Compact operators on the weighted Bergman space $A^1(\psi)$

by

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Abstract. We show that a bounded linear operator $S$ on the weighted Bergman space $A^1(\psi)$ is compact and the predual space $A_0(\varphi)$ of $A^1(\psi)$ is invariant under $S^*$ if and only if $Sk_z \to 0$ as $z \to \partial D$, where $k_z$ is the normalized reproducing kernel of $A^1(\psi)$. As an application, we give conditions for an operator in the Toeplitz algebra to be compact.

1. Introduction. Let $\varphi$ be a positive continuous function on $[0,1)$. We say that $\varphi$ is normal if there exist $0 < a < b$ and $r_0 < 1$ such that

\begin{align*}
\frac{\varphi(r)}{1-r^2} \to 0 \quad \text{and} \quad \frac{\varphi(r)}{(1-r^2)^b} \to \infty \quad (r_0 \leq r \to 1^-).
\end{align*}

The functions $\{\varphi, \psi\}$ will be called a normal pair if $\varphi$ is normal and if, for some $b$ satisfying (1), there exists $\alpha > b - 1$ such that $\varphi(r)\psi(r) = (1-r^2)^\alpha$.

Let $dA$ denote the normalized Lebesgue measure on the open unit disk $D$ in the complex plane, and $dA_\psi$ the measure on $D$ defined by $dA_\psi(z) = \psi(|z|)dA(z)$. The condition $\alpha > b - 1$ ensures that the measure $dA_\psi$ on $D$ is finite.

Let $L^1(\psi)$ denote the Banach space of measurable functions $f$ with norm $\|f\|_\psi = \int_D |f|dA_\psi < \infty$. Let $A^1(\psi)$ denote the closed subspace of $L^1(\psi)$ consisting of all analytic functions, which will be called the weighted Bergman space. In the case that $\varphi(r) = (1-r)^c$ for some constant $c > 0$, and that $\alpha = c$, $A^1(\psi)$ is the Bergman space $L^1_a(D)$.

Let $L^\infty(\varphi)$ denote the Banach space consisting of all measurable functions such that $f(z)\varphi(|z|)$ is essentially bounded on $D$ with norm $\|f\|_\varphi = \text{ess sup}_{z \in D} |f(z)|\varphi(|z|)$. Let $A^\infty(\varphi)$ denote the closed subspace of $L^\infty(\varphi)$ consisting of all analytic functions, and

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\[ A_0(\varphi) := \{ f \in A_\infty(\varphi) : \lim_{r \to 1^-} \sup_{|z| = r} f(z)\varphi(|z|) = 0 \}, \]
a closed subspace of \( A_\infty(\varphi) \).

Throughout this paper we shall use the following pairing between \( A^1(\psi) \) and \( A_\infty(\varphi) \):
\[
\langle f, g \rangle = \int_D f(z)\overline{g(z)}(1 - |z|^2)^\alpha \, dA(z).
\]
For \( z, w \in D \), let
\[
K_z(w) = \frac{1 + \alpha}{(1 - \overline{z}w)^{2+\alpha}},
\]
and define an operator \( Q \) on \( L^1(\psi) \) by
\[
(Qf)(z) = \langle f, K_z \rangle = \int_D f(w)\overline{K_z(w)}(1 - |w|^2)^\alpha \, dA(w).
\]
Then \( Q \) is a bounded projection from \( L^1(\psi) \) onto \( A^1(\psi) \) and \( K_z(w) \) is the reproducing kernel of \( A^1(\psi) \), that is, \( f(z) = \langle f, K_z \rangle \) for any \( f \in A^1(\psi) \). Since \( K_z \) is in \( A^1(\psi) \) for any \( z \in D \), the reproducing property \( f(z) = \langle f, K_z \rangle \) also holds for \( f \in A_\infty(\varphi) \) (see [6] for the details). The function \( k_z(w) := K_z(w)/\|K_z\|_\psi \) will be called the normalized reproducing kernel of \( A^1(\psi) \).

In the Bergman space \( L^2_\alpha(D) \) setting, Axler and Zheng [1] proved that an operator \( S \) which is a finite sum of finite products of Toeplitz operators, is compact if and only if \( \|Sk_z\| \to 0 \) as \( |z| \to 1^- \). This result also holds for the spaces \( L^p_\alpha(D) \) \( (1 < p < \infty) \) (see [7]), \( A^2_\psi(\Omega) \) with \( \Omega \) a regular bounded symmetric domain in \( C^n \) (see [2]), and \( H^2(\Omega, dv) \) with \( \Omega \) a smoothly bounded multiply connected domain in the complex plane (see [5]). Recently Miao and Zheng [4] proved that for a bounded operator \( S \) on \( L^p_\alpha(D) \) \( (1 < p < \infty) \) such that both \( \sup_{z \in \partial D} \|Sz\|_m \) and \( \sup_{z \in \partial D} \|S^*_z1\|_m \) are finite, \( S \) is a compact operator on \( L^p_\alpha(D) \) if and only if the Berezin transform of \( S \) tends to 0 at \( \partial D \).

In this note, we will obtain a similar result for \( A^1(\psi) \). More precisely, we show that a bounded linear operator \( S \) on \( A^1(\psi) \) is compact and \( A_0(\varphi) \) is an invariant subspace of \( S^* \) if and only if \( \|Sk_z\|_\psi \to 0 \) as \( z \to \partial D \).

2. Preliminaries. The following result can be found in [6].

**Lemma 1.** \( A^1(\psi)^* = A_\infty(\varphi) \) and \( A_0(\varphi)^* = A^1(\psi) \).

We also need some results about the reproducing kernel of \( A^1(\psi) \).

**Lemma 2.** There exist constants \( c \) and \( C \) such that
\[
c/\varphi(|z|) \leq \|K_z\|_\psi \leq C/\varphi(|z|).
\]

**Proof.** The second inequality can be derived from Lemmas 5 and 8 in [6].
Now we prove the first inequality. If $|z| \leq r_0$, the first inequality obviously holds. If $|z| > r_0$, using (1) and $\varphi(r)\psi(r) = (1 - r^2)^{\alpha}$, we have

$$
\frac{(1 - |w|^2)^{\alpha-a}}{\psi(|w|)} \leq \frac{(1 - |z|^2)^{\alpha-a}}{\psi(|z|)} \quad \text{when } 1 > |w| > |z|,
$$

$$
\frac{(1 - |w|^2)^{\alpha-b}}{\psi(|w|)} \leq \frac{(1 - |z|^2)^{\alpha-b}}{\psi(|z|)} \quad \text{when } r_0 \leq |w| < |z|.
$$

Hence

$$
\|K_z\|_{\psi} = \int_D \frac{(1 + \alpha)\psi(|w|)}{|1 - \bar{z}w|^{2+\alpha}} dA(w)
$$

$$
\geq \frac{(1 + \alpha)\psi(|z|)}{(1 - |z|^2)^{\alpha-b}} \left( \int_{r_0 \leq |w| \leq |z|} \frac{(1 - |w|^2)^{\alpha-b}}{|1 - \bar{z}w|^{2+\alpha}} dA(w) + \int_{|z| < |w| < 1} \frac{(1 - |w|^2)^{\alpha-a}}{|1 - \bar{z}w|^{2+\alpha}(1 - |z|^2)^{b-a}} dA(w) \right).
$$

Note that

$$
\frac{1}{(1 - |z|^2)^{b-a}} = \frac{1}{((1 + |z|)(1 - |z|))^{b-a}} \geq \frac{1}{(2|1 - \bar{z}w|)^{b-a}},
$$

and similarly

$$
\frac{1}{(1 - |w|^2)^{b-a}} \geq \frac{1}{(2|1 - \bar{z}w|)^{b-a}}.
$$

So there exists a positive constant $c_1$ such that

$$
\|K_z\|_{\psi} \geq \frac{c_1\psi(|z|)}{(1 - |z|^2)^{\alpha-b}} \left( \int_{r_0 \leq |w|} \frac{(1 - |w|^2)^{\alpha-a}}{|1 - \bar{z}w|^{2+\alpha+b-a}} dA(w) \right)
$$

$$
= \frac{c_1\psi(|z|)}{(1 - |z|^2)^{\alpha-b}} \left( \int_D - \int_{|w| \leq r_0} \frac{(1 - |w|^2)^{\alpha-a}}{|1 - \bar{z}w|^{2+\alpha+b-a}} dA(w) \right)
$$

$$
=: \frac{c_1\psi(|z|)}{(1 - |z|^2)^{\alpha-b}} (I_1(z) - I_2(z)).
$$

Now by Lemma 4.2.2 of [8], $I_1(z) \sim (1 - |z|^2)^{-b}$ as $z \to \partial D$; and $I_2$ is bounded. Thus it is easy to see that there exists $c > 0$ so that

$$
\|K_z\|_{\psi} \geq c \frac{\psi(|z|)}{(1 - |z|^2)^{\alpha}} = c/\varphi(|z|).
$$

The proof is now complete. $\blacksquare$

**Lemma 3.** The normalized reproducing kernel $k_z$ converges weakly* to 0 in $A^1(\psi)$ as $z \to \partial D$. 

Proof. For \( g \in A_0(\varphi) \), by the reproducing property of \( K_z \), we have
\[
\langle k_z, g \rangle = \frac{\langle K_z, g \rangle}{\|K_z\|_\psi} = \frac{\overline{g(z)}}{\|K_z\|_\psi}.
\]
Now it follows from the definition of \( A_0(\varphi) \) and Lemma 2 that \( \langle k_z, g \rangle \to 0 \) as \( z \to \partial D \).

3. Compactness

**Theorem 1.** Suppose that \( S \) is a bounded linear operator on \( A^1(\psi) \). Then \( S \) is compact and \( A_0(\varphi) \) is an invariant subspace of \( S^* \) if and only if \( \|Sk_z\|_\psi \to 0 \) as \( z \to \partial D \).

**Proof.** Necessity. Suppose that \( S \) is a compact operator and \( A_0(\varphi) \) is an invariant subspace of \( S^* \). If \( \|Sk_z\|_\psi \not\to 0 \) as \( z \to \partial D \), then there exist a constant \( \delta > 0 \) and a sequence \( \{z_n\} \) in \( D \) such that
\[
z_n \to \partial D \quad \text{and} \quad \|Sk_{z_n}\|_\psi > \delta.
\]
Since \( \{k_{z_n}\} \) is a bounded sequence in \( A^1(\psi) \) and \( S \) is compact, there exists a subsequence of \( \{k_{z_n}\} \), also denoted by \( \{k_{z_n}\} \), such that \( \{Sk_{z_n}\} \) converges in \( A^1(\psi) \). By Lemma 3, \( z_n \to \partial D \) implies that \( k_{z_n} \xrightarrow{w^*} 0 \). Since \( A_0(\varphi) \) is an invariant subspace of \( S^* \), we have, for any \( g \in A_0(\varphi) \),
\[
\langle Sk_{z_n}, g \rangle = \langle k_{z_n}, S^*g \rangle \to 0.
\]
Thus \( Sk_{z_n} \xrightarrow{w^*} 0 \). Since \( \{Sk_{z_n}\} \) converges in \( A^1(\psi) \), it must converge to its weak*-limit, that is, 0. This contradicts (3).

Sufficiency. Suppose that \( \|Sk_z\|_\psi \to 0 \) as \( z \to \partial D \). By the reproducing property of \( K_z \), one can see that
\[
(S^*K_w)(z) = \langle S^*K_w, K_z \rangle = \overline{\langle SK_z, K_w \rangle} = \overline{(SK_z)(w)}.
\]
So for \( f \in A^1(\psi) \),
\[
(Sf)(w) = \langle Sf, K_w \rangle = \langle f, S^*K_w \rangle = \int_D f(z)(S^*K_w)(z)(1 - |z|^2)^\alpha dA(z)
\]
\[
= \int_D f(z)(SK_z)(w)\varphi(|z|) dA_\psi(z).
\]
For \( 0 < t < 1 \), define a compact supporting continuous function \( \eta_t \) on \( D \) by
\[
\eta_t(z) = \begin{cases} 
1, & |z| \leq t, \\
\frac{1 + t}{1 - t} - \frac{2|z|}{1 - t}, & t < |z| \leq (1 + t)/2, \\
0, & (1 + t)/2 < |z| < 1.
\end{cases}
\]
For any $0 < r, t < 1$, define the integral operators $S_{[r]}$ on $A^1(\psi)$ and $S_{[r,t]}$ from $A^1(\psi)$ to $L^1(\psi)$ as follows:

\[
S_{[r]} f(w) = \int_D f(z)(SK_z)(w)\eta_r(z)\varphi(|z|) dA_\psi(z),
\]

\[
S_{[r,t]} f(w) = \int_D f(z)(SK_z)(w)\eta_r(z)\eta_t(w)\varphi(|z|) dA_\psi(z).
\]

Using Lemma 2, we have

\[
\|(S - S_{[r]})f\|_\psi \leq \int_D |f(z)| dA_\psi(z) \int_D |(SK_z)(w)\varphi(|z|)(1 - \eta_r(z))| dA_\psi(w)
\]

\[
\leq C\|f\|_\psi \sup_{z \in D} (1 - \eta_r(z))\|Sk_z\|_\psi.
\]

Since $\|Sk_z\|_\psi \to 0$ as $z \to \partial D$, we have $\sup_{z \in D} (1 - \eta_r(z))\|Sk_z\|_\psi \to 0$ as $r \to 1^-$. Thus

\[
(4) \quad \|S - S_{[r]}\| \to 0 \quad \text{as } r \to 1^-.
\]

Seeing $S_{[r]}$ as an operator from $A^1(\psi)$ to $L^1(\psi)$, if we prove that it is compact, then it is also compact as an operator on $A^1(\psi)$. Similar to the above, we have

\[
(5) \quad \|S_{[r]} - S_{[r,t]}\| \leq C \sup_{z \in \frac{1+r}{2}D} \int D |(Sk_z)(w)(1 - \eta_t(w))| dA_\psi(w).
\]

We will prove that

\[
\sup_{z \in \frac{1+r}{2}D} \int D |(Sk_z)(w)(1 - \eta_t(w))| dA_\psi(w) \to 0
\]

as $t \to 1^-$ for fixed $r < 1$.

Let $g_t(z) = \int_D |(Sk_z)(w)(1 - \eta_t(w))| dA_\psi(w)$. Firstly we will show that \{$g_t : 0 < t < 1$\} is equicontinuous and uniformly bounded on $\frac{1+r}{2}D$.

Since the Taylor expansion of $K_z$ is $K_z(w) = \sum_{m=0}^\infty \beta_m(wz)^m$, where $\beta_m = (\alpha + 1) \cdots (\alpha + m + 1)/m!$, for any $z_1, z_2 \in \frac{1+r}{2}D$ we have

\[
|K_{z_1}(w) - K_{z_2}(w)| = \left| \sum_{m=0}^\infty \beta_m(wz_1)^m - \sum_{m=0}^\infty \beta_m(wz_2)^m \right|
\]

\[
\leq \sum_{m=1}^\infty \beta_m |w|^m |z_1^m - z_2^m| \leq |z_1 - z_2| \sum_{m=1}^\infty \beta_m mr^{m-1}.
\]

The last series above is the derivative of the series $\sum_{m=0}^\infty \beta_m rm$ for $\frac{1+r}{2}D$ hence convergent in $|r| < 1$. So for any $\varepsilon > 0$, there exists a constant $\delta_1 > 0$ such that $|K_{z_1}(w) - K_{z_2}(w)| \leq \varepsilon$ for any $z_1, z_2 \in \frac{1+r}{2}D$ with $|z_1 - z_2| < \delta_1$, and so $\|K_z\|_\psi$ is uniformly continuous on $\frac{1+r}{2}D$. Then for any $\varepsilon > 0$, there exist
\( \delta_2 > 0 \) such that \( |k_{z_1}(w) - k_{z_2}(w)| < \varepsilon \) for \( z_1, z_2 \in \frac{1+r}{2}D \) with \( |z_1 - z_2| < \delta_2 \), whence

\[
|g_t(z_1) - g_t(z_2)| \leq \int_D |Sk_{z_1}(w) - Sk_{z_2}(w)|(1 - \eta_t(w)) \, dA(z)
\]

\[
\leq \|S\| \int_D |k_{z_1}(w) - k_{z_2}(w)| \, dA(z) \leq \varepsilon \|S\| \|1\|_\psi.
\]

Since \( \varepsilon \) is arbitrary, \( \{g_t : 0 < t < 1\} \) is equicontinuous. It is obvious that \( \{g_t : 0 < t < 1\} \) is uniformly bounded.

For \( z \in \frac{1+r}{2}D \), Lebesgue’s dominated convergence theorem implies that \( g_t(z) \to 0 \) as \( t \to 1^- \). It follows from Ascoli’s theorem that \( \{g_t : 0 < t < 1\} \) is relatively compact in \( C(\frac{1+r}{2}D) \), the Banach space of continuous functions on \( \frac{1+r}{2}D \), so has a unique accumulation point, 0. Therefore \( g_t \to 0 \) as \( t \to 1^- \).

So (5) implies that

\[
(6) \quad \|S_{[r]} - S_{[r,t]}\| \to 0 \quad \text{as} \quad t \to 1^-.
\]

Since the kernel of \( S_{[r,t]} \) is a compact supporting continuous function on \( D \times D \), it can be approximated uniformly by polynomials in \( z, \bar{z}, w, \bar{w} \) on \( D \times D \). Because a polynomial kernel induces a finite rank integral operator, \( S_{[r,t]} \) is a compact operator (cf. [3]). Thus (6) implies that \( S_{[r]} \) is compact, and (4) implies that \( S \) is compact.

Finally, we show that \( A_0(\varphi) \) is invariant under \( S^* \). Suppose that \( g \in A_0(\varphi) \) and \( S^*g \in A_\infty(\varphi) \setminus A_0(\varphi) \). By the definitions of \( A_0(\varphi) \) and \( A_\infty(\varphi) \), there exist some positive constant \( \varepsilon \) and a sequence \( \{z_n\} \) in \( D \) such that \( z_n \to \partial D \) as \( n \to \infty \) and \( |\langle S^*g(z_n), \varphi(z_n) \rangle| > \varepsilon \). Thus using Lemma 2 and the reproducing property of \( K_z \), we have

\[
|\langle Sk_{z_n}, g \rangle| = |\langle k_{z_n}, S^*g \rangle| = \frac{|\langle (S^*g)(z_n), \varphi(z_n) \rangle|}{\|K_{z_n}\|_\psi} \geq \frac{1}{C} \|S^*g(z_n)\| \|\varphi(z_n)\| > \frac{\varepsilon}{C}.
\]

This contradicts the hypothesis \( \|Sk_z\|_\psi \to 0 \) as \( z \to \partial D \), and completes the proof of Theorem 1. \( \blacksquare \)

In the last argument of the proof of Theorem 1, \( g \in A_\infty(\varphi) \) is only needed. So the following result holds.

**Corollary.** If \( S \) is a compact operator on \( A^1(\psi) \) and \( S^*A_0(\varphi) \subset A_0(\varphi) \), then \( S^*A_\infty(\varphi) \subset A_0(\varphi) \).

**Remark.** If \( S \) is a compact operator on \( L^2_a(D) \), then \( \|Sk_z\| \to 0 \) as \( z \to \partial D \). Examples in [1] and [4] show that the converse does not hold. However, in \( A^1(\psi) \), \( \|Sk_z\| \to 0 \) as \( z \to \partial D \) is a sufficient condition for the compactness of \( S \). Moreover, the following example shows that \( A_0(\varphi) \) being an invariant subspace of \( S^* \) in Theorem 1 is necessary.
EXAMPLE. Suppose that $f \in A_\infty(\varphi) \setminus A_0(\varphi)$. By the definitions of $A_\infty(\varphi)$ and $A_0(\varphi)$, there exists a sequence $\{z_n\}$ in $D$ such that $z_n \to \partial D$ and $|f(z_n)|\varphi(|z_n|) \to 0$.

Suppose that $0 \neq g \in A^1(\psi)$. Then there exists $h \in A_\infty(\varphi)$ such that $\langle g, h \rangle \neq 0$. Let $S = g \otimes f$. Then $S$ is a compact operator on $A^1(\psi)$. However,

$$|\langle Sk_z, h \rangle| = |\langle k_z, (f \otimes g^*)h \rangle| = |\langle k_z, (f \otimes g^*)h \rangle|$$

$$\geq C|\langle g, h \rangle||f(z_n)|\varphi(|z_n|) \to 0,$$

where the inequality comes from Lemma 2, and $g^{**}$ denotes the image of $g$ in the double dual space of $A^1(\psi)$. So we have $\|Sk_z\|_{\psi} \to 0$.

For $u \in L^\infty(D)$, define the Toeplitz operator $T_u$ by $T_u(f) = Q(uf)$, where $f \in A^1(\psi)$. Let $T(L^\infty)$ denote the closed subalgebra of $B(A^1(\psi))$ generated by $\{T_u : u \in L^\infty(D)\}$.

**Theorem 2.** Suppose that $S \in T(L^\infty)$. Then $S$ is compact if and only if $\|Sk_z\|_{\psi} \to 0$ as $z \to \partial D$.

**Proof.** By Theorem 1, it is sufficient to prove that $A_0(\varphi)$ is invariant under $S^*$ when $S \in T(L^\infty)$. By the definition of $T(L^\infty)$, it is sufficient to prove that $A_0(\varphi)$ is invariant under $T_u^*$ for any $u \in L^\infty(D)$. Let $g \in A_0(\varphi)$. Then

$$(T_u^*g)(z) = \langle K_z, T_u^*g \rangle = \langle T_uK_z, g \rangle = \langle uK_z, g \rangle.$$  

Hence

$$|(T_u^*g)(z)\varphi(|z|)| \leq \int_D |u(w)K_z(w)g(w)|(|1 - |w|^2)^a \varphi(|z|) dA(w)$$

$$\leq \|u\|_\infty \int_D \varphi(|z|)|K_z(w)g(w)|\varphi(|w|) dA_\psi(w)$$

$$= \|u\|_\infty \left( \int_{rD} \varphi(|z|)|K_z(w)g(w)|\varphi(|w|) dA_\psi(w) + \int_{D \setminus rD} \varphi(|z|)|K_z(w)g(w)|\varphi(|w|) dA_\psi(w) \right),$$

where $0 < r < 1$. Since $g \in A_0(\varphi)$, for any $\varepsilon > 0$, there exists $r$ such that $|g(w)|\varphi(|w|) \leq \varepsilon$ for $w \in D \setminus rD$. Then it follows from Lemma 2 that the second integral on the right hand side of (7) is $\leq \varepsilon \int_{D \setminus rD} \varphi(|z|)|K_z(w)| dA_\psi(w) \leq C\varepsilon$ for fixed $r$. Since $g \in A_0(\varphi)$, there exists a constant $M$ such that $\sup_{w \in D} |g(w)|\varphi(|w|) \leq M$. Since $\varphi(|z|)|K_z(w)|$ converges uniformly to 0 on $rD$ as $z \to \partial D$, the first integral on the right hand side of (7) converges to 0.
as \( z \to \partial D \). So we have
\[
\limsup_{z \to \partial D} |(T^*_u g)(z)| \varphi(|z|) \leq \|u\|_\infty C \varepsilon.
\]
Since \( \varepsilon \) is arbitrary, \( |(T^*_u g)(z)| \varphi(|z|) \to 0 \) as \( z \to \partial D \). Thus \( T^*_u g \in A_0(\varphi) \).

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