

Irreducible Markov systems on Polish spaces

by

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Abstract. Contractive Markov systems on Polish spaces which arise from graph directed constructions of iterated function systems with place dependent probabilities are considered. It is shown that their stability may be studied using the concentrating methods developed by the second author [Dissert. Math. 415 (2003)]. In this way Werner's results obtained in a locally compact case [J. London Math. Soc. 71 (2005)] are extended to a noncompact setting.

1. Introduction and notation. In [1] Barnsley *et al.* considered the evolution of distributions due to the action of randomly chosen transformations—so-called iterated function systems with complete connections. In their paper sufficient conditions for the existence of an invariant measure and stability are formulated.

In [7] A. Lasota and J. Yorke generalized the results mentioned above. The novelty of their study consists in the use of the lower bound technique developed earlier for Markov operators acting on L^1 spaces.

Recently Werner [13, 14] has extended iterated function systems with complete connections to a more general setting. Indeed, he has studied graph directed constructions on locally compact spaces with an open partition (see also [8]).

Barnsley *et al.* and Werner tackled the problem of the existence of an invariant measure from the probabilistic point of view. In this paper we aim to show that Werner's result may be obtained by functional analysis methods even in the general case of complete separable metric spaces. In fact, Werner's assumption looks somewhat restrictive. For instance it rules out a lot of processes defined on function spaces. Such spaces appear not only when one studies Markov systems generated by partial differential equations (see for instance [6]) but also in random processes describing various phenomena in physics and biology (see for instance [5]). Although there are a huge

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number of results on Markov processes associated with random iteration of maps (see [2–4, 9–11]), none of our results is covered by the existing theory.

Let (K, d) be a complete and separable metric space. Assume that $K = \bigcup_{j=1}^N K_j$, where K_1, \dots, K_N are such that $\inf\{d(x, y) : x \in K_m, y \in K_n\} > 0$ for $m \neq n$. We denote by $\mathcal{L}^0(K)$ the space of all bounded Borel measurable functions $f : K \rightarrow \mathbb{R}$, and by $C(K)$ the subspace of all bounded continuous functions. These spaces are equipped with the supremum norm. Moreover, $P(K)$ denotes the space of all Borel probability measures on K .

For each $j \in \{1, \dots, N\}$, let

$$w_{j1}, \dots, w_{jL_j} : K_j \rightarrow K$$

be a family of Borel measurable maps such that for each $m \in \{1, \dots, L_j\}$, there exists $n \in \{1, \dots, N\}$ such that $w_{jm}(K_j) \subset K_n$. Furthermore, for each $j \in \{1, \dots, N\}$, let

$$p_{j1}, \dots, p_{jL_j} : K_j \rightarrow [0, 1]$$

be a family of Borel measurable probability functions, that is, $p_{jm} > 0$ for all j, m and $\sum_{m=1}^{L_j} p_{jm}(x) = 1$ for all $x \in K_j$.

Following Werner (see [11]) we call $V = \{1, \dots, N\}$ the *set of vertices*, and the subsets K_1, \dots, K_N the *vertex sets*. Further, we call

$$E = \{(j, n_j) : j \in \{1, \dots, N\}, n_j \in \{1, \dots, L_j\}\}$$

the *set of edges* and we write

$$p_e := p_{jn} \quad \text{and} \quad w_e := w_{jn} \quad \text{for } e = (j, n) \in E.$$

For an edge $e \in E$ we denote by $i(e)$ the *initial vertex* of e , that is, $i(e) = j$ if and only if $e = (j, k)$ for some $k \in \{1, \dots, L_j\}$. The *terminal vertex* $t(e)$ for $e = (j, n) \in E$ is equal to k if and only if $w_e(K_j) \subset K_k$.

We recall that the quadruple $G = (V, E, i, t)$ is called a *directed multi-graph* or *digraph*. A sequence (finite or infinite) $(\dots, e_{-1}, e_0, e_1, \dots)$ of edges is called a *path* if $t(e_k) = i(e_{k+1})$ for all k .

For every $j \in V$ we denote by $l(j)$ the smallest number, say k , such that there is a path (e_1, \dots, e_k) with $i(e_1) = j$ and $t(e_k) = j$.

A path $\mathbf{c} = (e_1, \dots, e_m)$ is called a *cycle* if $i(e_1) = t(e_m)$. A cycle is called *simple* if it does not contain any other cycle. Let $l(\mathbf{c})$ denote the *length* of \mathbf{c} , i.e., $l(\mathbf{c}) = m$ if $\mathbf{c} = (e_1, \dots, e_m)$.

We denote by $[k_1, \dots, k_M]$ the greatest common divisor of $k_1, \dots, k_M \in \mathbb{N}$.

We call the family $(K_{i(e)}, w_e, p_e)_{e \in E}$ a *Markov system*. A Markov system is *irreducible* if and only if its directed graph is irreducible, that is, there is a path from any vertex to any other. An irreducible Markov system is said to have *period* p if the set of vertices can be partitioned into p nonempty subsets $\Omega_1, \dots, \Omega_p$ such that

$$i(e) \in \Omega_j \Rightarrow t(e) \in \Omega_{j+1} \pmod p$$

for all $e \in E$, and p is the largest number with this property. An irreducible Markov system with period 1 is called *aperiodic*.

Finally, $\mathbf{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_M\}$ denotes the set of all simple cycles in $(K_{i(e)}, w_e, p_e)_{e \in E}$.

REMARK 1. If an irreducible Markov system $(K_{i(e)}, w_e, p_e)_{e \in E}$ is aperiodic, then

$$[l(\mathbf{c}_1), \dots, l(\mathbf{c}_M)] = 1.$$

Indeed, assume contrary to our claim that $[l(\mathbf{c}_1), \dots, l(\mathbf{c}_M)] > 1$. Let $p > 1$ be a prime number dividing each $l(\mathbf{c}_j)$. By a simple induction on the number of cycles we then show that the system is periodic with period p .

To define a Markov operator on $\mathcal{L}^0(K)$ associated with the Markov system under consideration we extend p_{jm} onto the whole space K by zero; the maps w_{jm} are extended arbitrarily.

We define a Markov operator on $\mathcal{L}^0(K)$ by

$$(1.1) \quad Uf := \sum_{e \in E} p_e f \circ w_e \quad \text{for all } f \in \mathcal{L}^0(K)$$

and its adjoint operator on $P(K)$ by

$$(1.2) \quad \int_K f dU^*\nu = \int_K Uf d\nu \quad \text{for all } f \in \mathcal{L}^0(K) \text{ and } \nu \in P(K).$$

REMARK 2. Observe that by the Lebesgue monotone convergence theorem, U may be extended to all nonnegative Borel measurable functions (not necessarily bounded).

For K unbounded, a continuous function $L : K \rightarrow [0, \infty)$ is called a *Lyapunov function* if

$$\lim_{\varrho(x, x_0) \rightarrow \infty} L(x) = \infty$$

for some $x_0 \in X$.

An irreducible Markov system $(K_{i(e)}, w_e, p_e)_{e \in E}$ is called *weakly asymptotically stable* if there exists an invariant measure $\mu_0 \in P(K)$, i.e., $U^*\mu_0 = \mu_0$ and

$$\lim_{n \rightarrow \infty} \int_K U^n f(x) \mu(dx) = \int_K f(x) \mu_0(dx)$$

for all $f \in C(K)$ and $\mu \in P(K)$.

For $\mu \in P(K)$ we denote by $\omega(\mu)$ the set of all weak limits of subsequences of $(U^{*n}\mu)_{n \geq 1}$.

We will assume that the Markov system $(K_{i(e)}, w_e, p_e)$ is *contractive*, i.e., there exists $0 < a < 1$ such that

$$(1.3) \quad \sum_{e \in E} p_e(x) d(w_e(x), w_e(y)) \leq ad(x, y)$$

for all $x, y \in K_j$ and $j \in \{1, \dots, N\}$. Finally, recall that a function $f : (K, d) \rightarrow \mathbb{R}$ is called *Dini continuous* if there is $c > 0$ such that

$$\int_0^c \frac{\Phi(t)}{t} dt < \infty,$$

where Φ is the *modulus of uniform continuity* of f , that is,

$$\Phi(t) = \sup\{|f(x) - f(y)| : d(x, y) \leq t, x, y \in K\}.$$

Our main results concerning irreducible Markov systems defined on Polish spaces are the following:

THEOREM 1.1. *Let $(K_{i(e)}, w_e, p_e)_{e \in E}$ be a contractive Markov system such that every p_e is Dini continuous on $K_{i(e)}$. Then the system has an invariant measure $\mu_0 \in P(K)$. Moreover the set $\omega(\mu)$ is nonempty for each $\mu \in P(K)$ and the set $\bigcup_{\mu \in P(K)} \omega(\mu)$ is tight.*

THEOREM 1.2. *Let $(K_{i(e)}, w_e, p_e)_{e \in E}$ be an irreducible contractive Markov system such that $p_e|_{K_{i(e)}}$ is Dini continuous and there exists $\delta > 0$ such that $p_e|_{K_{i(e)}} \geq \delta$ for all $e \in E$. If, in addition, the system is aperiodic, then it is weakly asymptotically stable.*

2. Auxiliary lemmas. We start with two definitions. An irreducible Markov system $(K_{i(e)}, w_e, p_e)_{e \in E}$ is called *globally concentrating* if for every $\varepsilon > 0$ and every bounded Borel set $A \subset K$ there exist a bounded Borel set $B \subset K$ and $n_0 \in \mathbb{N}$ such that

$$(2.1) \quad U^{*n} \mu(B) \geq 1 - \varepsilon \quad \text{for } n \geq n_0, \mu \in P(A),$$

where U^* is given by (1.2).

The following lemma gives a condition which ensures that an irreducible Markov system is globally concentrating. Its proof may be found in [12, Lemma 2.4.2] where general Markov operators were considered.

LEMMA 2.1. *An irreducible Markov system $(K_{i(e)}, w_e, p_e)_{e \in E}$ is globally concentrating if there exists a Lyapunov function L , bounded on bounded sets, such that*

$$(2.2) \quad UL(x) \leq aL(x) + b \quad \text{for } x \in K,$$

where a, b are nonnegative constants and $a < 1$.

An irreducible Markov system $(K_{i(e)}, w_e, p_e)_{e \in E}$ is called *semi-concentrating* if for every $\varepsilon > 0$ there exist $\alpha > 0$ and $\{x_1, \dots, x_s\} \subset K$ such that

$$(2.3) \quad \liminf_{n \rightarrow \infty} U^{*n} \mu \left(\bigcup_{j=1}^s B(x_j, \varepsilon) \right) \geq \alpha \quad \text{for } \mu \in P(K).$$

LEMMA 2.2. *Suppose that $(K_{i(e)}, w_e, p_e)_{e \in E}$ is a contractive Markov system with an average contracting rate $0 < a < 1$ such that the family $\{K_1, \dots, K_N\}$ is an open partition of K . Then $(K_{i(e)}, w_e, p_e)_{e \in E}$ is globally concentrating. Moreover for every $\varepsilon > 0$ there exists a bounded Borel set $B \subset K$ such that*

$$\liminf_{n \rightarrow \infty} U^{*n} \mu(B) \geq 1 - \varepsilon \quad \text{for all } \mu \in P(K).$$

Proof. We show that the assumptions of Lemma 2.1 are satisfied. Choose $x_j \in K_j$ for $j = 1, \dots, N$. Set $L(x) = d(x, x_1)$ for $x \in K$. Then

$$\begin{aligned} UL(x) &= \sum_{e \in E} p_e(x) d(w_e(x), x_1) \\ &\leq \sum_{e \in E} p_e(x) (d(w_e(x), w_e(x_{i(e)})) + d(w_e(x_{i(e)}), x_1)) \\ &\leq ad(x, x_1) + a \max_{j \in V} d(x_j, x_1) + \max_{e \in E, j \in V} d(w_e(x_j), x_1) \quad \text{for } x \in K. \end{aligned}$$

Hence (2.2) holds with

$$b = a \max_{j \in V} d(x_j, x_1) + \max_{e \in E, j \in V} d(w_e(x_j), x_1). \blacksquare$$

Now we may formulate the following corollary:

COROLLARY 2.1. *Let $(K_{i(e)}, w_e, p_e)_{e \in E}$ satisfy the hypotheses of Lemma 2.2. If (K, d) is a metric space in which sets of finite diameter are relatively compact, then:*

- (i) *the sequence $(U^{*k} \mu)_{k \geq 1}$ is tight for all $\mu \in P(K)$;*
- (ii) *the Markov system $(K_{i(e)}, w_e, p_e)_{e \in E}$ has an invariant Borel probability measure μ_0 ;*
- (iii) *the invariant probability measure μ_0 is unique if and only if*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n U^k g(x) = \int_K g d\mu_0 \quad \text{for all } x \in K \text{ and } g \in C(K);$$

(iv) *we have*

$$\sum_{j=1}^N \int_K d(x, x_j) \mu_0(dx) < \infty \quad \text{for all } x_j \in K_j, j = 1, \dots, N.$$

Proof. (i)–(iii) easily follow from Lemma 2.1 and from Lemma 2.4.2 and Theorem 5.1 in [12]. On the other hand, from the proof of Lemma 2.1 and an induction argument it follows that for fixed $x_j \in K_j$,

$$U^n L(x) \leq a^n L(x) + \frac{b}{1 - a},$$

where $L(x) = d(x, x_j)$ and $n \in \mathbb{N}$. Let $\mu_0 \in P(K)$ be a unique invariant measure. Then

$$\begin{aligned} \int_K d(x, x_j) \mu_0(dx) &= \limsup_{n \rightarrow \infty} \int_K d(x, x_j) U^{*n} \mu_0(dx) \\ &= \limsup_{n \rightarrow \infty} \int_K U^n L(x) \mu_0(dx) \leq \limsup_{n \rightarrow \infty} \int_K \left[a^n L(x) + \frac{b}{1-a} \right] \mu_0(dx) \\ &\leq \frac{b}{1-a} < \infty, \end{aligned}$$

by the Lebesgue monotone convergence theorem. Hence (iv) follows. ■

From now on we will assume that $p_e, e \in E$, is Dini continuous and let Φ_e be its modulus of uniform continuity. Set $\Phi = \sum_{e \in E} \Phi_e$ and observe that

$$\int_0^c \frac{\Phi(t)}{t} dt < \infty \quad \text{for every } c > 0.$$

Then the function $\varphi : [0, \infty) \rightarrow [0, \infty)$ given by

$$\varphi(t) = t + \sum_{n=0}^{\infty} \Phi(a^n t)$$

satisfies

$$\Phi(t) + \varphi(at) \leq \varphi(t) \quad \text{for } t \geq 0.$$

Moreover, we easily check that:

- φ is continuous and $\varphi(0) = 0$;
- φ is nondecreasing and concave;
- $\varphi(t) > 0$ for $t > 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$.

We denote by \mathcal{F}_φ the family of all continuous functions $f : K \rightarrow \mathbb{R}$ such that $|f(x)| \leq 1$ and $|f(x) - f(y)| \leq \varphi(\tilde{d}(x, y))$ for all $x, y \in K$, where

$$\tilde{d}(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in K_j \text{ for } j \in \{1, \dots, N\}, \\ \max(c, d(x, y)) & \text{otherwise,} \end{cases}$$

with $c > 0$ such that $\varphi(c) > 2$. It is obvious that \tilde{d} is a metric on K equivalent to d . The definition of \tilde{d} looks somewhat sophisticated but it allows us to verify nonexpansiveness merely on one element of the partition of K .

LEMMA 2.3. *Let the assumptions of Theorem 1.1 hold. Then the operator U given by (1.1) satisfies $U(\mathcal{F}_\varphi) \subset \mathcal{F}_\varphi$.*

Proof. Fix $f \in \mathcal{F}_\varphi$. We have

$$|Uf(x)| = \left| \sum_{e \in E} p_e(x) f(w_e(x)) \right| \leq \sum_{e \in E} p_e(x) = 1 \quad \text{for } x \in K.$$

Further, from (1.3) it follows that

$$\begin{aligned} |Uf(x) - Uf(y)| &= \left| \sum_{e \in E} p_e(x) f(w_e(x)) - \sum_{e \in E} p_e(y) f(w_e(y)) \right| \\ &\leq \sum_{e \in E} |p_e(x) - p_e(y)| + \sum_{e \in E} p_e(x) |f(w_e(x)) - f(w_e(y))| \\ &\leq \Phi(d(x, y)) + \sum_{e \in E} p_e(x) \varphi(d(w_e(x), w_e(y))) \end{aligned}$$

for $x, y \in K_j, j = 1, \dots, N$. Since φ is concave and nondecreasing,

$$|Uf(x) - Uf(y)| \leq \Phi(d(x, y)) + \varphi(ad(x, y)) \leq \varphi(d(x, y)) = \varphi(\tilde{d}(x, y))$$

for $x, y \in K_j, j = 1, \dots, N$.

If x and y are in different K_j , then

$$|Uf(x) - Uf(y)| \leq 2 \leq \varphi(c) \leq \varphi(\tilde{d}(x, y)),$$

which finishes the proof that $U(\mathcal{F}_\varphi) \subset \mathcal{F}_\varphi$. ■

LEMMA 2.4. *Let the assumptions of Theorem 1.2 hold. Then the irreducible Markov system $(K_{i(e)}, w_e, p_e)_{e \in E}$ is semi-concentrating.*

Proof. Lemma 2.2 shows that there exists a bounded Borel set $B \subset K$ such that

$$\liminf_{n \rightarrow \infty} U^{*n} \mu(B) > 1/2 \quad \text{for all } \mu \in P(K).$$

Without loss of generality we may assume that $B_j = B \cap K_j \neq \emptyset$ for $j = 1, \dots, N$. Fix $\varepsilon > 0$. Choose an integer $m \in \mathbb{N}$ such that $a^m(\text{diam } B) < \varepsilon$. Further, let $\eta > 0$ be such that

$$(1 + \eta)^m a^m \text{diam } B \leq \varepsilon.$$

Fix $x_j \in B_j, j = 1, \dots, N$, and define $C \subset K$ by

$$C = \bigcup_{j=1}^N \bigcup_{e_1, \dots, e_m \in E} B(w_{e_m} \circ \dots \circ w_{e_1}(x_j), \varepsilon).$$

Now (1.3) implies that for every $y \in K_j$ there exists $I_y \subset E^m$ such that

$$d(w_{e_m} \circ \dots \circ w_{e_1}(x_j), w_{e_m} \circ \dots \circ w_{e_1}(y)) \leq (1 + \eta)^m a^m d(x_j, y)$$

for $(e_1, \dots, e_m) \in I_y$ and

$$\sum_{(e_1, \dots, e_m) \in I_y} p_{e_1}(y) p_{e_2}(w_{e_1}(y)) \dots p_{e_m}(w_{e_{m-1}} \circ \dots \circ w_{e_1}(y)) \geq \left(\frac{\eta}{1 + \eta} \right)^m.$$

This can be shown by an analysis similar to that in the proof of Lem-

ma 6.3.1 in [12]. Observe that for every $y \in B_j$ and $(e_1, \dots, e_m) \in I_y$ we have $w_{e_m} \circ \dots \circ w_{e_1}(y) \in C$. Set

$$\alpha = \frac{1}{2} \left(\frac{\eta}{1 + \eta} \right)^m.$$

By induction and the definition of C for each $n \in \mathbb{N}$ we obtain

$$\begin{aligned} U^{*(m+n)}\mu(C) &\geq \int_K \mathbb{1}_C(w_{e_m} \circ \dots \circ w_{e_1}(y)) \\ &\quad \times \sum_{(e_1, \dots, e_m) \in I_y} p_{e_1}(y) \cdots p_{e_m}(w_{e_{m-1}} \circ \dots \circ w_{e_1}(y)) U^{*n}\mu(dy) \\ &\geq \left(\frac{\eta}{1 + \eta} \right)^m U^{*n}\mu(B) \quad \text{for } \mu \in P(K) \end{aligned}$$

and consequently

$$\liminf_{n \rightarrow \infty} U^{*n}\mu(C) \geq \alpha \quad \text{for } \mu \in P(K),$$

which finishes the proof. ■

LEMMA 2.5. *If an irreducible Markov system is aperiodic, then for every $k, l \in V$ there exist $m \in \mathbb{N}$ and $(e_1, \dots, e_m), (\tilde{e}_1, \dots, \tilde{e}_m)$ such that $i(e_1) = k, i(\tilde{e}_1) = l$ and $t(e_m) = t(\tilde{e}_m)$.*

Proof. Fix $k, l \in V$. Let (e_1^k, \dots, e_p^k) and (e_1^l, \dots, e_q^l) be paths in (V, E, i, t) starting from k, l , respectively and containing all successive cycles from \mathbf{C} . Assume that $t(e_p^k) = t(e_q^l)$. If $p = q$, then the proof is complete. Conversely, assume that $p > q$. Since $[l(\mathbf{c}_1), \dots, l(\mathbf{c}_M)] = 1$, there exist integers m_1, \dots, m_M such that

$$\sum_{j=1}^M m_j l(\mathbf{c}_j) = p - q.$$

Let $J \subset \{1, \dots, M\}$ be such that $m_j < 0$ for $j \in J$ and $m_j \geq 0$ for $j \in \{1, \dots, M\} \setminus J$. Adding to (e_1^k, \dots, e_p^k) the cycle composed of the cycles \mathbf{c}_j taken m_j times for $j \in J$, and similarly adding to (e_1^l, \dots, e_q^l) the cycle composed of the cycles \mathbf{c}_j taken m_j times for $j \in \{1, \dots, M\} \setminus J$, we finish the proof of the lemma. ■

3. Proofs

Proof of Theorem 1.1. From Lemma 2.3 it follows that the Markov system $(K_{i(e)}, w_e, p_e)_{e \in E}$ is semi-concentrating. Thus the statement of Theorem 1.1 follows from Theorem 5.5 in [12] and Lemma 2.1. ■

Proof of Theorem 1.2. From Theorem 1.1 it follows that U^* admits an invariant probability measure. In view of Lemma 2.4, from Theorem 5.4 and Remark 5.1 in [12] it follows that to finish the proof of stability it remains to

show that for every $\varepsilon > 0$ there is $\beta > 0$ such that for every $\mu_1, \mu_2 \in P(K)$ there exist a bounded Borel set $A \subset K$ with $\text{diam } A \leq \varepsilon$ and $n \in \mathbb{N}$ satisfying

$$(3.1) \quad U^{*n} \mu_j(A) \geq \beta \quad \text{for } j = 1, 2.$$

Fix $\varepsilon > 0$. According to Theorem 1.1 there is a compact set $K_0 \subset K$ such that

$$\tilde{\mu}(K_0) \geq 4/5 \quad \text{for all } \tilde{\mu} \in \bigcup_{\mu \in P(K)} \omega(\mu).$$

By the Aleksandrov theorem there exists a sequence $(m_n)_{n \geq 1}$ such that for each open set G with $K_0 \subset G$,

$$\liminf_{n \rightarrow \infty} U^{*m_n} \mu_j(G) > 1/2 \quad \text{for } j = 1, 2.$$

Consequently, there exist $k, l \in V$ and a subsequence $(\tilde{m}_n)_{n \geq 1}$ of $(m_n)_{n \geq 1}$ such that

$$(3.2) \quad \liminf_{n \rightarrow \infty} U^{*\tilde{m}_n} \mu_1(G_1) > \frac{1}{2N}$$

and

$$(3.3) \quad \liminf_{n \rightarrow \infty} U^{*\tilde{m}_n} \mu_2(G_2) > \frac{1}{2N}$$

for arbitrary open neighbourhoods G_1, G_2 of $\tilde{K}_k = K_0 \cap K_k, \tilde{K}_l = K_0 \cap K_l$, respectively. By Lemma 2.3 we choose $\tilde{m} \in \mathbb{N}$ such that for $k, l \in V$ there exist paths $(\hat{e}_1, \dots, \hat{e}_m), (\tilde{e}_1, \dots, \tilde{e}_m)$ satisfying $i(\hat{e}_1) = k, i(\tilde{e}_1) = l, t(\hat{e}_m) = t(\tilde{e}_m)$ and $m \leq \tilde{m}$. Let $F_k = w_{\hat{e}_m} \circ \dots \circ w_{\hat{e}_1}(\tilde{K}_k)$ and $F_l = w_{\tilde{e}_m} \circ \dots \circ w_{\tilde{e}_1}(\tilde{K}_l)$. Set $F_0 = F_k \cup F_l$ and observe that $F_0 \subset K_{t(\hat{e}_m)}$. We easily check that

$$(3.4) \quad \liminf_{n \rightarrow \infty} U^{*m_n+m} \mu_j(\tilde{G}) > \frac{1}{2N} \delta^m \geq \frac{1}{2N} \delta^{\tilde{m}} \quad \text{for } j = 1, 2$$

and for every open neighbourhood \tilde{G} of F_0 .

Choose an integer $n \in \mathbb{N}$ such that

$$a^n \text{diam } F_0 \leq \varepsilon/3.$$

For every path (e_1, \dots, e_n) in the given diagraph define

$$\pi_{(e_1, \dots, e_n)}(x) = p_{e_1}(x) \cdots p_{e_n}(w_{e_{n-1}} \circ \dots \circ w_{e_1}(x)) \quad \text{for } x \in K.$$

For $x \in F_0$ and (e_1, \dots, e_n) such that $i(e_1) = t(\hat{e}_m)$ we define

$$O_{(e_1, \dots, e_n)}(x) = \{y \in K_{t(e_m)} : d(w_{e_n} \circ \dots \circ w_{e_1}(x), w_{e_n} \circ \dots \circ w_{e_1}(y)) < \varepsilon/3\}.$$

Define

$$O_x = \bigcap O_{(e_1, \dots, e_n)}(x) \quad \text{for } x \in F_0,$$

where the intersection is taken over all paths (e_1, \dots, e_n) in the directed graph $(K_{i(e)}, w_e, p_e)$ starting from $t(e_m)$. Since F_0 is a compact set there is

a finite covering

$$F_0 \subset \bigcup_{j=1}^q O_{x_j}.$$

Set $\tilde{G} = \bigcup_{j=1}^q O_{x_j}$. We are going to show that (3.1) holds with $\beta = \delta^{n+\tilde{m}}/(2Nq)$. Indeed, by (3.4) there exists $M \in \mathbb{N}$ such that

$$U^{*M+m} \mu_j(\tilde{G}) > \frac{1}{2N} \delta^{\tilde{m}} \quad \text{for } j = 1, 2.$$

Therefore there exist $O_1 = O_{x_k}$ and $O_2 = O_{x_l}$ such that

$$U^{*M+m} \mu_j(O_j) > \frac{1}{2N} \frac{\delta^{\tilde{m}}}{q}.$$

By (1.3) and the definition of O_1 and O_2 we find a path (e'_1, \dots, e'_n) such that

$$A = w_{e'_n} \circ \dots \circ w_{e'_1}(O_1) \cup w_{e'_n} \circ \dots \circ w_{e'_1}(O_2)$$

satisfies $\text{diam } A < \varepsilon$. Finally,

$$\begin{aligned} &U^{*M+m+n} \mu_j(A) \\ &= \sum_{(e_1, \dots, e_n) \in K} \int \mathbb{1}_A(w_{e_n} \circ \dots \circ w_{e_1}(y)) \pi_{(\tilde{e}_1, \dots, \tilde{e}_n)}(y) U^{*M+m} \mu_j(dy) \\ &\geq \delta^n \cdot \frac{1}{2N} \delta^{\tilde{m}}/q = \frac{\delta^{n+\tilde{m}}}{2Nq} = \beta \quad \text{for } j = 1, 2, \end{aligned}$$

which finishes the proof. ■

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