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Bounded holomorphic functions with multiple sheeted pluripolar hulls

by

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Abstract. We describe compact subsets K of $\partial\mathbb{D}$ and $\mathbb{R}$ admitting holomorphic functions f with the domains of existence equal to $\mathbb{C}\setminus K$ and such that the pluripolar hulls of their graphs are infinitely sheeted. The paper is motivated by a recent paper of Poletsky and Wiegerinck.

1. Introduction. The paper is motivated by a recent paper of E. Poletsky and J. Wiegerinck (see [8]).

Let $E\subset\mathbb{C}^n$ be any subset. We say that E is *pluripolar* if for any $z\in E$ there exist a connected neighborhood U_z of z and a plurisubharmonic function u_z defined on U_z, $u_z\not\equiv-\infty$, such that

$$E\cap U_z\subset\{w\in U_z: u_z(w)=-\infty\}.$$

By Josefson's theorem (see [5]) the set E is pluripolar if and only if there exists a plurisubharmonic function u, $u\not\equiv-\infty$, defined on $\mathbb{C}^n$ such that $E\subset\{w\in\mathbb{C}^n: u(w)=-\infty\}$. It is well known (see e.g. [6]) that $\max\{\log|f_1|,\dots,\log|f_m|\}$ is a plurisubharmonic function for any holomorphic functions $f_1,\dots,f_m$. Therefore, the pluripolar sets are a generalization of analytic ones. So, it seems natural to study, with regard to pluripolar sets, problems similar to those studied in the case of analytic sets. In particular, we may study extension type problems. For this we use the notion of a pluripolar hull.

We denote by $\mathrm{PSH}(\Omega)$ the set of all plurisubharmonic functions defined on an open set $\Omega\subset\mathbb{C}^n$. For a pluripolar set $E\subset\Omega$ we define its *pluripolar*

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hull as

$$E^*_\Omega = \bigcap_{h\in \mathrm{PSH}(\Omega),\, h|_E\equiv -\infty} \{w\in \Omega : h(w) = -\infty\}.$$

For properties of the hull see e.g. [13, 7],

Let D be a domain in $\mathbb{C}$ and let f be a holomorphic function on D. For any subset $A\subset D$ we denote the graph of f over A by

$$\Gamma_f(A) = \{(z, f(z)) : z\in A\}.$$

Note that if A is a non-polar subset of D then $(\Gamma_f(A))^*_{\mathbb{C}^2} = (\Gamma_f(D))^*_{\mathbb{C}^2}$. Our aim is to find conditions on K which would imply the existence of a bounded holomorphic function on $\mathbb{C}\setminus K$ such that $\mathbb{C}\setminus K$ is the domain of existence of the function and the pluripolar hull of its graph is at least two-sheeted. It follows from [3] that there are no holomorphic functions on $\mathbb{C}\setminus K$ whose graphs have multiple sheeted pluripolar hulls if the set K is polar. In [8] the authors constructed a Cantor type set K on the real line and a holomorphic function f with the domain of existence $\mathbb{C}\setminus K$ such that $(\Gamma_f(\mathbb{C}\setminus K))^*_{\mathbb{C}^2}$ is double sheeted over $\mathbb{C}\setminus K$.

Let us fix some notations which will be used throughout the paper. We put $\mathbb{D}_R = \{z\in\mathbb{C} : |z| < R\}$, where $R > 0$. We denote by $\mathbb{D}$ the unit disc $\mathbb{D}_1$ and $\mathbb{D}^{\mathrm{e}} := \mathbb{C}\setminus\overline{\mathbb{D}}$. We also denote by $\mathbb{D}(p, r)$ the open disc with center at $p\in\mathbb{C}$ and radius $r > 0$ and $H_- := \{z\in\mathbb{C} : \operatorname{Im} z < 0\}$. Let "cap" denote the logarithmic capacity and let $\omega(z, A, D)$ denote the harmonic measure for the domain $D\subset\mathbb{C}$ of the Borel set $A\subset\partial D$, $z\in D$ (for definition and basic properties see e.g. [9]). Let m denote the normalized Lebesgue measure on the circle $\partial\mathbb{D}(p, r)$ (i.e. $m(\partial\mathbb{D}(p, r)) = r$) or the Lebesgue measure on $\mathbb{R}$.

The main result of the paper is the following.

THEOREM 1. *Let I denote the unit circle in $\mathbb{C}$ or an open interval (a, b), where $-\infty < a < b < \infty$. Let K be a compact subset of I such that* $\operatorname{cap}(I\setminus K) < \operatorname{cap}(I)$. *Then the following conditions are equivalent*:

(1) $m(K\cap U) > 0$ *for any $z\in K$ and for any neighborhood U of z.*
(2) *There exists a bounded holomorphic function with the domain of existence equal to $\mathbb{C}\setminus K$.*
(3) *There exist a bounded holomorphic function f whose domain of existence is $\mathbb{C}\setminus K$ and a set $K_0\subset K$ with $m(K_0) > 0$ such that the non-tangential limits for $f|_{\mathbb{D}}$ (respectively, $f|_{H_-}$) exist on K_0 and for any $z\in(\mathbb{C}\setminus K)\cup K_0$ the set $\{w\in\mathbb{C} : (z, w)\in(\Gamma_f(\mathbb{C}\setminus K))^*_{\mathbb{C}^2}\}$ is infinite.*

It will follow from the proof that in the case $K\neq\partial\mathbb{D}$ we shall be able to choose f so that the pluripolar hull $(\Gamma_f(\mathbb{C}\setminus K))^*_{\mathbb{C}^2}$ contains the set

$$\{(z, e^{2k} f(z)) : z \in \mathbb{C} \setminus K\} \cup \{(z, e^{2k} f^*(z)) : z \in K_0\}$$

for any $k \in \mathbb{Z}$, where $f^*(z)$ denotes the non-tangential limit of $f|_{\mathbb{D}}$ (respectively, $f|_{H_-}$) at $z \in K_0$.

The condition $\operatorname{cap}(I \setminus K) < \operatorname{cap}(I)$ implies, in particular, that K is not empty, so condition (1) is not trivial.

It will follow from the proof that the assumption $\operatorname{cap}(I \setminus K) < \operatorname{cap}(I)$ may be replaced with some condition on the existence of a set G such that $I \setminus G$ is thin at each point of a set of positive measure (cf. Theorem 7).

It is worth noting that the existence of non-constant holomorphic mappings on $\mathbb{C} \setminus K$ implies that K is of positive analytic capacity (see e.g. [12]).

The implication (2)⇒(1) in Theorem 1 is well known (see e.g. [1, Appendix A 1.4]). The implication (3)⇒(2) is trivial. So, the main result of the paper is the implication (1)⇒(3).

Note that in the case $K = \partial\mathbb{D}$ the Blaschke products from [14] help us to produce examples similar to that from (3)—the functions have, however, a little weaker properties than the ones claimed in Theorem 1. In fact, one may take the function f defined as $f(z) := \log(2 + z)B(z)$ for $z \in \mathbb{D}$ and $f(z) := f(1/z)$, $z \in \mathbb{D}^{\mathrm{e}}$. Then the pluripolar hull $(\Gamma_f(\mathbb{C}\setminus\partial\mathbb{D}))^*_{\mathbb{C}^2}$ is generically (with the possible exception of at most countably many points in $\mathbb{C} \setminus \partial\mathbb{D}$) infinitely sheeted. The methods from [10] could also help to exhibit a set K_0 as in Theorem 1.

In our paper except for pluripolar hulls we use mainly results and notions from potential theory on the complex plane. Apart from the notions of logarithmic capacity and harmonic measure we use, among others, the notions of logarithmic potential, equilibrium measure and Green function. The book [9] may serve as a good reference for the notions from potential theory used in this paper.

We mostly prove our results for subsets of $\partial\mathbb{D}$. The case of subsets of $\mathbb{R}$, when handled along the same lines, is sometimes only sketched or some details are omitted.

2. Useful results on the Green function and harmonic measure. Let $K \subset \mathbb{C}$ be a non-polar compact set in the complex plane. We define its Green function as

$$g_K(z) = p_\mu(z) - \log \operatorname{cap}(K),$$

where μ is the equilibrium measure of K and $p_\mu(z) = \int_K \log|z - w|\, d\mu(w)$ (for properties of the Green function see [9]). We extend the definition of the Green function to bounded non-polar F_σ-sets. Assume that $F = \bigcup K_j$, where K_j is an increasing sequence of non-polar compact sets. Then $\{g_{K_j}\}$ is a decreasing sequence of subharmonic functions. Put $g_F(z) = \lim_{j\to\infty} g_{K_j}(z)$.

The following properties are well known:

- $g_F \in \mathrm{SH}(\mathbb{C}) \cap (\mathbb{C} \setminus \overline{F})$;
- $g_F \geq 0$ on $\mathbb{C}$;
- $g_F = 0$ on $F \setminus E$, where E is a Borel polar set;
- if $F \subset \partial\mathbb{D}$ then $g_F(0) = -\log \operatorname{cap}(F)$.

LEMMA 2. *Let $F \subset \partial\mathbb{D}$ be an F_σ-set such that* $\operatorname{cap}(F) < 1$. *Then there exists a Borel set $L \subset \partial\mathbb{D} \setminus F$ of positive measure such that F is thin at every point of L.*

Let $F \subset (a,b)$, where $-\infty < a < b < \infty$, be an F_σ-set such that $\operatorname{cap}(F) < \operatorname{cap}((a,b)) = (b-a)/4$. *Then there exists a subset L of $(a,b) \setminus F$ of positive measure such that F is thin at every point of L.*

Proof. Without loss of generality we may assume in both cases that F is not polar.

Assume that $F \subset \partial\mathbb{D}$. The properties of g_F imply that

$$0 < -\log \operatorname{cap}(F) = g_F(0) = \frac{1}{2\pi} \int_0^{2\pi} g_F(e^{i\theta})\, d\theta.$$

Note that the Borel set $L := \{z \in \partial\mathbb{D} \setminus F : g_F(z) > 0\} \subset \partial\mathbb{D} \setminus F$ is of positive measure. We claim that F is thin at every point of L. Indeed, $g_F = 0$ on $F \setminus E$, where $E \subset F$ is a Borel polar set. Then $F \setminus E$ and E (and, therefore, $F = (F \setminus E) \cup E)$) are thin at every point of L.

In the case $F \subset (a,b)$, we may assume that $a = -1$ and $b = 1$. Put $I = [-1,1]$. Then the function

$$u(z) = g_F(z) - g_I(z), \qquad u(\infty) = \log \frac{b-a}{4 \operatorname{cap} F},$$

is a positive harmonic bounded function on $\widehat{\mathbb{C}} \setminus I$. The function $v(\zeta) = u\big(\frac{1}{2}(\zeta + 1/\zeta)\big)$ is harmonic in the unit disc and $\lim_{r\to 1^-} v(re^{it}) = v^*(e^{it}) = g_F(\cos t)$ for almost all $t \in [0, 2\pi)$. Therefore,

$$0 < u(\infty) = v(0) = \frac{1}{2\pi} \int_0^{2\pi} v^*(e^{it}) dt = \frac{1}{\pi} \int_0^{\pi} g_F(\cos t)\, dt.$$

Hence, the set $L = \{z \in (a,b) \setminus F : g_F(z) > 0\}$ has positive Lebesgue measure and so, as in the previous case, F is thin at every point of L. ■

Now we are going to give some estimate of the harmonic measure at thin points. In the context of pluripolar hulls it was proved in [2]. In [4] the authors observed that from the proof it follows that the estimate holds in some fine neighborhood of a thin point. So, let $A \subset \mathbb{C}$ be a set thin at a point $p \in \mathbb{C}$. It is well known that there exists a sequence $\varrho_k \searrow 0$ such that $A \cap \partial\mathbb{D}(p, \varrho_k) = \emptyset$ (see [9, Theorem 5.4.2]).

LEMMA 3. *Let $A \subset \mathbb{C}$ be a set thin at a point $p \in \mathbb{C}$. Then for any $\varepsilon > 0$ there exist $\varrho \in (0,\varepsilon)$, $r \in (0,\varrho)$, and an open set $U_1 \supset (A \cap \mathbb{D}(p,\varrho)) \setminus \{p\}$ such that*

- $A \cap \partial\mathbb{D}(p,\varrho) = \emptyset$, $\partial\mathbb{D}(p,r) \cap U_1 = \emptyset$;
- U_1 *is thin at* p;
- $\omega(z,J,\mathbb{D}(p,\varrho) \setminus F) \geq m(J)/\varrho - \varepsilon$ *for any* $z \in \overline{\mathbb{D}(p,r)} \setminus U_1$, *where* J *is any closed subarc of* $\partial\mathbb{D}(p,\varrho)$ *and* F *is any closed* (*in* $\mathbb{C}$) *subset of* A.

Proof. Note that if $p \notin \overline{A \setminus \{p\}}$ then the assertion is trivial (recall that $\omega(0,J,\mathbb{D}(p,\varrho)) = m(J)/\varrho$, take r sufficiently small and $U_1 = \emptyset$). So, we may assume that $p \in \overline{A \setminus \{p\}}$.

Let $\mathcal{U} \in \mathrm{SH}(\mathbb{C})$ be an entire subharmonic function such that

$$\lim_{z \to p,\, z \in A} \mathcal{U}(z) = -\infty$$

and $\mathcal{U}(p) > -\infty$. By adding a constant to $\mathcal{U}$ we may assume that $\mathcal{U} < 0$ on $\overline{D(p,\varrho)}$. Multiplying $\mathcal{U}$ by a positive constant we may assume that $\mathcal{U}(p) > -\varepsilon/2$. Taking $0 < \varrho < \varepsilon$ small enough, we may assume that $A \cap \partial\mathbb{D}(p,\varrho) = \emptyset$ and $\mathcal{U}(z) < -1$ on $(A \cap \overline{\mathbb{D}(p,\varrho)}) \setminus \{p\}$. Then for any arc $J \subset \partial\mathbb{D}(p,\varrho)$ and any closed $F \subset A$,

$$(1)\qquad \omega(z,J,\mathbb{D}(p,\varrho) \setminus F) \geq \omega(z,J,\mathbb{D}(p,\varrho)) + \mathcal{U}(z), \quad z \in \mathbb{D}(p,\varrho) \setminus F.$$

Put $U_1 := \{z \in \mathbb{D}(p,\varrho) : \mathcal{U}(z) < -\varepsilon/2\}$. Then U_1 is open, $U_1 \supset (A \cap \mathbb{D}(p,\varrho)) \setminus \{p\}$, and since $\mathcal{U}(p) > -\varepsilon/2$ the set U_1 is thin at p, with $p \in \overline{U_1 \setminus \{p\}}$. We know that $\omega(p,J,\mathbb{D}(p,\varrho)) = m(J)/\varrho$. Hence, we may choose $0 < r < \varrho$ with $U_1 \cap \partial\mathbb{D}(p,r) = \emptyset$ and $\omega(z,J,\mathbb{D}(p,\varrho)) > m(J)/\varrho - \varepsilon/2$ for $z \in \overline{\mathbb{D}(p,r)}$. Then by (1),

$$(2)\qquad \omega(z,J,\mathbb{D}(p,\varrho) \setminus F) \geq \frac{m(J)}{\varrho} - \varepsilon, \quad z \in \overline{\mathbb{D}(p,r)} \setminus U_1. \ \blacksquare$$

Now we study the case when $A \subset \partial\mathbb{D}$ is thin at $p \in \partial\mathbb{D}$. First note that for every ϱ with $0 < \varrho < 1$ there exists a closed subarc J of the arc $\mathbb{D} \cap \partial\mathbb{D}(p,\varrho)$ (respectively, $\mathbb{D}^{\mathrm{e}} \cap \partial\mathbb{D}(p,\varrho)$) with $m(J) \geq \varrho/3$. So, from Lemma 3 we get the following.

COROLLARY 4. *Let $A \subset \partial\mathbb{D}$ be a set thin at a point $p \in \partial\mathbb{D}$. Then for any $\varepsilon > 0$ there exist $\varrho \in (0,\varepsilon)$, $r \in (0,\varrho)$, and an open set $U_1 \supset A \cap \mathbb{D}(p,\varrho) \setminus \{p\}$ such that $A \cap \partial\mathbb{D}(p,\varrho) = \emptyset$, $\partial\mathbb{D}(p,r) \cap U_1 = \emptyset$, U_1 is thin at p, and*

$$(3)\qquad \omega(z,J,\mathbb{D}(p,\varrho) \setminus F) \geq 1/4, \quad z \in \overline{\mathbb{D}(p,r)} \setminus U_1,$$

where J is any closed subarc of the arc $\mathbb{D} \cap \partial\mathbb{D}(p,\varrho)$ (respectively, $\mathbb{D}^{\mathrm{e}} \cap \partial\mathbb{D}(p,\varrho)$) such that $m(J) \geq \varrho/3$ and F is any closed (in $\mathbb{C}$) subset of A.

3. Construction of a decreasing sequence of open sets

PROPOSITION 5. *Let I denote either the unit circle $\partial\mathbb{D}$ or an interval (a,b), where $-\infty < a < b < \infty$. Let $K \subset I$ be a non-empty compact set such that $m(K \cap U) > 0$ for any open set U such tha $U \cap K \neq \emptyset$. Then there is a decreasing sequence $(G_n)_{n=1}^{\infty}$ of open sets in I with the following properties*:

- $0 < m(U \cap G) < m(G)$ *for any open set $U \subset I$ such that $U \cap K \neq \emptyset$, where $G = \bigcap G_n$*;
- $\bigcap \bar{G}_n = K$;
- *there exists a Borel set $L \subset G$ of positive measure such that $I \setminus G$ is thin at every point of L.*

It follows from Lemma 2 that for the existence of L as in the proposition it is sufficient to construct G so that $\operatorname{cap}(I \setminus G) < \operatorname{cap}(I)$.

Denote by $\mathcal{C}_1$ the family of non-trivial connected components of K and by $\mathcal{C}_2$ the family of one-point connected components of K. Then $\mathcal{C}_1$ is at most countable (it may be empty). It will follow from the proof that in the case $\mathcal{C}_1 = \emptyset$ the equality $G = K$ holds.

LEMMA 6. *Let I be either an open arc in $\partial\mathbb{D}$ (we allow I to be equal to $\partial\mathbb{D}$) or an interval (a,b), where $-\infty < a < b < \infty$, and $\varepsilon > 0$. Then there is a decreasing sequence (G_n) of open sets in I, dense in I, such that*

$$0 < m(J \cap G) < m(J) \tag{4}$$

for any non-trivial open arc $J \subset I$, where $G := \bigcap_{n \geq 1} G_n$ and

$$\operatorname{cap}(I \setminus G) < \varepsilon. \tag{5}$$

Assume for a while that the lemma holds. Then we may complete the proof of the proposition.

Proof of Proposition 5. We define G_n to be the union of $G_n^{\operatorname{int} P}$ over all $P \in \mathcal{C}_1$ ($G_n^{\operatorname{int} P}$ is the set constructed as in the lemma with $I = \operatorname{int} P$) and $\{z \in I : \operatorname{dist}(z, \bigcup \mathcal{C}_2) < 1/n\}$. Note that if $\mathcal{C}_1$ is empty then $G = K$. For the last statement it suffices to use Lemma 2 (in the case $\mathcal{C}_1 \neq \emptyset$ we need to use, additionally, the previous lemma for one $P \in \mathcal{C}_1$). ■

Therefore, it is sufficient to show the above lemma.

Proof of Lemma 6. For an arbitrary non-trivial closed arc (interval) P we make the following construction of Cantor-type sets (we follow the notation from [9]). Below, we use the word arc (or subarc) for interval (subinterval). For a sequence $\mathbf{s} := (s_n)_{n=1}^{\infty}$ of numbers such that $0 < s_n < 1$ we define $C(s_1, P)$ to be the set obtained by removing from P the concentric (in the case $I = \partial\mathbb{D}$ the choice is arbitrary) arc whose length is an s_1 fraction of the length of the whole arc. At the nth stage let $C(s_1, \ldots, s_n, P)$ be the set obtained by removing from each connected component of $C(s_1, \ldots, s_{n-1}, P)$

the open concentric arc whose length is an s_n fraction of the whole component. We define

$$C(\mathbf{s},P) := \bigcap_{n=1}^{\infty} C(s_1,\dots,s_n,P).$$

Recall that (see e.g. [9]) the Lebesgue measure of $C(\mathbf{s},P)$ equals $m(P)t(\mathbf{s}) := m(P)\prod_{n=1}^{\infty}(1-s_n)$. Below we choose a sequence of sequences $\mathbf{s}$ satisfying additional assumptions. For $\theta \in (0,1)$ we define $\mathbf{s}(\theta) = (s_1(\theta), s_2(\theta), \dots)$ as follows:

$$s_j(\theta) := \begin{cases} 3^{-j} & \text{for } j \neq 2, \\ \theta & \text{for } j = 2. \end{cases}$$

Note that $t(\theta) := t(\mathbf{s}(\theta)) = C_0(1-\theta) > 0$ for some constant $C_0 \in (0,1)$.

Below we choose a sequence (θ_N) of numbers from $(0,1)$ such that $0 < \theta_1 < \theta_2 < \dots < 1$, $\lim_{N\to\infty}\theta_N = 1$ and $\prod_{N=1}^{\infty}(1-t^N) > 0$, where $t^N := t(\theta_N)$.

For a relatively open (in I) subset Ω and $\theta \in (0,1)$ we construct an open set $\Omega(\theta)$ as follows. Let us represent Ω as the union of at most countably many open, disjoint arcs I_j—these are all connected components of Ω. Then we define

$$\Omega(\theta) := \bigcup_j (I_j \setminus C(\mathbf{s}(\theta), \bar{I}_j)) = \bigcup_j (\bar{I}_j \setminus C(\mathbf{s}(\theta), \bar{I}_j)). \tag{6}$$

Note that $m(\Omega(\theta)) = m(\Omega)(1-t(\theta))$ and $\Omega(\theta) \nearrow_{\theta\to 1^-} \Omega \setminus A$, where A is at most countable.

Now we come back to the proof of the lemma. We make an inductive construction based on a proper choice of a sequence (θ_N) with the above mentioned properties. We define, simultaneously, two sequences of sets.

We start with $G_0 := I$, $K_0 := \emptyset$. Then we define $K_1 := C(\mathbf{s}(\theta_1), \bar{I})$, $G_1 = I(\theta_1) = I \setminus C(\mathbf{s}(\theta_1), \bar{I})$, where $\theta_1 \in (0,1)$ is chosen so that $\operatorname{cap}(I \setminus G_1) < \varepsilon/2$. Then we define inductively sets G_{N+1}, K_{N+1} and a number θ_N as follows. If G_N, K_N and the numbers $0 < \theta_1 < \dots < \theta_N < 1$ are already constructed so that $\operatorname{cap}(I \setminus G_N) < \varepsilon(1/2 + \dots + 1/2^N)$, we define $G_{N+1} := G_N(\theta_{N+1})$, $K_{N+1} := K_{N+1}(\theta_{N+1}) := \bigcup_j C(\mathbf{s}(\theta_{N+1}), \bar{I}_j^N)$, where $(I_j^N)_j$ are the connected components of G_N, and $1 > \theta_{N+1} > \theta_N$ is chosen so that $\operatorname{cap}(I \setminus G_{N+1}) = \operatorname{cap}((I \setminus G_N) \cup K_{N+1}) < \varepsilon(1/2 + \dots + 1/2^{N+1})$; note that $K_{N+1}(\theta_{N+1}) \searrow B$ as $\theta_{N+1} \to 1^-$, where B is at most countable, so when $\theta_{N+1} \in (\theta_N, 1)$ is sufficiently close to 1, $\operatorname{cap}(K_{N+1}(\theta_{N+1}))$ may be chosen to be arbitrarily small. It is no problem to choose $(\theta_N)_N$ so that $\lim_{N\to\infty}\theta_N = 1$ and $\prod_{N=1}^{\infty}(1-t^N) > 0$.

The assumption $s_1(\theta) = 1/3$ implies that for any N the set $\bar{I} \setminus G_N$ is totally disconnected (and certainly compact). Moreover, the inequality

$m(I_j^N) \le 1/3^N$ holds for all possible j, N, where $G_N = \bigcup_j I_j^N$ and $(I_j^N)_j$ is the family of connected components of G_N.

Now we prove that for a non-trivial open arc $J \subset I$ the inequalities $0 < m(J \cap G) < m(J)$ hold.

Fix a non-trivial open arc $J \subset I$. Then there are N and j such that $\bar{I}_j^N \subset J$. Hence

$$m(J \cap G) \ge m(I_j^N \cap G) = m(I_j^N) \prod_{k=N+1}^{\infty} (1 - t^k) > 0.$$

On the other hand, the inequality $m(J \cap G) < m(J)$ is equivalent to the inequality $m(J \setminus G) > 0$. But

$$m(J\setminus G) \ge m(I_j^N \setminus G) \ge m(I_j^N \setminus G_{N+1}) = m(\bar{I}_j^N \setminus G_{N+1}) = m(I_j^N) t^{N+1} > 0. \blacksquare$$

4. Construction of holomorphic functions. In the whole section we fix a non-empty compact set $K \subset \partial\mathbb{D}$ and a decreasing sequence of open sets $(G_n)_{n\in\mathbb{N}} \subset \partial\mathbb{D}$ such that $\bigcap_{n=1}^{\infty} \overline{G}_n = K$. We are going to construct a holomorphic function f on $\mathbb{C} \setminus K$ with special properties.

Put $G = \bigcap G_n$. Let u_n (resp. u) denote the solution of the Dirichlet problem on $\mathbb{D}$ with the boundary data χ_{G_n} (resp. χ_G). Let v_n (resp. v) denote the conjugate to u_n (resp. u) with $v_n(0) = 0$ (resp. $v(0) = 0$). Put $\varphi_n = u_n + iv_n$ and $\varphi = u + iv$.

First recall the following result related to the solution of the Dirichlet problem on the unit disc.

THEOREM 7 (see [9, Theorem 1.2.4], [11, Theorem IV.1]). *Let* $\psi \in L^1(\partial\mathbb{D})$. *Put*

$$u_\psi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \psi(e^{i\theta})\, d\theta.$$

Then

(a) u_ψ *is a harmonic function on* $\mathbb{D}$, $\inf_{\partial\mathbb{D}} \psi \le u_\psi \le \sup_{\partial\mathbb{D}} \psi$;
(b) $u_\psi^*(e^{i\theta}) = \lim_{r\to 1} u_\psi(re^{i\theta})$ *exists for almost all* $\theta \in [0, 2\pi)$; *moreover*, $u_\psi^* = \psi$ *a.e. on* $\partial\mathbb{D}$;
(c) *if* ψ *is continuous at* $\zeta_0 \in \partial\mathbb{D}$, *then* u_ψ *extends continuously to* ζ_0.

From Theorem 7 we obtain

$$\varphi_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \chi_{G_n}(e^{i\theta})\, d\theta.$$

An analogous formula holds for φ. Note that $\operatorname{Re}\varphi_n \in C(\overline{\mathbb{D}} \setminus \partial G_n)$. Moreover, $\operatorname{Re}\varphi_n = 1$ on G_n and $\operatorname{Re}\varphi_n = 0$ on $\partial\mathbb{D} \setminus \overline{G}_n$. Here the boundary of G_n is taken with respect to the topology of $\partial\mathbb{D}$.

LEMMA 8.

(a) *φ_n extends continuously to $\partial\mathbb{D} \setminus \partial G_n$.*
(b) *The sequence $\{\varphi_n\}_{n=1}^{\infty}$ tends to φ locally uniformly on $\mathbb{D}$.*

Proof. (a) Let $I \subset G_n$ be an open arc. Without loss of generality we may assume that $I = \{e^{i\theta} : 0 < \theta < \theta_I\}$, where $\theta_I \in (0, 2\pi)$. Then

$$\varphi_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\chi_{G_n\setminus I}(e^{i\theta})d\theta + \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\chi_I(e^{i\theta})d\theta =: I_1 + I_2.$$

Fix $\theta_0 \in (0, \theta_I)$. Note that I_1 extends continuously to $e^{i\theta_0}$ and

$$I_2 = \frac{1}{\pi i}\operatorname{Log}\biggl(e^{-\theta_I/2}\,\frac{z-e^{i\theta}}{z-1}\biggr),$$

where $\operatorname{Log} : \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \to \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$ is such that $\operatorname{Log} i = \pi i/2$, so it also extends continuously to $e^{i\theta_0}$.

(b) Fix $r \in (0, 1)$. Then for any $z \in \mathbb{D}_r$ we have

$$(7) \qquad |\varphi_n(z) - \varphi(z)| = \biggl|\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\chi_{G_n\setminus G}(e^{i\theta})\,d\theta\biggr| \le \frac{2}{1-r}\,m(G_n \setminus G).$$

The last expression tends to 0 as $n \to \infty$. ■

Put $f_n = \exp(\varphi_n)$ and $f = \exp(\varphi)$. Note that f_n extends to a function from $\mathcal{O}(\mathbb{C} \setminus \bar{G}_n)$, denoted also by f_n. Even more, the extension is defined by the formula $f_n(z) = 1/\overline{f_n(1/\bar{z})}$, $z \in \mathbb{D}^{\mathrm{e}}$. For $z \in \mathbb{D}^{\mathrm{e}}$ we also put $f(z) = 1/\overline{f(1/\bar{z})}$. From Lemma 8 we see that the sequence $(f_n)_{n=1}^{\infty}$ is convergent to f locally uniformly on $\mathbb{D} \cup \mathbb{D}^{\mathrm{e}}$.

Note that $e^{-1} < |f_n| < e$ on $\mathbb{C} \setminus \bar{G}_n$ and $e^{-1} < |f| < e$ on $\mathbb{C} \setminus K$. Note also that $\lim_{r\to 1^-} |f(rz)| = e$ for almost all $z \in G$ (consequently, $\lim_{r\to 1^+} |f(rz)| = e^{-1}$ for almost all $z \in G$) and $\lim_{r\to 1^-} |f(rz)| = 1$ for almost all $z \in \partial\mathbb{D} \setminus G$ (hence $\lim_{r\to 1^+} |f(rz)| = 1$ for almost all $z \in \partial\mathbb{D} \setminus G$).

It is easy to see that if $0 < m(U \cap G) < m(U)$ for any non-empty open set U such that $U \cap K \neq \emptyset$ then $\mathbb{C} \setminus K$ is the domain of existence of f.

5. Proof of Theorem 1. By Lemma 2 to prove Theorem 1 it suffices to prove the following result. We shall prove it for $K \subset \partial\mathbb{D}$. The proof for $K \subset \mathbb{R}$ follows exactly the same lines, so we omit it.

THEOREM 9. *Let I denote either the unit circle or an interval $(a, b) \subset \mathbb{R}$, where $a < b$. Let $K \subset I$ be a compact subset of I for which there exists a decreasing sequence $\{G_n\}$ of open subsets of I with the following properties*:

- *$0 < m(G \cap U) < m(U)$ for every open set $U \subset I$ with $U \cap K \neq \emptyset$, where $G := \bigcap_{n=1}^{\infty} G_n$;*
- *$K = \bigcap_{n=1}^{\infty} \bar{G}_n$;*

- *there exists a set* $L \subset G$ *such that* $m(L) > 0$ *and* $I \setminus G$ *is thin at every point of* L (*which holds*, *for instance*, *when* $\operatorname{cap}(I \setminus G) < \operatorname{cap} I$—*see Lemma* 2).

Put

$$f(z) := f_K(z) := \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \chi_G(e^{it})\, dt,$$

respectively,

$$f(z) := f_K(z) := \exp \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\chi_G(t)}{t - z}\, dt,$$

for $z \in \mathbb{C} \setminus K$. *Then* f_K *is a bounded holomorphic function on* $\mathbb{C} \setminus K$ *whose maximal domain of existence is* $\mathbb{C} \setminus K$ *and* $1/e < |f_K| < e$ *on* $\mathbb{C} \setminus K$.

In the case $K \neq \partial\mathbb{D}$ *there exists a set* $K_0 \subset K$ *with* $m(K_0) > 0$ *such that the non-tangential limits of* $f|_{\mathbb{D}}$ (*respectively*, $f|_{H_-}$) *exist on* K_0 *and the pluripolar hull* $(\Gamma_f(\mathbb{C} \setminus K))^*_{\mathbb{C}^2}$ *contains the set*

$$\{(z, e^{2k} f(z)) : z \in \mathbb{C} \setminus K\} \cup \{(z, e^{2k} f^*(z)) : z \in K_0\}$$

for any $k \in \mathbb{Z}$, *where* $f^*(z)$ *denotes the non-tangential limit of* $f|_{\mathbb{D}}$ (*respectively*, $f|_{H_-}$) *at* $z \in K_0$.

In the case $K = \partial\mathbb{D}$ *we define* $K^1 := \partial\mathbb{D} \cap \overline{H}_-$, $K^2 := \partial\mathbb{D} \cap \overline{H}_+$ *and* $f := f_1 + f_2 := f_{K^1} + f_{K^2}$. *Then* f *is a bounded holomorphic function on* $\mathbb{C} \setminus \partial\mathbb{D}$ *whose maximal domain of existence is* $\mathbb{C} \setminus \partial\mathbb{D}$ *and there exist sets* $K_0^1 \subset K^1$, $K_0^2 \subset K^2$ *with* $m(K_0^j) > 0$ *such that the non-tangential limits of* $f_j|_{\mathbb{D}}$ *exist on* K_0^j *and the pluripolar hull* $(\Gamma_f(\mathbb{C} \setminus \partial\mathbb{D}))^*_{\mathbb{C}^2}$ *contains the sets*

$$\{(z, e^{2k} f_1(z) + e^{-2k-2} f_2(z)) : z \in \mathbb{D}\} \cup \{(z, e^{2k} f_1(z) + e^{-2k} f_2(z)) : z \in \mathbb{D}^{\mathrm{e}}\}$$

and

$$\{(z, e^{2k} f_1^*(z) + e^{-2k-2} f_2^*(z)) : z \in K_0^1 \cup K_0^2\}$$

for any $k \in \mathbb{Z}$.

Proof. As already mentioned, we restrict ourselves to the case $K \subset \partial\mathbb{D}$. Recall that $f = f_K$ is constructed in Section 4. Put

$$K_0 = \{a \in L : f|_{\mathbb{D}} \text{ has the angular limit value } f^*(a) \text{ at } a\}. \tag{8}$$

By the Fatou theorem the set K_0 has positive Lebesgue measure.

Now we are left with the proof of the properties of pluripolar hulls. Let us start with the case $K \neq \partial\mathbb{D}$. In particular, $\mathbb{C} \setminus K$ is connected. Let $p \in \mathrm{PSH}(\mathbb{C}^2)$ be such that

$$p(z, f(z)) = -\infty \quad \text{for any } z \in \mathbb{C} \setminus K. \tag{9}$$

We need to show that

$$p(z, e^{2k} f(z)) = -\infty \quad \text{for any } z \in (\mathbb{C} \setminus K) \cup K_0,\ k \in \mathbb{Z}, \tag{10}$$

where we put $f(a) := f^*(a)$ for $a \in K_0$.

To prove (10) it is sufficient to show that

$$p(z, F_k(z)) = -\infty \quad \text{for any } z \in \mathbb{D} \cup K_0 \cup \mathbb{D}^{\mathrm{e}},\ k \in \mathbb{Z}, \tag{11}$$

where F_k is given by

$$F_k(z) = \begin{cases} e^{2k-2} f(z), & z \in \mathbb{D}, \\ e^{2k-2} f^*(z), & z \in K_0, \\ e^{2k} f(z), & z \in \mathbb{D}^{\mathrm{e}}. \end{cases}$$

We shall argue by induction with respect to k.

First let us show (11) for $k = 0$. By the assumption $p(z, f(z)) = -\infty$ for $z \in \mathbb{D}^{\mathrm{e}}$. Let c be a fixed point of K_0. We now apply Corollary 4 to get

$$\omega(z, J, (\mathbb{D}(c, \varrho) \setminus \partial\mathbb{D}) \cup (G_n \cap \partial\mathbb{D})) \geq 1/4, \qquad |z - c| \leq r_1,\ z \notin U_1,$$

where J is some arc in $\mathbb{D}(c, \varrho) \cap \mathbb{D}^{\mathrm{e}}$ such that $m(J) \geq \varrho/3$.

The functions

$$F_{n,0}(z) = \begin{cases} e^{-2} f_n(z), & z \in \mathbb{D} \cup G_n, \\ f_n(z), & z \in \mathbb{D}^{\mathrm{e}}, \end{cases}$$

are holomorphic and uniformly bounded on $\mathbb{D} \cup G_n \cup \mathbb{D}^{\mathrm{e}}$, and $F_{n,0} \to F_0$ locally uniformly on $\mathbb{D} \cup \mathbb{D}^{\mathrm{e}}$. Consider two cases:

(α) $z_0 \in \mathbb{D}$ and $|z_0 - c| = r_1$;
(β) $z_0 = c$.

In case (α) we put

$$u_n(z) = p(z, F_{n,0}(z) + F_0(z_0) - F_{n,0}(z_0))$$

and

$$M = \sup_{n \geq 1} \sup\{u_n(z) : z \in (\overline{\mathbb{D}(c, \varrho)} \setminus \partial\mathbb{D}) \cup (\mathbb{D}(c, \varrho) \cap G_n)\}.$$

Let $M_n := \sup\{u_n(z) : z \in J\}$. Then $M_n \to -\infty$ and the two constants theorem gives

$$u_n(z) \leq M + (M_n - M)\omega(z, J, (\mathbb{D}(c, \varrho) \setminus \partial\mathbb{D}) \cup (\mathbb{D}(c, \varrho) \cap G_n))$$

for all $z \in \mathbb{D}(c, r_1) \setminus U_1$, $n \geq n_0$, where n_0 is so large that $M_n - M < 0$ for $n \geq n_0$. It follows that

$$p(z_0, F_0(z_0)) = u_n(z_0) \leq M + (M_n - M)\frac{1}{4} \to -\infty$$

as $n \to \infty$. Since z_0 was chosen arbitrarily,

$$p(z, F_0(z)) = p(z, e^{-2} f(z)) = -\infty$$

when $|z - c| = r_1$, $z \in \mathbb{D}$. Consequently, $p(z, F_0(z)) = -\infty$ for all $z \in \mathbb{D} \cup \mathbb{D}^{\mathrm{e}}$.

In case (β) let U_1 be the set from Corollary 4 thin at the point c, and let $\{z_k\}$ be a sequence in $\mathbb{D} \setminus U_1$ such that $z_k \to c$. Let $\{n_k\}$ be a strictly increasing sequence of positive numbers such that

$$|F_{n_k,0}(z_k) - F_0(z_k)| < 1/k, \quad k = 1, 2, \ldots.$$

Put

$$u_k(z) = p(z + c - z_k, F_{n_k,0}(z) + F_0(c) - F_{n_k,0}(z_k))$$

and

$$\begin{aligned} M &:= \sup_{k \geq 1} \sup\{u_k(z) : z \in (\overline{\mathbb{D}(c, \varrho)} \setminus \partial\mathbb{D}) \cup (\mathbb{D}(c, \varrho) \cap G_{n_k})\}, \\ M_k &:= \sup\{u_k(z) : z \in J\}. \end{aligned}$$

It is clear that $M_k \to -\infty$ and proceeding as in case (α) we get

$$p(c, F_0(c)) = u_k(z_k) \leq M + (M_k - M)\frac{1}{4}, \quad k \geq k_0,$$

which implies that $p(c, F_0(c)) = -\infty$. Since $c \in K_0$ was chosen arbitrarily, we have proved that (11) is true for $k = 0$, i.e.,

$$p(z, F_0(z)) = -\infty \quad \text{on } \mathbb{D} \cup K_0 \cup \mathbb{D}^{\mathrm{e}}.$$

Assume that

$$p(z, F_k(z)) = -\infty \quad \text{on } \mathbb{D} \cup K_0 \cup \mathbb{D}^{\mathrm{e}}$$

for all $k \in \mathbb{Z}$ with $|k| \leq m$, where $m \geq 0$. It remains to show that (11) is true for all $k \in \mathbb{Z}$ with $|k| = m + 1$.

First, let $k = m + 1$. By the induction assumption

$$p(z, e^{2m} f(z)) = -\infty \quad \text{on } \mathbb{D}^{\mathrm{e}}.$$

Since $e^{2m} f(z)$ is holomorphic on $\mathbb{C} \setminus K$ and $\mathbb{C} \setminus K$ is connected, we get

$$p(z, e^{2m} f(z)) = -\infty \quad \text{on } \mathbb{D}.$$

Hence, by Corollary 4 with J in $\mathbb{D}$ and proceeding analogously as in the case $k = 0$ we get

$$p(z, e^{2m+2} f(z)) = -\infty \quad \text{on } \mathbb{D}^{\mathrm{e}} \cup K_0.$$

Therefore,

$$p(z, F_{m+1}(z)) = -\infty \quad \text{on } \mathbb{D} \cup K_0 \cup \mathbb{D}^{\mathrm{e}}.$$

Now, set $k = -m - 1$. We know that

$$p(z, e^{-2m-2} f(z)) = -\infty \quad \text{on } \mathbb{D}^{\mathrm{e}}.$$

Hence, by the analyticity of f on $\mathbb{C} \setminus K$ and the fact that $\mathbb{C} \setminus K$ is connected, we get

$$p(z, e^{-2m-2} f(z)) = -\infty \quad \text{on } \mathbb{D}.$$

So, by Corollary 4 we get similarly

$$p(z, e^{-2m-4} f(z)) = -\infty \quad \text{on } \mathbb{D} \cup K_0 \cup \mathbb{D}^{\mathrm{e}}.$$

Consequently,

$$p(z, F_{-m-1}(z)) = -\infty \quad \text{on } \mathbb{D} \cup K_0 \cup \mathbb{D}^{\mathrm{e}}.$$

Let us now consider the case $K = \partial\mathbb{D}$. We shall follow the same idea as in the previous case. First recall that f_j is holomorphic on $\mathbb{C} \setminus K^j$. Let $p \in \mathrm{PSH}(\mathbb{C}^2)$ be such that $p(z, f_1(z)+f_2(z)) = -\infty$, $z \in \mathbb{D}^{\mathrm{e}}$. Then apply approximation of f_2 as in the previous case (with $K = K^2$). Then the sums of f_1 (holomorphic on $\mathbb{C} \setminus K^1$) with the elements of the approximating sequence of f_2 play the same role as the functions f_n previously. Therefore, we get the equality $p(z, f_1(z) + e^{-2}f_2(z)) = -\infty$, $z \in \mathbb{D} \cup K_0^2$. Then applying the same reasoning for the function $f_1 + e^{-2}f_2$ restricted this time to $\mathbb{D}$ (with $K = K^1$ and applying approximation of f_1) we get $p(z, e^2 f_1(z) + e^{-2}f_2(z)) = -\infty$, $z \in \mathbb{D}^{\mathrm{e}}$, and $p(z, f_1(z) + e^{-2}f_2(z)) = -\infty$, $z \in K_0^1$. Proceeding inductively we get

$$\begin{aligned} p(z, e^{2k}f_1(z) + e^{-2k}f_2(z)) &= -\infty, \quad z \in \mathbb{D}^{\mathrm{e}},\ k = 0, 1, \ldots, \\ p(z, e^{2k}f_1(z) + e^{-2k-2}f_2(z)) &= -\infty, \quad z \in K_0^1 \cup K_0^2 \cup \mathbb{D},\ k = 0, 1, \ldots. \end{aligned}$$

But proceeding exactly in the same way with f_1 replaced with f_2 (and vice versa) we get the desired equality for all $k \in \mathbb{Z}$. ∎

REMARK 10. Let f be as in the theorem for $K = \partial\mathbb{D}$. For $k \in \mathbb{Z}$ put

$$F_k(z) := \begin{cases} e^{2k}f_1(z) + e^{-2k-2}f_2(z), & z \in \mathbb{D}, \\ e^{2k}f_1^*(z) + e^{-2k-2}f_2^*(z), & z \in K_0^1 \cup K_0^2, \\ e^{2k}f_1^*(z) + e^{-2k}f_2(z), & z \in \mathbb{D}^{\mathrm{e}}. \end{cases}$$

Then $\Gamma_{F_k}(\mathbb{D} \cup K \cup \mathbb{D}^{\mathrm{e}}) \subset (\Gamma_f(\mathbb{C} \setminus \partial\mathbb{D}))^*_{\mathbb{C}^2}$ and the functions $F_k|_{\mathbb{D}}$ and $F_k|_{\mathbb{D}^{\mathrm{e}}}$ are pseudocontinuations of each other across $K_0^1 \cup K_0^2$, i.e. $(F_k|_{\mathbb{D}})^*(a) = (F_k|_{\mathbb{D}^{\mathrm{e}}})^*(a)$, $a \in K_0^1 \cup K_0^2$.

6. Examples. It is natural to ask whether the assumption in Theorem 1 on the capacity of K is essential, more precisely, whether there are compact sets $K \subset \partial\mathbb{D}$ such that $m(K) > 0$ and $\mathrm{cap}(\partial\mathbb{D} \setminus K) = 1$. Below we shall produce such a set. First we prove

LEMMA 11. *There exists an F_σ-set $F \subset \partial\mathbb{D}$ such that $m(F) = 0$ and $\mathrm{cap}(F) = 1$.*

Proof. First recall that for any Borel subset $I \subset \partial\mathbb{D}$ and all $n \in \mathbb{N}$, if we set $\widetilde{I} := \{\lambda \in \partial\mathbb{D} : \lambda^n \in I\}$, then

$$m(\widetilde{I}) = m(I), \qquad \mathrm{cap}(\widetilde{I}) = \sqrt[n]{\mathrm{cap}\, I}.$$

There exists a compact set $K \subset \partial\mathbb{D}$ such that $m(K) = 0$ and $\mathrm{cap}(K) > 0$ (take for example a Cantor type set on the unit circle). Put $I_k = \{z \in \partial\mathbb{D} : z^k \in K\}$. It is sufficient to define $F = \bigcup_{k=1}^{\infty} I_k$. ∎

Example 12. *Let $0 < \delta < 1$. Then there is a compact set $K \subset \partial\mathbb{D}$ such that $m(\partial\mathbb{D} \setminus K) < \delta$ (equivalently, $m(K) > 1 - \delta$) and* $\operatorname{cap}(\partial\mathbb{D} \setminus K) = 1$.

Proof. Take an F_σ-set $F \subset \partial\mathbb{D}$ such that $m(F) = 0$ and $\operatorname{cap}(F) = 1$. For any $\delta > 0$ there exists an open set $U_\delta \subset \partial\mathbb{D}$ such that $F \subset U_\delta$ and $m(U_\delta) < \delta$. Put $K_\delta = \partial\mathbb{D} \setminus U_\delta$. ■

It seems to us that the most interesting examples of compact sets satisfying the conditions of Theorem 1 are ones which are totally disconnected. Below we show the existence of such subsets of $\partial\mathbb{D}$.

Example 13. There is a totally disconnected compact set $K \subset \partial\mathbb{D}$ as desired in Theorem 1, i.e. such that $\operatorname{cap}(\partial\mathbb{D} \setminus K) < 1$ and $m(K \cap U) > 0$ for any $z \in K$ and any neighborhood U of z.

The construction of K is inductive and standard. For example we may define $K := \partial\mathbb{D} \setminus \bigcup_{n=1}^{\infty} L_n$, where L_n is a union of disjoint open arcs with centers at $e^{ij\pi/2^n}$, where $j = -(2^n - 1), -(2^n - 3), \ldots, 2^n - 1$ with m-measure of each arc smaller than $1/2^{2n}$ and with $\operatorname{cap}(L_1 \cup \cdots \cup L_n) < 1/2$.

Then $\operatorname{cap}(\partial\mathbb{D} \setminus K) \le 1/2$ and the choice may be done so that $m(U \cap K) > 0$ for any $z \in K$ and for any neighborhood U of z.

In fact, we may even specify effectively how large the open arcs deleted in the above process may be. These lengths may be deduced from the following example.

Example 14. Let $\{I_j\}_{j=1}^{\infty}$ be a sequence of disjoint open arcs in $\partial\mathbb{D}$ such that

$$\sum_{j=1}^{\infty} \frac{1}{\log(2/m(I_j))} < \infty$$

(for instance $m(I_j) = \exp(-j^2)$). Then there is a j_0 such that $U := \bigcup_{j=j_0}^{\infty} I_j$ has the properties

$$\operatorname{cap}(U) < 1, \quad m(\partial\mathbb{D} \setminus U) > 0.$$

In fact, the result follows immediately from the subadditivity property of the logarithmic capacity and the fact that $\operatorname{cap}(I_j) = \sin(m(I_j)\pi/4)$.

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