g)(x_1, x_2) \mathcal{M}h(x_1, x_2),$

(2.11) $\quad \left| \int_* \eta_8(y_1, y_2) h(y_1, y_2) \, dy_1 \, dy_2 \right| \leq c \widetilde{H}_1^*(\widetilde{H}_2^* g)(x_1, x_2) \mathcal{M}h(x_1, x_2),$

where $\int_*$ denotes $\iint_{\Omega_2}$. To estimate the remaining terms, we fix r and s such that $1/q' < 1/r < 1/q' + \varepsilon$ and $1/r' = 1/s - \varepsilon$, where $1/r + 1/r' = 1$. Then by the Hölder inequality and the Hardy–Littlewood–Sobolev theorem,

$$(2.12) \quad \left| \iint_{\Omega_2} \eta_9(y_1, y_2) h(y_1, y_2) \, dy_1 \, dy_2 \right|$$

$$\leq \frac{c}{t_2^{1+\varepsilon}} \int_{|y_2 - x_2| \leq 4t_2} M^{(1)} h(x_1, y_2) I_\varepsilon^{(2)}(M^{(1)}(g) \phi'(|x_2 - \cdot|/16t_2))(x_1, y_2) \, dy_2$$

$$\leq \frac{c}{t_2^{1+\varepsilon}} \left(\int_{|y_2 - x_2| \leq 4t_2} |M^{(1)} h(x_1, y_2)|^r \, dy_2 \right)^{1/r}$$

$$\times \left(\int_{|y_2 - x_2| \leq 4t_2} |I_\varepsilon^{(2)}(M^{(1)}(g) \phi'(|x_2 - \cdot|/16t_2))(x_1, y_2)|^{r'} \, dy_2 \right)^{1/r'}$$

$$\leq \frac{c}{t_2^{1+\varepsilon}} \left(\int_{|y_2 - x_2| \leq 4t_2} |M^{(1)} h(x_1, y_2)|^r \, dy_2 \right)^{1/r}$$

$$\times \left(\int |M^{(1)}(g)(x_1, y_2) \phi'(|x_2 - y_2|/16t_2)|^s \, dy_2 \right)^{1/s}$$

$$\leq c M_r^{(2)}(M^{(1)} h)(x_1, x_2) M_s^{(2)}(M^{(1)} g)(x_1, x_2),$$

and

$$(2.13) \quad \left| \iint_{\Omega_2} \eta_{10}(y_1, y_2) h(y_1, y_2) \, dy_1 \, dy_2 \right|$$

$$\leq \frac{c}{t_2^{1+\varepsilon}} \int_{|y_2 - x_2| \leq 4t_2} M^{(1)} h(x_1, y_2) I_\varepsilon^{(2)}(\widetilde{H}_1^*(g) \phi'(|x_2 - \cdot|/16t_2))(x_1, y_2) \, dy_2$$

$$\leq \frac{c}{t_2^{1+\varepsilon}} \left(\int_{|y_2 - x_2| \leq 4t_2} |M^{(1)} h(x_1, y_2)|^r \, dy_2 \right)^{1/r}$$

$$\times \left(\int |\widetilde{H}_1^*(g)(x_1, y_2) \phi'(|x_2 - y_2|/16t_2)|^s \, dy_2 \right)^{1/s}$$

$$\leq c M_r^{(2)}(M^{(1)} h)(x_1, x_2) M_s^{(2)}(\widetilde{H}_1^* g)(x_1, x_2).$$

Similarly, we obtain estimates for the last three terms:

$$(2.14) \quad \left| \iint_{\Omega_2} \eta_{11}(y_1, y_2) h(y_1, y_2) \, dy_1 \, dy_2 \right|$$

$$\leq c M_r^{(1)}(M^{(2)} h)(x_1, x_2) M_s^{(1)}(M^{(2)} g)(x_1, x_2),$$

$$(2.15) \quad \left| \iint_{\Omega_2} \eta_{12}(y_1, y_2) h(y_1, y_2) \, dy_1 \, dy_2 \right|$$

$$\leq c M_r^{(1)}(M^{(2)} h)(x_1, x_2) M_s^{(1)}(\widetilde{H}_2^* g)(x_1, x_2),$$

and

$$(2.16) \quad \left| \iint_{\Omega_2} \eta_{13}(y_1,y_2) h(y_1,y_2)\, dy_1\, dy_2 \right|$$

$$\leq \frac{c}{t_1^{1+\varepsilon} t_2^{1+\varepsilon}} \int_{|y_2-x_2|\leq 4t_2} \left(\int_{|y_1-x_1|\leq 4t_1} |h(y_1,y_2)|^r\, dy_1 \right)^{1/r} \times$$

$$\left(\int_{|y_1-x_1|\leq 4t_1} |I_\varepsilon^{(1)}(I_\varepsilon^{(2)}(|g|\phi'(|x_2-\cdot|/16t_2))\phi'(|x_1-\cdot|/16t_1))(y_1,y_2)|^{r'}\, dy_1 \right)^{1/r'} dy_2$$

$$\leq \frac{c}{t_1^{1+\varepsilon} t_2^{1+\varepsilon}} \int_{|y_2-x_2|\leq 4t_2} \left(\int_{|y_1-x_1|\leq 4t_1} |h(y_1,y_2)|^r\, dy_1 \right)^{1/r}$$

$$\times \left(\int_{|y_1-x_1|\leq 16t_1} |I_\varepsilon^{(2)}(|g|\phi'(|x_2-\cdot|/16t_2))(y_1,y_2)|^s\, dy_1 \right)^{1/s} dy_2$$

$$\leq \frac{c}{t_1^{1+\varepsilon} t_2^{1+\varepsilon}} \left(\iint_{\Omega_2} |h(y_1,y_2)|^r\, dy_1 \right)^{1/r}$$

$$\times \left(\int_{|y_2-x_2|\leq 4t_2} \left(\int_{|y_1-x_1|\leq 16t_1} |I_\varepsilon^{(2)}(|g|\phi'(|x_2-\cdot|/16t_2))(y_1,y_2)|^s\, dy_1 \right)^{r'/s} dy_2 \right)^{1/r'}$$

$$\leq \frac{c}{t_1^{1+\varepsilon} t_2^{1+\varepsilon}} \left(\iint_{\Omega_2} |h(y_1,y_2)|^r\, dy_1 \right)^{1/r}$$

$$\times \left(\int_{|y_1-x_1|\leq 16t_1} \left(\int_{|y_2-x_2|\leq 4t_2} |I_\varepsilon^{(2)}(|g|\phi'(|x_2-\cdot|/16t_2))(y_1,y_2)|^{r'} dy_2 \right)^{s/r'} dy_1 \right)^{1/s}$$

$$\leq \frac{c}{t_1^{1+\varepsilon} t_2^{1+\varepsilon}} \left(\iint_{\Omega_2} |h(y_1,y_2)|^r\, dy_1 \right)^{1/r}$$

$$\times \left(\int_{|y_1-x_1|\leq 16t_1} \int_{|y_2-x_2|\leq 16t_2} |g(y_1,y_2)|^s\, dy_2\, dy_1 \right)^{1/s}$$

$$\leq c M_r^{(1)}(M_r^{(2)}h)(x_1,x_2) M_s^{(1)}(M_s^{(2)}g)(x_1,x_2).$$

CASE III: $|x_1 - y_1| > 4t_1$, $|x_2 - y_2| \leq 4t_2$. Define

$$\Omega_3 = \{(y_1,y_2) \in \mathbb{R} \times \mathbb{R} : |x_1 - y_1| > 4t_1,\ |x_2 - y_2| \leq 4t_2\}.$$

If $(y_1,y_2) \in \Omega_3$, then $\psi_{t_1}^{(1)}(x_1 - y_1) = 0$ and hence

$$(2.17) \quad \iint \eta(y_1,y_2,z_1,z_2)\, dz_1\, dz_2$$

$$= \iint \frac{g(z_1,z_2)}{(y_1-z_1)(y_2-z_2)}\, \psi_{t_1}^{(1)}(x_1-z_1)[\psi_{t_2}^{(2)}(x_2-z_2) - \psi_{t_2}^{(2)}(x_2-y_2)] dz_1 dz_2$$

$$= \iint \frac{g(z_1,z_2)}{(y_1-z_1)(y_2-z_2)} \psi_{t_1}^{(1)}(x_1-z_1)[\psi_{t_2}^{(2)}(x_2-z_2)-\psi_{t_2}^{(2)}(x_2-y_2)]$$

$$\times [\phi(|x_2-z_2|/16t_2)+\phi'(|x_2-z_2|/16t_2)]dz_1\,dz_2$$

$$= [-\psi_{t_2}^{(2)}(x_2-y_2)]\iint \left[\frac{1}{y_1-z_1}-\frac{1}{y_1-x_1}\right]\left[\frac{1}{y_2-z_2}-\frac{1}{x_2-z_2}\right]$$

$$\times \psi_{t_1}^{(1)}(x_1-z_1)\phi(|x_2-z_2|/16t_2)g(z_1,z_2)\,dz_1\,dz_2$$

$$+ [-\psi_{t_2}^{(2)}(x_2-y_2)]\iint \frac{1}{y_1-x_1}\left[\frac{1}{y_2-z_2}-\frac{1}{x_2-z_2}\right]$$

$$\times \psi_{t_1}^{(1)}(x_1-z_1)\phi(|x_2-z_2|/16t_2)g(z_1,z_2)\,dz_1\,dz_2$$

$$+ [-\psi_{t_2}^{(2)}(x_2-y_2)]\iint \left[\frac{1}{y_1-z_1}-\frac{1}{y_1-x_1}\right]\frac{1}{x_2-z_2}$$

$$\times \psi_{t_1}^{(1)}(x_1-z_1)\phi(|x_2-z_2|/16t_2)g(z_1,z_2)\,dz_1\,dz_2$$

$$+ [-\psi_{t_2}^{(2)}(x_2-y_2)]\iint \frac{g(z_1,z_2)}{(y_1-x_1)(x_2-z_2)}\psi_{t_1}^{(1)}(x_1-z_1)\phi(|x_2-z_2|/16t_2)dz_1dz_2$$

$$+ \iint \left[\frac{1}{y_1-z_1}-\frac{1}{y_1-x_1}\right]\frac{1}{y_2-z_2}\psi_{t_1}^{(1)}(x_1-z_1)[\psi_{t_2}^{(2)}(x_2-z_2)-\psi_{t_2}^{(2)}(x_2-y_2)]$$

$$\times \phi'(|x_2-z_2|/16t_2)g(z_1,z_2)\,dz_1\,dz_2$$

$$+ \iint \frac{1}{(y_1-x_1)(y_2-z_2)}\psi_{t_1}^{(1)}(x_1-z_1)[\psi_{t_2}^{(2)}(x_2-z_2)-\psi_{t_2}^{(2)}(x_2-y_2)]$$

$$\times \phi'(|x_2-z_2|/16t_2)g(z_1,z_2)\,dz_1\,dz_2$$

$$= \sum_{j=14}^{19} \eta_j(y_1,y_2).$$

If $\psi_{t_1}^{(1)}(x_1-z_1) \neq 0$ then $|x_1-z_1| \leq t_1$ and $|x_1-y_1| > 4t_1$, hence $|x_1-z_1| < |x_1-y_1|/2$. Similarly, if $\phi(|x_2-z_2|/16t_2) \neq 0$, then $|x_2-y_2| \leq |x_2-z_2|/2$. Thus

$$|\eta_{14}(y_1,y_2)| \leq c|\psi_{t_2}^{(2)}(x_2-y_2)| \frac{t_1}{(t_1+|y_1-x_1|)^2}$$

$$\times \iint |\psi_{t_1}^{(1)}(x_1-z_1)| \frac{t_2}{(t_2+|x_2-z_2|)^2} |g(z_1,z_2)|\,dz_1\,dz_2$$

$$\leq c|\psi_{t_2}^{(2)}(x_2-y_2)| \frac{t_1}{(t_1+|y_1-x_1|)^2} \mathcal{M}g(x_1,x_2).$$

Integrating $\eta_{14}(y_1,y_2)$ against $h(y_1,y_2)$ over Ω_3 with respect to y_1 and y_2, we easily get

(2.18) $\left|\iint\limits_{\Omega_3} \eta_{14}(y_1, y_2) h(y_1, y_2)\, dy_1\, dy_2\right| \leq c\mathcal{M}g(x_1, x_2)\mathcal{M}h(x_1, x_2).$

To estimate $\eta_{15}(y_1, y_2)$, we integrate it against $h(y_1, y_2)$ over Ω_3 directly:

(2.19) $\left|\iint\limits_{\Omega_3} \eta_{15}(y_1, y_2) h(y_1, y_2)\, dy_1\, dy_2\right|$

$$\leq c\mathcal{M}g(x_1, x_2) \int\limits_{|y_2 - x_2| \leq 4t_2} H_1^* h(x_1, y_2)|\psi_{t_2}^{(2)}(x_2 - y_2)|\, dy_2$$

$$\leq c\mathcal{M}g(x_1, x_2) M^{(2)}(H_1^* h)(x_1, x_2).$$

Similarly, we obtain an estimate for $\eta_{16}(y_1, y_2)$:

$$|\eta_{16}(y_1, y_2)| \leq c|\psi_{t_2}^{(2)}(x_2 - y_2)| \frac{t_1}{(t_1 + |y_1 - x_1|)^2} \widetilde{H}_2^*(M^{(1)}g)(x_1, x_2),$$

which yields

(2.20) $\left|\iint\limits_{\Omega_3} \eta_{16}(y_1, y_2) h(y_1, y_2)\, dy_1\, dy_2\right| \leq c\widetilde{H}_2^*(M^{(1)}g)(x_1, x_2)\mathcal{M}h(x_1, x_2).$

To estimate $\eta_{17}(y_1, y_2)$, we integrate it against $h(y_1, y_2)$ over Ω_3 directly:

(2.21) $\left|\iint\limits_{\Omega_3} \eta_{17}(y_1, y_2) h(y_1, y_2)\, dy_1\, dy_2\right|$

$$\leq c\widetilde{H}_2^*(M^{(1)}g)(x_1, x_2) \int\limits_{|y_2 - x_2| \leq 4t_2} H_1^* h(x_1, y_2)|\psi_{t_2}^{(2)}(x_2 - y_2)|\, dy_2$$

$$\leq c\widetilde{H}_2^*(M^{(1)}g)(x_1, x_2) M^{(2)}(H_1^* h)(x_1, x_2).$$

To estimate $\eta_{18}(y_1, y_2)$ and $\eta_{19}(y_1, y_2)$, we use the Lipschitz continuity of $\psi_{t_1}^{(1)}$ and $\psi_{t_2}^{(2)}$:

$|\eta_{18}(y_1, y_2)|$

$$\leq c \frac{t_1}{(t_1 + |y_1 - x_1|)^2} \iint \frac{|\psi_{t_1}^{(1)}(x_1 - z_1)|}{t_2^{1+\varepsilon}} \frac{\phi'(|x_2 - z_2|/16t_2)}{|y_2 - z_2|^{1-\varepsilon}} |g(z_1, z_2)|\, dz_1\, dz_2$$

$$\leq c \frac{t_1}{(t_1 + |y_1 - x_1|)^2} \frac{1}{t_2^{1+\varepsilon}} I_\varepsilon^{(2)}(M^{(1)}(g)\phi'(|x_2 - \cdot|/16t_2))(x_1, y_2).$$

By the Hölder inequality and the Hardy–Littlewood–Sobolev theorem,

(2.22) $\left|\iint\limits_{\Omega_3} \eta_{18}(y_1, y_2) h(y_1, y_2)\, dy_1\, dy_2\right|$

$$\leq \frac{c}{t_2^{1+\varepsilon}} \int\limits_{|y_2 - x_2| \leq 4t_2} M^{(1)}h(x_1, y_2) I_\varepsilon^{(2)}(M^{(1)}(g)\phi'(|x_2 - \cdot|/16t_2))(x_1, y_2)\, dy_2$$

$$\leq \frac{c}{t_2^{1+\varepsilon}} \left(\int\limits_{|y_2-x_2|\leq 4t_2} |M^{(1)}h(x_1,y_2)|^r \, dy_2 \right)^{1/r}$$

$$\times \left(\int\limits_{|y_2-x_2|\leq 4t_2} |I_\varepsilon^{(2)}(M^{(1)}(g)\phi'(|x_2-\cdot|/16t_2))(x_1,y_2)|^{r'} \, dy_2 \right)^{1/r'}$$

$$\leq \frac{c}{t_2^{1+\varepsilon}} \left(\int\limits_{|y_2-x_2|\leq 4t_2} |M^{(1)}h(x_1,y_2)|^r \, dy_2 \right)^{1/r}$$

$$\times \left(\int |M^{(1)}(g)(x_1,y_2)\phi'(|x_2-y_2|/16t_2)|^s \, dy_2 \right)^{1/s}$$

$$\leq cM_r^{(2)}(M^{(1)}h)(x_1,x_2)M_s^{(2)}(M^{(1)}g)(x_1,x_2).$$

To estimate $\eta_{19}(y_1,y_2)$, using a method similar to that for $\eta_{18}(y_1,y_2)$, we integrate it against $h(y_1,y_2)$ over Ω_3 directly to obtain

$$(2.23) \quad \left| \iint\limits_{\Omega_3} \eta_{19}(y_1,y_2)h(y_1,y_2) \, dy_1 \, dy_2 \right|$$

$$\leq \frac{c}{t_2^{1+\varepsilon}} \int\limits_{|y_2-x_2|\leq 4t_2} H_1^*h(x_1,y_2) \left(\int \frac{\phi'(|x_2-z_2|/16t_2)}{|y_2-z_2|^{1-\varepsilon}} M^{(1)}g(x_1,z_2) \, dz_2 \right) dy_2$$

$$\leq \frac{c}{t_2^{1+\varepsilon}} \int\limits_{|y_2-x_2|\leq 4t_2} H_1^*h(x_1,y_2)I_\varepsilon^{(2)}(M^{(1)}(g)\phi'(|x_2-\cdot|/16t_2))(x_1,y_2) \, dy_2$$

$$\leq cM_r^{(2)}(H_1^*h)(x_1,x_2)M_s^{(2)}(M^{(1)}g)(x_1,x_2).$$

CASE IV: $|x_1-y_1| \leq 4t_1$, $|x_2-y_2| > 4t_2$. Define

$$\Omega_4 = \{(y_1,y_2) \in \mathbb{R} \times \mathbb{R} : |x_1-y_1| \leq 4t_1, \ |x_2-y_2| > 4t_2\}.$$

The estimates in this case are similar to those of Case III. We omit the details.

By (2.1), we have

$$(2.24) \quad \iint D(g,h)(y_1,y_2)\psi_{t_1}^{(1)}(x_1-y_1)\psi_{t_2}^{(2)}(x_2-y_2) \, dy_1 \, dy_2$$

$$= \iint \eta(y_1,y_2,z_1,z_2) \, dz_1 \, dz_2 \, h(y_1,y_2) \, dy_1 \, dy_2$$

$$= \iint\limits_{\Omega_1} \eta(y_1,y_2,z_1,z_2) \, dz_1 \, dz_2 \, h(y_1,y_2) \, dy_1 \, dy_2$$

$$+ \iint\limits_{\Omega_2} \eta(y_1,y_2,z_1,z_2) \, dz_1 \, dz_2 \, h(y_1,y_2) \, dy_1 \, dy_2$$

$$+ \iint\limits_{\Omega_3} \eta(y_1,y_2,z_1,z_2) \, dz_1 \, dz_2 \, h(y_1,y_2) \, dy_1 \, dy_2$$

$$+ \iint\limits_{\Omega_4} \eta(y_1, y_2, z_1, z_2)\, dz_1\, dz_2\, h(y_1, y_2)\, dy_1\, dy_2$$

$$= I(x_1, x_2) + II(x_1, x_2) + III(x_1, x_2) + IV(x_1, x_2).$$

By (2.2)–(2.6), we get

(2.25) $\quad |I(x_1, x_2)| \le c\mathcal{M}g(x_1, x_2)\{\mathcal{M}h(x_1, x_2) + M^{(2)}(H_1^*h)(x_1, x_2)$
$$+ M^{(1)}(H_2^*h)(x_1, x_2) + H_1^*(H_2^*h)(x_1, x_2)\}.$$

Corresponding to (2.7), $II(x_1, x_2)$ consists of nine terms. By the estimates (2.8)–(2.16), we obtain an estimate for $II(x_1, x_2)$:

(2.26) $\quad |II(x_1, x_2)| \le c\mathcal{M}h(x_1, x_2)\{\mathcal{M}g(x_1, x_2) + M^{(2)}(\widetilde{H}_1^*g)(x_1, x_2)$
$$+ M^{(1)}(\widetilde{H}_2^*g)(x_1, x_2) + \widetilde{H}_1^*(H_2^*g)(x_1, x_2)\}$$
$$+ cM_r^{(2)}(M^{(1)}h)(x_1, x_2)\{M_s^{(2)}(M^{(1)}g)(x_1, x_2) + M_s^{(2)}(\widetilde{H}_1^*g)(x_1, x_2)\}$$
$$+ cM_r^{(1)}(M^{(2)}h)(x_1, x_2)\{M_s^{(1)}(M^{(2)}g)(x_1, x_2) + M_s^{(1)}(\widetilde{H}_2^*g)(x_1, x_2)\}$$
$$+ cM_r^{(1)}(M_r^{(2)}h)(x_1, x_2)M_s^{(1)}(M_s^{(2)}g)(x_1, x_2).$$

Corresponding to (2.17), $III(x_1, x_2)$ has six terms. Collecting the estimates (2.18)–(2.23) for $III(x_1, x_2)$, we have

(2.27) $\quad |III(x_1, x_2)| \le c\mathcal{M}g(x_1, x_2)\{\mathcal{M}h(x_1, x_2) + M^{(2)}(H_1^*h)(x_1, x_2)\}$
$$+ c\widetilde{H}_2^*(M^{(1)}g)(x_1, x_2)\{\mathcal{M}h(x_1, x_2) + M^{(2)}(H_1^*h)(x_1, x_2)\}$$
$$+ cM_s^{(2)}(M^{(1)}g)(x_1, x_2)\{M_r^{(2)}(M^{(1)}h)(x_1, x_2) + M_r^{(2)}(H_1^*h)(x_1, x_2)\}.$$

The same argument gives estimates for $IV(x_1, x_2)$. Taking the supremum in (2.24) for $t_1, t_2 > 0$ together with (2.25)–(2.27), we obtain a maximal function estimate for the bilinear form:

$$[D(g, h)]^*(x_1, x_2) \le c\mathcal{M}g(x_1, x_2)\{\mathcal{M}h(x_1, x_2) + M^{(2)}(H_1^*h)(x_1, x_2)$$
$$+ M^{(1)}(H_2^*h)(x_1, x_2) + H_1^*(H_2^*h)(x_1, x_2)\}$$
$$+ c\mathcal{M}h(x_1, x_2)\{\mathcal{M}g(x_1, x_2) + M^{(2)}(\widetilde{H}_1^*g)(x_1, x_2)$$
$$+ M^{(1)}(\widetilde{H}_2^*g)(x_1, x_2) + \widetilde{H}_1^*(H_2^*g)(x_1, x_2)\}$$
$$+ cM_r^{(2)}(M^{(1)}h)(x_1, x_2)\{M_s^{(2)}(M^{(1)}g)(x_1, x_2) + M_s^{(2)}(\widetilde{H}_1^*g)(x_1, x_2)\}$$
$$+ cM_r^{(1)}(M^{(2)}h)(x_1, x_2)\{M_s^{(1)}(M^{(2)}g)(x_1, x_2) + M_s^{(1)}(\widetilde{H}_2^*g)(x_1, x_2)\}$$
$$+ cM_r^{(1)}(M_r^{(2)}h)(x_1, x_2)M_s^{(1)}(M_s^{(2)}g)(x_1, x_2)$$
$$+ c\widetilde{H}_2^*(M^{(1)}g)(x_1, x_2)\{\mathcal{M}h(x_1, x_2) + M^{(2)}(H_1^*h)(x_1, x_2)\}$$
$$+ cM_s^{(2)}(M^{(1)}g)(x_1, x_2)\{M_r^{(2)}(M^{(1)}h)(x_1, x_2) + M_r^{(2)}(H_1^*h)(x_1, x_2)\}$$
$$+ \text{ other terms corresponding to Case IV.}$$

We now show that all terms on the right hand side belong to $L^1(\mathbb{R} \times \mathbb{R})$. By the boundedness of the maximal functions and the maximal operators of the Hilbert transforms H_1 and H_2, we can easily see that the $L^1(\mathbb{R} \times \mathbb{R})$ norm of the first two terms is controlled by $c\|g\|_{L^q(\mathbb{R} \times \mathbb{R})}\|h\|_{L^{q'}(\mathbb{R} \times \mathbb{R})}$.

For the fifth term on the right hand side, we have $s < q$ and $r < q'$ by the choice of r and s. So by the Hölder inequality,

$$\iint M_r^{(1)}(M_r^{(2)}h)(x_1,x_2)M_s^{(1)}(M_s^{(2)}g)(x_1,x_2)\,dx_1\,dx_2$$

$$\leq \int\limits_{\mathbb{R}} \Big(\int\limits_{\mathbb{R}}|M_r^{(1)}(M_r^{(2)}h)(x_1,x_2)|^{q'}\,dx_1\Big)^{1/q'}\Big(\int\limits_{\mathbb{R}}|M_s^{(1)}(M_s^{(2)}g)(x_1,x_2)|^q\,dx_1\Big)^{1/q}\,dx_2$$

$$\leq c\int\limits_{\mathbb{R}} \Big(\int\limits_{\mathbb{R}}|M_r^{(2)}h(x_1,x_2)|^{q'}\,dx_1\Big)^{1/q'}\Big(\int\limits_{\mathbb{R}}|M_s^{(2)}g)(x_1,x_2)|^q\,dx_1\Big)^{1/q}\,dx_2$$

$$\leq c\Big(\iint|M_r^{(2)}h(x_1,x_2)|^{q'}\,dx_1\,dx_2\Big)^{1/q'}\Big(\iint|M_s^{(2)}g(x_1,x_2)|^q\,dx_1\,dx_2\Big)^{1/q}$$

$$\leq c\Big(\iint|h(x_1,x_2)|^{q'}\,dx_1\,dx_2\Big)^{1/q'}\Big(\iint|g(x_1,x_2)|^q\,dx_1\,dx_2\Big)^{1/q}$$

$$\leq c\|g\|_{L^q(\mathbb{R} \times \mathbb{R})}\|h\|_{L^{q'}(\mathbb{R} \times \mathbb{R})}.$$

The other terms can be dealt with similarly. Hence we finally obtain

$$\iint [D(g,h)]^*(x_1,x_2)\,dx_1\,dx_2 \leq c\|g\|_{L^q(\mathbb{R} \times \mathbb{R})}\|h\|_{L^{q'}(\mathbb{R} \times \mathbb{R})}.$$

This completes the proof of Theorem 1.

From Theorem 1, we can easily deduce Theorem 2 as follows.

Proof of Theorem 2. To obtain the factorization theorem, consider the Banach space

$$B = \Big\{f \in L^1(\mathbb{R} \times \mathbb{R}) : f = \sum_{i=1}^{\infty} D(g_i,h_i),\ g_i, h_i \in L^2(\mathbb{R} \times \mathbb{R}),\ i = 1,2,\dots\Big\}$$

and define

$$\|f\|_B = \inf \sum_{i=1}^{\infty}\|g_i\|_2\|h_i\|_2,$$

where the infimum is taken over all decompositions of f.

By Theorem 1,

$$\|f\|_{H^1} \leq \sum_{i=1}^{\infty}\|D(g_i,h_i)\|_{H^1} \leq c\sum_{i=1}^{\infty}\|g_i\|_2\|h_i\|_2.$$

So, B is a subspace of $H^1(\mathbb{R} \times \mathbb{R})$. On the other hand, if b is in BMO, then formally

$$\langle b, D(g,h)\rangle = \langle [[b,H_1],H_2](g),h\rangle.$$

By Theorem 1.2 in [5],

$$\|b\|_{\mathrm{BMO}} \cong \|[[b, H_1], H_2]\|_{L^2 \to L^2} \cong \sup \left\{ \left| \iint fb \right| : \|f\|_B = 1 \right\}.$$

This gives the equivalence of the H^1 norm and $\| \cdot \|_B$, showing that the two spaces are the same. The proof is complete.

References

[1]   S. A. Chang and R. Fefferman, *A continuous version of the duality of H^1 and* BMO *on the bi-disc*, Ann. of Math. 112 (1980), 179–201.

[2]   —, —, *Some recent developments in Fourier analysis and H^p theory on product domains*, Bull. Amer. Math. Soc. 12 (1985), 1–43.

[3]   R. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. 103 (1976), 611–635.

[4]   R. Fefferman, *Calderón–Zygmund theory for product domains: H^p spaces*, Proc. Natl. Acad. Sci. USA 83 (1986), 840–843.

[5]   S. H. Ferguson and M. T. Lacey, *A characterization of product* BMO *by commutators*, Acta Math. 189 (2002), 143–160.

[6]   S. H. Ferguson and C. Sadosky, *Characterizations of bounded mean oscillation on the polydisk in terms of Hankel operators and Carleson measures*, J. Anal. Math. 81 (2000), 239–267.

[7]   A. Uchiyama, *The factorization of H^p on the space of homogeneous type*, Pacific J. Math. 92 (1981), 453–468.

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