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Somewhere dense Cesàro orbits and rotations of Cesàro hypercyclic operators

by

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Abstract. Let T be a continuous linear operator acting on a Banach space X. We examine whether certain fundamental results for hypercyclic operators are still valid in the Cesàro hypercyclicity setting. In particular, in connection with the somewhere dense orbit theorem of Bourdon and Feldman, we show that if for some vector $x \in X$ the set $\{Tx, \frac{T^2}{2}x, \frac{T^3}{3}x, \ldots\}$ is somewhere dense then for every $0 < \varepsilon < 1$ the set $(0,\varepsilon)\{Tx, \frac{T^2}{2}x, \frac{T^3}{3}x, \ldots\}$ is dense in X. Inspired by a result of Feldman, we also prove that if the sequence $\{n^{-1}T^n x\}$ is d-dense then the operator T is Cesàro hypercyclic. Finally, following the work of León-Saavedra and Müller, we consider rotations of Cesàro hypercyclic operators and we establish that in certain cases, for any λ with $|\lambda| = 1$, T and λT share the same sets of Cesàro hypercyclic vectors.

1. Introduction. Let X be an infinite-dimensional topological vector space over the field $\mathbb{C}$ or $\mathbb{R}$. A continuous linear operator T acting on X is said to be *hypercyclic* if there exists a vector $x \in X$ whose orbit under T, $\mathrm{Orb}(T,x) = \{x, Tx, T^2x, \ldots\}$, is dense in X. Such a vector x is called *hypercyclic* for T, and $\mathrm{HC}(T)$ denotes the set of hypercyclic vectors for T. In the literature one may find many examples of hypercyclic operators. For an introduction to this subject and an informative account of results we refer to the very nice review article [19]; see also the more recent review articles [8], [26], [32], [20].

In the present paper we mainly examine whether certain properties of hypercyclic operators are still valid in the context of Cesàro hypercyclicity. The notion of Cesàro hypercyclicity was introduced by León-Saavedra in [22]. An operator $T : X \to X$ is called *Cesàro hypercyclic* if there exists a vector $x \in X$ so that the sequence

$$\frac{I + T + \cdots + T^{n-1}}{n}\, x, \qquad n = 1, 2, \ldots,$$

is dense in X. In [22] León-Saavedra showed that T is Cesàro hypercyclic if

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and only if there exists $x \in X$ so that the weighted orbit

$$\left\{Tx, \frac{T^2}{2}x, \frac{T^3}{3}x, \dots\right\}$$

is dense in X. He also established that a hypercyclic operator need not be Cesàro hypercyclic and vice versa.

In this paper when we refer to the *Cesàro orbit* of a vector x under an operator T we mean the sequence $\{n^{-1}T^n x : n = 1, 2, \dots\}$. Also a weaker notion of hypercyclicity which will be of use to us is that of supercyclicity. An operator T acting on X over $\mathbb{C}$ (or $\mathbb{R}$) is said to be *supercyclic* if there exists a vector $x \in X$ so that the set $\mathbb{C}\,\mathrm{Orb}(T, x)$ (respectively $\mathbb{R}\,\mathrm{Orb}(T, x)$) is dense in X. We shall also be using the following notation: if $A \subset \mathbb{C}$ and $Y \subset X$ the set AY is defined by $AY = \{ay : a \in A, y \in Y\}$.

The first problem which concerns us is the somewhere dense orbit theorem due to Bourdon and Feldman. In [9] Bourdon and Feldman answered a question of Peris [29] by proving the following deep theorem.

Theorem (Bourdon–Feldman). *Let X be a locally convex topological vector space over $\mathbb{C}$ (or $\mathbb{R}$) and let T be a continuous linear operator on X.*

(i) *If for some $x \in X$ the set* $\mathrm{Orb}(T, x)$ *is somewhere dense then it is everywhere dense.*
(ii) *If for some $x \in X$ the set* $\mathbb{C}\,\mathrm{Orb}(T, x)$ (*or* $\mathbb{R}\,\mathrm{Orb}(T, x)$) *is somewhere dense then it is everywhere dense.*

In fact Wengenroth [33] has shown that this theorem holds in arbitrary linear topological spaces without the assumption of local convexity. A natural question to ask is whether the above theorem holds for the Cesàro orbit $\{n^{-1}T^n x\}$. Although we are not able to prove a complete analogue of the Bourdon–Feldman theorem in the Cesàro hypercyclicity setting, in Section 2 (Theorem 2.9) we establish that if the set $\{n^{-1}T^n x\}$ is somewhere dense then for every $0 < \varepsilon < 1$ the set $(0, \varepsilon)\{n^{-1}T^n x\}$ is everywhere dense. Furthermore under the extra assumption that T is hypercyclic we prove that $\{n^{-1}T^n x\}$ is dense in X (Theorem 2.5).

In Section 3 we focus on the problem of d-dense Cesàro orbits. Our results are inspired by the recent work of Feldman [15] in which he proved the following.

Theorem (Feldman). *Let X be a Banach space and let T be a bounded linear operator acting on X. Suppose there exists $x \in X$ so that* $\mathrm{Orb}(T, x)$ *is d-dense for some $d > 0$, that is, for every $y \in X$ there exists $n \in \mathbb{N}$ so that $\|T^n x - y\| < d$. Then T is hypercyclic, but in general x need not be a hypercyclic vector.*

By a careful adaptation of Feldman's method and slightly modifying his arguments we prove the corresponding result for Cesàro orbits. In addition, we provide some results, with simple proofs, concerning perturbations of certain dense sets.

In Section 4 we present some generalizations related to the Bourdon–Feldman theorem. Actually, we show that Theorem 2.9 still holds if the sequence of weights $\{n^{-1}\}$ is replaced by more general sequences.

The last problem we deal with comes from an elegant result of León-Saavedra and Müller and is directly connected to rotations of hypercyclic operators. Bès asked whether T hypercyclic implies that λT is also hypercyclic for every λ with $|\lambda| = 1$ (see also [31] for a related question). León-Saavedra and Müller [24], using semigroups of operators, gave an affirmative answer by proving the following stronger result.

THEOREM (León-Saavedra–Müller). *Let X be a complex Banach space and let T be a bounded linear operator on X. If T is hypercyclic then λT is hypercyclic for every λ with $|\lambda| = 1$ and in addition* $\mathrm{HC}(T) = \mathrm{HC}(\lambda T)$.

In Section 5 (Theorem 5.1) an analogue of the above theorem is established for Cesàro hypercyclic operators, under the assumption that they are also hypercyclic. The proof of Theorem 5.1 is based on three ingredients: the T-invariance of certain limit sets, Baire's category theorem and Cantor's theorem.

Finally, Section 6 contains some remarks and open problems.

2. Somewhere dense Cesàro orbits. In this section T will be a bounded linear operator acting on a (separable) complex Banach space X. We introduce the following notation. For $x \in X$ define the sets $\mathrm{Orb}_{\mathrm{C}}(T, x)$ and $\omega_{\mathrm{C}}(x)$ by

$$\mathrm{Orb}_{\mathrm{C}}(T, x) = \{n^{-1}T^n x : n = 1, 2, \ldots\},$$
$$\omega_{\mathrm{C}}(x) = \{y \in X : \exists n_k \to \infty,\ n_k^{-1}T^{n_k}x \to y\}.$$

The sets $\mathrm{Orb}_{\mathrm{C}}(T, x)$, $\omega_{\mathrm{C}}(x)$ will be called the *Cesàro T-orbit* of x and the *ω_{C}-limit set* of x respectively. Observe that the latter notion resembles that of ω-limit set from dynamical systems.

Let us start with a simple lemma that can be found in [12]; for completeness we include its proof.

LEMMA 2.1. *Let $x \in X$ be a non-zero vector. The following are equivalent.*

(i) *The Cesàro T-orbit of x, $\mathrm{Orb}_{\mathrm{C}}(T, x)$, is dense in X.*
(ii) *The ω_{C}-limit set of x, $\omega_{\mathrm{C}}(x)$, is the whole space X.*

Proof. Since $\overline{\mathrm{Orb}_{\mathrm{C}}(T, x)} = \omega_{\mathrm{C}}(x) \cup \mathrm{Orb}_{\mathrm{C}}(T, x)$ it follows that (ii) implies (i).

To show the converse we first observe that (i) gives that for every $j = 1, 2, \ldots$,

$$j^{-1}T^j x \in \overline{\{n^{-1}T^n x : n > m\}} = X, \qquad \forall m = 1, 2, \ldots,$$

which in turn implies that for every $j = 1, 2, \ldots$ there is a sequence $\{n_k\}$ of natural numbers such that $n_k \to \infty$ and $n_k^{-1}T^{n_k}x \to j^{-1}T^j x$. Therefore,

$$\mathrm{Orb}_{\mathrm{C}}(T, x) \subset \omega_{\mathrm{C}}(x).$$

Since the set $\omega_{\mathrm{C}}(x)$ is closed, the proof is finished. ∎

We shall also need the following two lemmata. The first one gives a characterization of Cesàro hypercyclic operators and is due to León-Saavedra [22]. The second is a simple topological lemma which enables us to work with invariant sets under T. This turns out to be crucial to our approach.

LEMMA 2.2. *An operator $T : X \to X$ is Cesàro hypercyclic if and only if there exists a non-zero vector $x \in X$ such that $\mathrm{Orb}_{\mathrm{C}}(T, x)$ is dense in X.*

LEMMA 2.3. *Let x be a non-zero vector. Then $\mathrm{Orb}_{\mathrm{C}}(T, x)$ is somewhere dense if and only if $\omega_{\mathrm{C}}(x)$ is somewhere dense.*

Proof. It suffices to show that if for some $y \in X$ and $\varepsilon > 0$ the ball $B(y, \varepsilon) = \{z \in X : \|z - y\| < \varepsilon\}$ is contained in the closure of $\mathrm{Orb}_{\mathrm{C}}(T, x)$ then $B(y, \varepsilon) \subset \omega_{\mathrm{C}}(x)$.

Let $z \in B(y, \varepsilon)$. Then there exists a sequence of positive integers $\{n_k\}$ such that $\|z - n_k^{-1}T^{n_k}x\| < 1/k$ for $k = 1, 2, \ldots$. If $z \notin \mathrm{Orb}_{\mathrm{C}}(T, x)$ then obviously $n_k \to \infty$. In case $z \in \mathrm{Orb}_{\mathrm{C}}(T, x)$, we may approximate z by a sequence $\{w_k\}$ of vectors not belonging to $\mathrm{Orb}_{\mathrm{C}}(T, x)$. Hence, we are in the previous situation where each w_k can be approximated by some $n_k^{-1}T^{n_k}x$ as close as we want with $n_k \to \infty$. The other implication is obvious. ∎

LEMMA 2.4. *The set $\omega_{\mathrm{C}}(x)$ is invariant under T.*

Proof. Take $y \in \omega_{\mathrm{C}}(x)$. Then there exist $n_k \to \infty$ such that $n_k^{-1}T^{n_k}x \to y$. Applying T we get $n_k^{-1}T^{n_k+1}x \to Ty$. Then

$$\begin{aligned}\left\|\frac{T^{n_k+1}}{n_k+1}x - Ty\right\| &\le \left\|\frac{T^{n_k+1}}{n_k+1}x - \frac{T^{n_k+1}}{n_k}x\right\| + \left\|\frac{T^{n_k+1}}{n_k}x - Ty\right\| \\ &\le \left|\frac{n_k}{n_k+1} - 1\right| \left\|\frac{T^{n_k+1}}{n_k}x\right\| + \left\|\frac{T^{n_k+1}}{n_k}x - Ty\right\|.\end{aligned}$$

Since the sequence $\|n_k^{-1}T^{n_k+1}x\|$ is bounded, the above inequality implies that

$$\frac{T^{n_k+1}}{n_k+1}x \to Ty$$

as $k \to \infty$. Thus $Ty \in \omega_{\mathrm{C}}(x)$ and $\omega_{\mathrm{C}}(x)$ is invariant under T. ∎

The next theorem is a slight generalization of Theorem 1.2 in [12].

THEOREM 2.5. *T is hypercyclic and Cesàro hypercyclic if and only if there exist $x, y \in X$ so that both $\mathrm{Orb}(T, x)$ and $\mathrm{Orb}_{\mathrm{C}}(T, y)$ are somewhere dense.*

Proof. Assume first that T is hypercyclic and Cesàro hypercyclic. According to the definition of hypercyclicity there is some vector x whose orbit $\mathrm{Orb}(T, x)$ is dense. Since T is Cesàro hypercyclic, by (León-Saavedra's characterization) Lemma 2.2 it follows that there is some vector y such that $\mathrm{Orb}_{\mathrm{C}}(T, y)$ is dense. Thus for the one direction we are done.

Assume that for some vectors x, y both $\mathrm{Orb}(T, x)$ and $\mathrm{Orb}_{\mathrm{C}}(T, y)$ are somewhere dense. Using Lemma 2.3, we conclude that $\omega_{\mathrm{C}}(y)$ has non-empty interior. By the Bourdon–Feldman theorem we immediately see that the closure of $\mathrm{Orb}(T, x)$ is the whole space X, hence T is hypercyclic. Let us denote by U the interior of $\omega_{\mathrm{C}}(y)$, which is non-empty as we showed above. Since T is hypercyclic, there is a hypercyclic vector z such that $z \in U$ (the set of hypercyclic vectors is G_δ and dense in X). Using the invariance of $\omega_{\mathrm{C}}(y)$ under T (Lemma 2.4), we get $\mathrm{Orb}(T, z) \subset \omega_{\mathrm{C}}(y)$ and since z is hypercyclic for T, it follows that $\omega_{\mathrm{C}}(y) = X$. In view of Lemma 2.1 the Cesàro T-orbit of y is dense in X and finally Lemma 2.2 implies that T is Cesàro hypercyclic. ∎

REMARK 2.6. Of course the previous theorem still holds if X is a locally convex topological vector space.

REMARK 2.7. The proof of Theorem 2.5 implies that if T is hypercyclic and $\mathrm{Orb}_{\mathrm{C}}(T, x)$ is somewhere dense then it is everywhere dense.

In order to give an application of Theorem 2.5 let us introduce some terminology and standard notation. Let $H(\mathbb{C})$ be the Fréchet space of entire functions endowed with the topology of uniform convergence on compact subsets of $\mathbb{C}$. Birkhoff [7] showed that the translation operator $T_1 : H(\mathbb{C}) \to H(\mathbb{C})$, $T_1(f)(z) = f(z+1)$ for $f \in H(\mathbb{C})$, is hypercyclic, and MacLane [25] showed the same for the differentiation operator $D : H(\mathbb{C}) \to H(\mathbb{C})$, $Df = f'$ for $f \in H(\mathbb{C})$. We establish the following.

COROLLARY 2.8.

(i) *If for some $f \in H(\mathbb{C})$ the set $\mathrm{Orb}_{\mathrm{C}}(D, f)$ is somewhere dense then it is dense in $H(\mathbb{C})$.*
(ii) *If for some $f \in H(\mathbb{C})$ the set $\mathrm{Orb}_{\mathrm{C}}(T_1, f)$ is somewhere dense then it is dense in $H(\mathbb{C})$.*
(iii) *Let T be a unilateral weighted backward shift, with weight sequence $\{w_n\}$, acting on the Hilbert space $l^2(\mathbb{N})$ of square summable sequences. Suppose that there exists an increasing sequence $\{n_k\}$ of*

positive integers such that

$$\lim_{k\to\infty} \frac{\prod_{i=1}^{n_k} w_{i+q}}{n_k} = \infty$$

for every non-negative integer q. If for some $x \in l^2(\mathbb{N})$ the set $\mathrm{Orb}_{\mathrm{C}}(T,x)$ is somewhere dense then it is dense in $l^2(\mathbb{N})$.

Proof. It is well known that D and T_1 are hypercyclic operators. By applying Remark 2.6, (i) and (ii) follow. Observe that it is also easy to see that both D and T_1 are Cesàro hypercyclic. In order to prove (iii) we use some results due to León-Saavedra. Since the weights $\{w_n\}$ satisfy the condition in (iii), T is Cesàro hypercyclic (see Proposition 3.1 in [22]). Since every Cesàro hypercyclic unilateral weighted shift is hypercyclic (see Corollary 3.3 in [22]) using Remark 2.7 we get the desired result. ∎

It is plausible that in Remark 2.7 the assumption that T is hypercyclic can be dropped. Unfortunately we do not have a proof for this. However, as already mentioned, we can prove the following.

THEOREM 2.9. *Suppose there is $x \in X$ such that $\mathrm{Orb}_{\mathrm{C}}(T,x)$ is somewhere dense. Then for every $0 < \varepsilon < 1$ the set $(0,\varepsilon)\,\mathrm{Orb}_{\mathrm{C}}(T,x)$ is dense in X. In particular the last assertion implies that $\overline{(0,\varepsilon)\,\mathrm{Orb}(T,x)} = X$.*

Proof. Since $\mathrm{Orb}_{\mathrm{C}}(T,x)$ is somewhere dense, Lemma 2.3 implies that $\omega_{\mathrm{C}}(x)$ has non-empty interior. At this point we introduce the following notation. For every $x \in X$ the symbol U_x stands for the interior of $\omega_{\mathrm{C}}(x)$, i.e.

$$U_x = \omega_{\mathrm{C}}(x)^\circ.$$

LEMMA 2.10. *If for some $x, y \in X \setminus \{0\}$, $U_x \cap U_y \neq \emptyset$ then*

$$\overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)} = \overline{(0,\varepsilon)\omega_{\mathrm{C}}(y)} \quad \textit{for every } 0 < \varepsilon < 1.$$

Proof. We know that $\omega_{\mathrm{C}}(x)$ is invariant under T (Lemma 2.4). Fix any $0 < \varepsilon < 1$. It is not difficult to see that there exists $k \in \{1, 2, \ldots\}$ such that

$$\frac{T^k}{k}\, x \in U_x \cap U_y.$$

Applying the T-invariance of $\omega_{\mathrm{C}}(x)$, $\omega_{\mathrm{C}}(y)$ we get

$$\frac{T^{n+k}}{n+k}\, x \in \frac{k}{n+k}\,\omega_{\mathrm{C}}(x) \cap \frac{k}{n+k}\,\omega_{\mathrm{C}}(y), \qquad \forall n = 1, 2, \ldots.$$

Choose $n_0 \in \mathbb{N}$ such that

$$\frac{k}{n+k} < \varepsilon, \qquad \forall n \geq n_0.$$

Hence,

$$\frac{T^{n+k}}{n+k}\, x \in (0,\varepsilon)\omega_{\mathrm{C}}(x) \cap (0,\varepsilon)\omega_{\mathrm{C}}(y), \qquad \forall n \geq n_0.$$

Let now $z \in \omega_{\mathrm{C}}(x)$. Then there exists a sequence $\{n_m\}$ of natural numbers such that $n_m \to \infty$ as $m \to \infty$, $n_m \geq n_0 + k$ and

$$\frac{T^{n_m}}{n_m} x \to z \qquad \text{as } m \to \infty.$$

Observe that

$$\frac{T^{n_m}}{n_m} x \in (0,\varepsilon)\omega_{\mathrm{C}}(x) \cap (0,\varepsilon)\omega_{\mathrm{C}}(y), \qquad \forall m \in \mathbb{N}.$$

Therefore,

$$z \in \overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)} \cap \overline{(0,\varepsilon)\omega_{\mathrm{C}}(y)},$$

and thus,

$$\omega_{\mathrm{C}}(x) \subset \overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)} \cap \overline{(0,\varepsilon)\omega_{\mathrm{C}}(y)}.$$

From the last inclusion it follows that

$$\overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)} \subset \overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)} \cap \overline{(0,\varepsilon)\omega_{\mathrm{C}}(y)} \subset \overline{(0,\varepsilon)\omega_{\mathrm{C}}(y)}.$$

By a similar argument we get

$$\overline{(0,\varepsilon)\omega_{\mathrm{C}}(y)} \subset \overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)}.$$

Hence $\overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)} = \overline{(0,\varepsilon)\omega_{\mathrm{C}}(y)}$. ∎

LEMMA 2.11. *If* $U_x \neq \emptyset$, *then* $\overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)} = \overline{(0,\varepsilon)\omega_{\mathrm{C}}(\lambda x)}$ *for every* $\lambda \in \mathbb{C} \setminus \{0\}$ *and every* $0 < \varepsilon < 1$.

Proof. Since $U_x \neq \emptyset$, there are $y \in X \setminus \{0\}$ and $\delta > 0$ such that $B(y,\delta) \subset U_x$. Without loss of generality we may assume that $\|y\| > \delta$. First suppose that $\lambda > 1$. Then for $\mu > 1$, $B(\mu y, \mu\delta) \subset \mu U_x$. It is easy to check that $\mu\omega_{\mathrm{C}}(x) \subset \omega_{\mathrm{C}}(\mu x)$. Hence $\mu U_x \subset U_{\mu x}$ and therefore $B(\mu y, \mu\delta) \subset U_{\mu x}$. For $\mu > 1$ and $k = 1, 2, \ldots$ the balls $B(\mu^{k-1} y, \mu^{k-1}\delta)$ and $B(\mu^k y, \mu^k \delta)$ intersect if and only if $1 < \mu < (\delta + \|y\|)/(\|y\| - \delta)$. Therefore, if we can find $1 < \mu < (\delta + \|y\|)/(\|y\| - \delta)$ and $n \in \mathbb{N}$ such that $\mu^n = \lambda$ then for every $k = 1, \ldots, n$ we have

$$B(\mu^{k-1} y, \mu^{k-1}\delta) \cap B(\mu^k y, \mu^k \delta) \neq \emptyset.$$

The last relation implies that

$$U_{\mu^{k-1} x} \cap U_{\mu^k x} \neq \emptyset$$

and applying the previous lemma we conclude that for every $0 < \varepsilon < 1$,

$$\overline{(0,\varepsilon)\omega_{\mathrm{C}}(\mu^{k-1} x)} = \overline{(0,\varepsilon)\omega_{\mathrm{C}}(\mu^k x)}, \qquad \forall k = 1, \ldots, n.$$

Hence,

$$\overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)} = \overline{(0,\varepsilon)\omega_{\mathrm{C}}(\lambda x)}.$$

It remains to show the existence of $1 < \mu < (\delta + \|y\|)/(\|y\| - \delta)$ and $n \in \mathbb{N}$ such that $\mu^n = \lambda$. But this is possible, since $\lambda^{1/n} \to 1$ as $n \to \infty$ and

$\lambda^{1/n} > 1$. The case $0 < \lambda < 1$ can be handled in a similar way and it is left to the reader. This completes the proof of the lemma for λ positive.

To finish the proof consider $\lambda = e^{i\theta}$ for $\theta > 0$. If $\phi > 0$ then for every $k = 1, 2, \ldots$ the balls $B(e^{i(k-1)\phi}y, \delta)$ and $B(e^{ik\phi}y, \delta)$ intersect if and only if $|1 - e^{i\phi}| < 2\delta/\|y\|$. Working as before (for λ positive), we will be done if we can find $\phi > 0$ and $n \in \mathbb{N}$ such that $n\phi = \theta$ and $|1 - e^{i\phi}| < 2\delta/\|y\|$. But it is obvious that such ϕ, n can be found. Therefore $\overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)} = \overline{(0,\varepsilon)\omega_{\mathrm{C}}(e^{i\theta}x)}$. ∎

Let us now continue with the proof of Theorem 2.9. Since $\omega_{\mathrm{C}}(x)$ has non-empty interior, it follows that $\overline{\mathbb{C}\operatorname{Orb}(T,x)}$ has non-empty interior. Using the Bourdon–Feldman theorem we conclude that $\mathbb{C}\operatorname{Orb}(T,x)$ is dense in X, hence T is supercyclic. It is well known that the set of supercyclic vectors is residual. Therefore, there is a supercyclic vector $z \in X$ such that $z \in U_x$ (since U_x is open and non-empty). Since $\omega_{\mathrm{C}}(x)$ is T-invariant, $\operatorname{Orb}(T,z) \subset \omega_{\mathrm{C}}(x)$. Thus we get $(0,\varepsilon)\operatorname{Orb}(T,z) \subset \overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)}$. The last relation and Lemma 2.11 yield

$$\bigcup_{\lambda\in\mathbb{C}\setminus\{0\}} \lambda(0,\varepsilon)\operatorname{Orb}(T,z) \subset \bigcup_{\lambda\in\mathbb{C}\setminus\{0\}} \overline{(0,\varepsilon)\lambda\omega_{\mathrm{C}}(x)} \subset \bigcup_{\lambda\in\mathbb{C}\setminus\{0\}} \overline{(0,\varepsilon)\omega_{\mathrm{C}}(\lambda x)} = \overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)}.$$

Since it can be easily checked that $0 \in \overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)}$, we arrive at

$$\overline{\mathbb{C}\operatorname{Orb}(T,z)} \subset \overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)}.$$

But z is a supercyclic vector and this implies $\overline{(0,\varepsilon)\omega_{\mathrm{C}}(x)} = X$. Now it is straightforward that $\overline{(0,\varepsilon)\operatorname{Orb}_{\mathrm{C}}(T,x)} = X$. The proof of Theorem 2.9 is complete. ∎

Remark 2.12. At this point we would like to point out how the previous argument is connected to the Bourdon–Feldman theorem. To be precise, we shall prove that statement (ii) of the Bourdon–Feldman theorem implies (i) of that theorem. Indeed, suppose $\operatorname{Orb}(T,x)$ is somewhere dense. Using (ii) it follows that $\mathbb{C}\operatorname{Orb}(T,x)$ is dense in X. Hence T is supercyclic and since the set of supercyclic vectors is G_δ and dense in X, there exists a supercyclic vector $z \in X$ so that

$$z \in \overline{\operatorname{Orb}(T,x)}^{\circ}$$

where D° denotes the interior of the set D. A. Peris established in [29] that

$$\text{if} \quad \overline{\operatorname{Orb}(T,x)}^{\circ} \cap \overline{\operatorname{Orb}(T,y)}^{\circ} \neq \emptyset \quad \text{then} \quad \overline{\operatorname{Orb}(T,x)}^{\circ} = \overline{\operatorname{Orb}(T,y)}^{\circ}.$$

Using the previously mentioned result and following the proof of Lemma 2.11, one deduces that

$$\text{if} \quad \overline{\operatorname{Orb}(T,x)}^{\circ} \neq \emptyset \quad \text{then} \quad \overline{\operatorname{Orb}(T,x)}^{\circ} = \overline{\operatorname{Orb}(T,\lambda x)}^{\circ} \quad \forall\lambda \in \mathbb{C}\setminus\{0\}.$$

Therefore $(\mathbb{C} \setminus \{0\}) \operatorname{Orb}(T, z) \subset \overline{\operatorname{Orb}(T, x)}^{\circ}$ and since z is supercyclic we conclude that $\operatorname{Orb}(T, x)$ is dense in X.

3. d-dense Cesàro orbits. Let X be a separable Banach space and T be a continuous linear operator on X. In [15] N. Feldman introduced the notion of a d-dense orbit. Recall that if d is a given positive number and $x \in X$ then the orbit $\operatorname{Orb}(T, x) = \{x, Tx, T^2x, \dots\}$ is called d-dense if for every $y \in X$ there is a positive integer n so that $\|T^n x - y\| < d$. As already mentioned in the introduction, Feldman found a connection between hypercyclicity and d-dense orbits by proving the following.

THEOREM 3.1. *If* $\operatorname{Orb}(T, x)$ *is d-dense for some $x \in X$ and $d > 0$ then T is hypercyclic.*

Let us now proceed by giving a definition.

DEFINITION 3.2. For a given positive number d, a subset A of X will be called *d-dense* if for every $x \in X$ there exists $a \in A$ such that $\|a - x\| < d$.

In this section we extend Feldman's result to the context of Cesàro hypercyclicity. More precisely, we establish the following result.

THEOREM 3.3. *If* $\operatorname{Orb}_{\mathrm{C}}(T, x)$ *is d-dense for some $x \in X$ and $d > 0$ then T is Cesàro hypercyclic.*

Proof. Let $\{x_j\}$ be a countable dense set in X. Define

$$E(j, s, n) = \left\{ x \in X : \left\| \frac{T^n}{n} x - x_j \right\| < s \right\}$$

for every $j, n \in \{1, 2, \dots\}$ and any $s > 0$.

LEMMA 3.4. *If* $\operatorname{Orb}_{\mathrm{C}}(T, x)$ *is d-dense for some $x \in X$ and $d > 0$, then $B(y, 3d) \cap \operatorname{Orb}_{\mathrm{C}}(T, x)$ is infinite for every $y \in X$.*

Proof. The proof is similar to the proof of Lemma 2.4 from [15] but we give it for completeness. Fix $y \in X$. Since X is infinite-dimensional, there exists an infinite sequence $\{y_n : n = 1, 2, \dots\}$ such that $\|y_n - y\| = 2d$ and $\|y_n - y_m\| \geq 2d$ for every $n \neq m$. As $B(y_n, d) \subset B(y, 3d)$ and $\operatorname{Orb}_{\mathrm{C}}(T, x)$ intersects every member of the sequence of pairwise disjoint open balls $\{B(y_n, d)\}$, we conclude that $B(y, 3d)$ contains infinitely many elements of the Cesàro T-orbit of x. ∎

LEMMA 3.5. *If* $\operatorname{Orb}_{\mathrm{C}}(T, x)$ *is d-dense for some $x \in X$ and $d > 0$, then for every $\varepsilon > 0$ and every $m \in \{1, 2, \dots\}$ the set* $\operatorname{Orb}_{\mathrm{C}}(T, T^m x)$ *is $3d + \varepsilon$-dense.*

Proof. Fix $\varepsilon > 0$ and $m \in \{1, 2, \ldots\}$. Let $y \in X$. By Lemma 3.4 there is a sequence $\{n_k\}$ of positive integers with $n_k \to \infty$ as $k \to \infty$ such that

$$\left\| \frac{T^{n_k+m}}{n_k+m} x - y \right\| < 3d, \qquad \forall k = 1, 2, \ldots. \tag{1}$$

Using (1) we have $\left\| \frac{T^{n_k+m}}{n_k+m} x \right\| \le \|y\| + 3d$ and hence

$$\left\| \frac{T^{n_k+m}}{n_k} x - \frac{T^{n_k+m}}{n_k+m} x \right\| = \left(\frac{n_k+m}{n_k} - 1 \right) \left\| \frac{T^{n_k+m}}{n_k+m} x \right\| \to 0 \tag{2}$$

as $k \to \infty$. By (1) and (2) Lemma 3.5 follows. ∎

LEMMA 3.6. *If $x \in \bigcap_j \bigcup_n E(j, s, n)$ for some $s > 0$ then*

$$x/l \in \bigcap_j \bigcup_n E(j, s/l, n) \quad \textit{for every } l \in \{1, 2, \ldots\}.$$

Proof. If $l = 1$ there is nothing to prove. Suppose $l \in \{2, 3, \ldots\}$ and fix x_j. Then there is $j' \in \{1, 2, \ldots\}$ such that $\|x_{j'} - l x_j\| < s$ or equivalently

$$\left\| \frac{x_{j'}}{l} - x_j \right\| < \frac{s}{l}. \tag{3}$$

By our hypothesis there is a positive integer n such that $\left\| \frac{T^n}{n} x - x_{j'} \right\| < s$, which trivially implies

$$\left\| \frac{T^n}{n} \left(\frac{x}{l} \right) - \frac{x_{j'}}{l} \right\| < \frac{s}{l}. \tag{4}$$

By (3), (4) and since $l \ge 2$ the result follows. ∎

LEMMA 3.7. *If $\mathrm{Orb}_{\mathrm{C}}(T, x)$ is d-dense for some $x \in X$ and $d > 0$, then $\bigcap_j \bigcup_n E(j, 4d, n)$ is G_δ and dense in X.*

Proof. Lemma 3.5 implies that $\mathrm{Orb}_{\mathrm{C}}(T, T^m x)$ is $3d + \varepsilon$-dense for every $m \in \{1, 2, \ldots\}$ and every $\varepsilon > 0$. Hence, $T^m x \in \bigcap_j \bigcup_n E(j, 3d+\varepsilon, n)$ for every $m \in \{1, 2, \ldots\}$ and every $\varepsilon > 0$. Take $\varepsilon = d$. Then $T^m x \in \bigcap_j \bigcup_n E(j, 4d, n)$ for every $m \in \{1, 2, \ldots\}$. By Lemma 3.6 we conclude that

$$\frac{T^m}{m\varrho} x \in \bigcap_j \bigcup_n E(j, 4d, n) \tag{5}$$

for every $m, \varrho \in \{1, 2, \ldots\}$. We claim that the set

$$\left\{ \frac{T^m}{m\varrho} x : m, \varrho = 1, 2, \ldots \right\}$$

is dense in X. Indeed, let $y \in X$ and fix $\varepsilon > 0$. There exists a positive integer j such that

$$\|y - x_j\| < \varepsilon/2. \tag{6}$$

Fix a positive integer ϱ such that $d/\varrho < \varepsilon/2$. Then there is $m \in \{1,2,\ldots\}$ such that

$$\left\|\frac{T^m}{m}x - \varrho x_j\right\| < d.$$

Hence, by the choice of ϱ we get

$$\left\|\frac{T^m}{m\varrho}x - x_j\right\| < \frac{\varepsilon}{2}. \tag{7}$$

Combining now (6) and (7) we arrive at

$$\left\|\frac{T^m}{m\varrho}x - y\right\| < \varepsilon,$$

proving the claim. From (5), it follows that $\bigcap_j \bigcup_n E(j,4d,n)$ is dense in X and since $E(j,4d,n)$ is open in X we conclude that the set $\bigcap_j \bigcup_n E(j,4d,n)$ is G_δ and dense in X. ∎

Let us now proceed with the proof of Theorem 3.3. Since $\mathrm{Orb}_{\mathrm{C}}(T,x)$ is d-dense, it is easy to see that $\mathrm{Orb}_{\mathrm{C}}(T,x/s)$ is d/s-dense for every $s \in \{1,2,\ldots\}$. Lemma 3.7 shows that $\bigcap_j \bigcup_n E(j,4d/s,n)$ is G_δ and dense in X for every $s \in \{1,2,\ldots\}$. Hence, Baire's category theorem implies that

$$\bigcap_s \bigcap_j \bigcup_n E(j,4d/s,n)$$

is G_δ and dense in X. It is plain that

$$\bigcap_s \bigcap_j \bigcup_n E(j,4d/s,n) = \{x \in X : \overline{\mathrm{Orb}_{\mathrm{C}}(T,x)} = X\}.$$

Therefore using León-Saavedra's characterization (Lemma 2.2) we conclude that T is Cesàro hypercyclic. ∎

Below we include some simple observations on perturbations of certain dense sets. In the following, $T : X \to X$ will be a continuous linear operator.

PROPOSITION 3.8. *If* $\overline{\{m^{-1}\,\mathrm{Orb}(T,x) : m = 1,2,\ldots\}} = X$ *and* $\|T^n y\| < M$ *for all* $n = 0,1,2,\ldots$ *for some positive number* M, *then*

$$\overline{\{m^{-1}\,\mathrm{Orb}(T,x+y) : m = 1,2,\ldots\}} = X.$$

Proof. Let $z \in X$ and $\varepsilon > 0$. There are positive integers N and n_0 such that

$$1/n_0 < \varepsilon/2, \qquad M < N.$$

Then

$$\left\|\frac{1}{n_0 N}T^n y\right\| < \frac{\varepsilon}{2}, \qquad \forall n = 1,2,\ldots. \tag{8}$$

There exist positive integers m, n_1 so that

$$\left\| \frac{1}{m} T^{n_1} x - n_0 N z \right\| < N. \tag{9}$$

Now (9) implies that

$$\left\| \frac{1}{m n_0 N} T^{n_1} x - z \right\| < \frac{1}{n_0}. \tag{10}$$

From (8) and (10) we get

$$\left\| \frac{1}{m n_0 N} T^{n_1} (x + y) - z \right\| < \varepsilon. \quad \blacksquare$$

An immediate consequence is the following

Corollary 3.9. *If* $\overline{\mathrm{Orb}(T, x)} = X$ *and there is some positive number* M *such that* $\|T^n y\| < M$ *for every* $n = 0, 1, 2, \ldots$, *then*

$$\overline{\{m^{-1} \mathrm{Orb}(T, x + y) : m = 1, 2, \ldots\}} = X.$$

Observe that under the hypothesis of Corollary 3.9 it is not true in general that $\overline{\mathrm{Orb}(T, x + y)} = X$ (see [15]). In Proposition 3.8 one can also replace the sequence $\{m^{-1}\}$ by the interval $(0, 1)$ and get a similar result.

Proposition 3.10. *If* $\overline{(0, 1)\, \mathrm{Orb}(T, x)} = X$ *and there is some positive number* M *such that* $\|T^n y\| < M$ *for every* $n = 0, 1, 2, \ldots$, *then*

$$\overline{(0, 1)\, \mathrm{Orb}(T, x + y)} = X.$$

The proof of Proposition 3.10 is similar to that of Proposition 3.8 and is omitted. On the other hand as we show below, the last proposition is not true if we replace $(0, 1)$ by $\mathbb{R}^+ = (0, \infty)$.

Proposition 3.11. *If* $\overline{\mathbb{R}^+ \mathrm{Orb}(T, x)} = X$ *and there is some positive number* M *such that* $\|T^n y\| < M$ *for every* $n = 0, 1, 2, \ldots$, *then it is not true in general that* $\overline{\mathbb{R}^+ \mathrm{Orb}(T, x + y)} = X$.

Proof. The proof is based on the work of Müller [27], [28] and on a result due to Bermúdez, Bonilla and Peris [2]. In order to provide a counterexample, let a candidate operator be any supercyclic and power bounded operator T with spectral radius $r(T) = 1$, acting on a real Hilbert space X. Recall that T is *power bounded* if $\sup_n \|T^n\| < \infty$. Let $x \in X$ be such that $\overline{\mathbb{R}\, \mathrm{Orb}(T, x)} = X$. If for every $z \in X \setminus \{-x\}$ we have $\overline{\mathbb{R}\, \mathrm{Orb}(T, x + z)} = X$, then for every $z \in X \setminus \{0\}$ it follows that $\overline{\mathbb{R}\, \mathrm{Orb}(T, z)} = X$. The last equality contradicts Theorem 3 in [27] which states that if T satisfies the above requirements then there is a non-zero vector which is not supercyclic. Therefore there exists $y \in X$ so that $\overline{\mathbb{R}\, \mathrm{Orb}(T, x + y)} \neq X$ and obviously $\sup_n \|T^n y\| < \infty$. Using now the result of Bermúdez, Bonilla and Peris [2] that for any $x \in X$, $\overline{\mathbb{R}\, \mathrm{Orb}(T, x)} = X$ if and only if $\overline{\mathbb{R}^+ \mathrm{Orb}(T, x)} = X$, the conclusion follows. It is also possible to provide a similar example for Banach spaces. In this case one has to use the main result from [28] and then the argument is the same as in the Hilbert case. ∎

PROPOSITION 3.12. *If* $(0,1)\operatorname{Orb}(T,x)$ *is* d*-dense for some* $d>0$ *then it is also dense in* X.

Proof. Let $y\in X$ and $\varepsilon>0$. Without loss of generality we may assume that $\varepsilon<d$. Then there exist $n\in\mathbb{N}$ and $\lambda\in(0,1)$ such that

$$\left\|\lambda T^n x-\frac{d}{\varepsilon}\,y\right\|<d$$

or equivalently

$$\left\|\frac{\varepsilon}{d}\,\lambda T^n x-y\right\|<\varepsilon. \ \blacksquare$$

The next proposition shows that Proposition 3.12 does not remain true if $(0,1)$ is replaced by $(1,\infty)$.

PROPOSITION 3.13. *Let* B *be the backward shift on* $l^2(\mathbb{N})$. *Then there exist* $d>0$ *and* $x\in l^2(\mathbb{N})$ *such that* $(1,\infty)\operatorname{Orb}(2B,x)$ *is* d*-dense but not dense in* X.

Proof. Take x to be the vector constructed in the proof of Theorem 2.6 in [15]. Then the first coordinate of $(2B)^n x$ belongs to the set $F=\{z\in\mathbb{C}:|z|\geq 1\}\cup\{0\}$. Therefore for every $\lambda>1$ the first coordinate of $\lambda(2B)^n x$ belongs to F. Hence, by taking $y=(1/2,0,0,\dots)$ we have

$$\|\lambda(2B)^n x-y\|\geq 1/2 \quad \text{for all } \lambda>1$$

and for all non-negative integers n. This yields $\overline{(1,\infty)\operatorname{Orb}(2B,x)}\neq X$ and by Feldman's construction the set $\operatorname{Orb}(2B,x)$ is 4-dense in $l^2(\mathbb{N})$. The proof is complete. ∎

4. Generalizations. The purpose of this section is to provide generalizations of Theorems 2.5 and 2.9. Let T be a continuous linear operator on a Banach space X. Roughly speaking, we shall show that regarding Theorems 2.5 and 2.9, the particular sequence of weights $\{n^{-1}\}$ in the Cesàro orbit of a vector $x\in X$ under T does not play any significant role in our approach. We shall need the following well known more general notion of hypercyclicity. Let $\{T_n\}$ be a sequence of continuous linear operators on X. The sequence $\{T_n\}$ is said to be *hypercyclic* provided there exists a vector $x\in X$ so that the sequence $\{T_n x\}$ is dense in X, and the vector x is called *hypercyclic for the sequence* $\{T_n\}$. Following standard notation we denote by $\mathrm{HC}(\{T_n\})$ the set of hypercyclic vectors for the sequence $\{T_n\}$. We establish the following.

THEOREM 4.1. *Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\lambda_{n+1}/\lambda_n \to 1$ as $n \to \infty$. Then T is hypercyclic and the sequence $\{\lambda_n T^n\}$ is hypercyclic if and only if there exist $x, y \in X$ so that both sets $\{T^n x : n = 0, 1, 2, \ldots\}$, $\{\lambda_n T^n y : n = 0, 1, 2, \ldots\}$ are somewhere dense.*

The proof is similar to the proof of Theorem 2.5 and is omitted. A consequence of the proof of Theorem 4.1 is the following.

PROPOSITION 4.2. *Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\lambda_{n+1}/\lambda_n \to 1$ as $n \to \infty$. Suppose that T is hypercyclic. If the set $\{\lambda_n T^n x\}$ is somewhere dense then it is everywhere dense.*

THEOREM 4.3. *Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\lambda_{n+1}/\lambda_n \to \alpha$ as $n \to \infty$ for some $\alpha > 0$. Suppose that for some $x \in X$ the sequence $\{\lambda_n T^n x\}$ is somewhere dense.*

(i) *If*
$$\lim_{n\to\infty} \frac{\lambda_n}{\alpha^n} = 0$$
then $(0, \varepsilon)\{\lambda_n T^n x : n = 1, 2, \ldots\}$ is dense in X for every $0 < \varepsilon < 1$.

(ii) *If*
$$\lim_{n\to\infty} \frac{\lambda_n}{\alpha^n} = \infty$$
then $(M, \infty)\{\lambda_n T^n x : n = 1, 2, \ldots\}$ is dense in X for every $M > 1$.

(iii) *If*
$$\lim_{n\to\infty} \frac{\lambda_n}{\alpha^n} = b$$
for some $b \in (0, \infty)$ then $\{\lambda_n T^n x : n = 1, 2, \ldots\}$ is dense in X, hence the sequence $\{\lambda_n T^n\}$ is hypercyclic.

Proof. Since the proof is similar to the proof of Theorem 2.9 we shall only sketch it. For every $x \in X$ define
$$Z_x = \{y \in X : \exists n_k \to \infty,\ \lambda_{n_k} T^{n_k} x \to y\}$$
and let
$$W_x = Z_x^\circ,$$
the interior of Z_x. Arguing as in the proof of Lemma 2.4 it is easy to show that
$$T(Z_x) \subset \frac{1}{\alpha} Z_x,$$
and hence
$$T^n(Z_x) \subset \frac{1}{\alpha^n} Z_x, \qquad \forall n = 1, 2, \ldots.$$
Since the proofs of statements (i), (ii) and (iii) are quite similar we shall only sketch the proof of (ii). As in Section 2, using the above invariance property of Z_x we can prove the following two lemmata.

LEMMA 4.4. *If for some $x, y \in X$, $Z_x \cap Z_y \neq \emptyset$ then $\overline{(M,\infty)Z_x} = \overline{(M,\infty)Z_y}$ for every $M > 1$.*

Proof. Fix $M > 1$. There exists $k \in \{1, 2, \ldots\}$ such that

$$\lambda_k T^k x \in W_x \cap W_y.$$

Applying T n times we get

$$\lambda_{n+k} T^{n+k} x \in \frac{\lambda_{n+k}}{\lambda_k \alpha^n} Z_x \cap \frac{\lambda_{n+k}}{\lambda_k \alpha^n} Z_y, \quad \forall n = 1, 2, \ldots.$$

Since $\lim_{n\to\infty} \lambda_n/\alpha^n = \infty$, there is $n_0 \in \mathbb{N}$ so that

$$\frac{\lambda_{n+k}}{\lambda_k \alpha^n} > M, \quad \forall n \geq n_0.$$

Therefore

$$\lambda_{n+k} T^{n+k} x \in (M,\infty)Z_x \cap (M,\infty)Z_y, \quad \forall n \geq n_0.$$

Let $z \in Z_x$. Then there exists a sequence $\{n_m\}$ of natural numbers such that $n_m \to \infty$ as $m \to \infty$, $n_m \geq n_0 + k$ and

$$\lambda_{n_m} T^{n_m} x \to z \quad \text{as } m \to \infty.$$

It is plain that

$$\lambda_{n_m} T^{n_m} x \in (M,\infty)Z_x \cap (M,\infty)Z_y, \quad \forall m \in \mathbb{N}.$$

Hence $z \in \overline{(M,\infty)Z_x} \cap \overline{(M,\infty)Z_y}$, which gives

$$Z_x \subset \overline{(M,\infty)Z_x} \cap \overline{(M,\infty)Z_y}.$$

This implies

$$\overline{(M,\infty)Z_x} \subset \overline{(M,\infty)Z_x} \cap \overline{(M,\infty)Z_y} \subset \overline{(M,\infty)Z_y}.$$

In a similar manner it follows that $\overline{(M,\infty)Z_y} \subset \overline{(M,\infty)Z_x}$ and we conclude $\overline{(M,\infty)Z_y} = \overline{(M,\infty)Z_x}$. ∎

LEMMA 4.5. *If $W_x \neq \emptyset$, then $\overline{(M,\infty)Z_x} = \overline{(M,\infty)Z_{\lambda x}}$ for every $\lambda \in \mathbb{C} \setminus \{0\}$ and every $M > 1$.*

The proof of this lemma is exactly the same as that of Lemma 2.11 and is omitted. To finish the proof of (ii) one proceeds as in the proof of Theorem 2.9. The details are left to the reader. ∎

5. Rotations of Cesàro hypercyclic operators. Recently León-Saavedra and Müller [24] employing some methods of semigroups of operators showed that if T is a hypercyclic operator acting on a complex Banach space, then for every λ with $|\lambda| = 1$, λT is hypercyclic and $\mathrm{HC}(T) = \mathrm{HC}(\lambda T)$. This result gave an answer to a question raised by Bès. For related questions and partial results in this direction we also refer to [14], [11] and [31]. In connection with Bès' question and inspired by the notion of

Cesàro hypercyclicity we investigate whether it is possible to obtain a result similar to that of León-Saavedra and Müller for more general sequences of operators. Using in an essential way Cantor's theorem along with arguments that have been widely used in connection with hypercyclicity (as Baire's category theorem and invariance of certain sets) we prove the following.

THEOREM 5.1. *Let X be a topological vector space whose topology is induced by a complete and translation invariant metric, i.e. X is a Fréchet space. Fix a complex number λ with $|\lambda| = 1$. Let $\{\lambda_n\}$ be a sequence of positive numbers such that*

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$$

and let T be a continuous linear operator acting on X.

(i) *If T is hypercyclic and the sequence $\{\lambda_n(\lambda T)^n\}$ is hypercyclic then so is $\{\lambda_n T^n\}$ and*

$$\mathrm{HC}(\{\lambda_n(\lambda T)^n\}) \subset \mathrm{HC}(\{\lambda_n T^n\}).$$

(ii) *If λT is hypercyclic and the sequence $\{\lambda_n T^n\}$ then so is $\{\lambda_n(\lambda T)^n\}$ is hypercyclic and*

$$\mathrm{HC}(\{\lambda_n T^n\}) \subset \mathrm{HC}(\{\lambda_n(\lambda T)^n\}).$$

Proof. The proofs of (i) and (ii) are similar, hence we shall only prove (i). The symbol $\mathbb{T}$ stands for the set of complex numbers of modulus 1. Define

$$K_{\mu,x} = \{y \in X : \exists n_k \to \infty,\ \lambda_{n_k}(\mu T)^{n_k} x \to y\}$$

for every $x \in X$ and every μ with $|\mu| = 1$.

Suppose $x \in \mathrm{HC}(\{\lambda_n(\lambda T)^n\})$. Using a similar argument to the proof of Lemma 2.1 we get

$$X = \overline{K}_{\lambda,x}.$$

It follows that

$$X = \overline{K}_{\lambda,x} \subset \overline{\mathbb{T}K_{1,x}}.$$

Since $\mathbb{T}$ is closed, bounded and bounded away from zero, it is easy to see that $\overline{\mathbb{T}K_{1,x}} \subset \mathbb{T}\overline{K}_{1,x}$. Hence,

$$X = \mathbb{T}\overline{K}_{1,x}.$$

Divide $\mathbb{T}$ in two closed subarcs A_1, A_2 of length π such that $\mathbb{T} = A_1 \cup A_2$. Therefore

$$X = A_1\overline{K}_{1,x} \cup A_2\overline{K}_{1,x}$$

and Baire's category theorem yields $(A_i\overline{K}_{1,x})^\circ \neq \emptyset$ for some $i \in \{1,2\}$. Define

$$C_1 = A_i.$$

Using the hypothesis $\lim_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$, one can easily check the T-invariance of $C_1\overline{K}_{1,x}$, i.e.

$$T(C_1\overline{K}_{1,x}) \subset C_1\overline{K}_{1,x}.$$

Since T is hypercyclic and $C_1\overline{K}_{1,x}$ is somewhere dense, there exists $z \in \mathrm{HC}(T)$ such that $z \in C_1\overline{K}_{1,x}$. Therefore $\overline{\mathrm{Orb}(T,z)} \subset C_1\overline{K}_{1,x}$, which in turn implies that

$$X = C_1\overline{K}_{1,x}.$$

Divide C_1 in two closed subarcs $C_{1,1}$, $C_{1,2}$ of length $\pi/2$ such that $C_1 = C_{1,1} \cup C_{1,2}$. Using a similar procedure as above we conclude that

$$X = C_{1,i}\overline{K}_{1,x}$$

for some $i \in \{1, 2\}$. Defining $C_2 = C_{1,i}$ it follows that

$$X = C_2\overline{K}_{1,x}.$$

Proceeding inductively, we obtain a decreasing sequence $\{C_n\}$ of closed subarcs of $\mathbb{T}$ so that the length of C_n is $\pi/2^{n-1}$ and

$$X = C_n\overline{K}_{1,x}, \quad \forall n = 1, 2, \ldots. \tag{11}$$

By Cantor's theorem we have $\bigcap_n C_n = \{\alpha\}$ for some $\alpha \in \mathbb{T}$. One can easily verify that

$$\bigcap_n C_n\overline{K}_{1,x} = \alpha\overline{K}_{1,x}$$

and using (11) we get $X = \alpha\overline{K}_{1,x}$ or equivalently $X = \overline{K}_{1,x}$. This trivially implies that

$$X = \overline{\{\lambda_n T^n x : n = 1, 2, \ldots\}},$$

hence $x \in \mathrm{HC}(\{\lambda_n T^n\})$. ∎

As an application we provide a version of León-Saaevedra and Müller's theorem for sequences of operators of the form $\{\lambda_n T^n\}$ acting on Fréchet spaces, under the assumption that T satisfies the hypercyclicity criterion. This criterion was first discovered by C. Kitai [21] and later rediscovered by Gethner and Shapiro [16]. Since then there has been a lot of progress on problems related to this criterion, but the question of whether the hypercyclicity criterion is equivalent to hypercyclicity remains one of the major open problems in this area. For partial results in this direction and equivalent formulations of the hypercyclicity criterion we refer to [6], [4], [17], [23], [18].

Let us first recall this criterion.

Hypercyclicity Criterion. Let $\{T_n\}$ be a sequence of bounded continuous operators acting on a Fréchet space X. Suppose there exist an increasing sequence $\{n_k\}$ of positive integers, dense subsets X_1, X_2 of X and maps $S_{n_k} : X_2 \to X_2$ such that

(i) $T_{n_k} x \to 0$ as $k \to \infty$ for all $x \in X_1$,
(ii) $S_{n_k} y \to 0$ as $k \to \infty$ for all $y \in X_2$,
(iii) $T_{n_k} S_{n_k} y \to y$ as $k \to \infty$ for all $y \in X_2$.

Then the sequence $\{T_n\}$ is hypercyclic.

In the above criterion if $T_n = T^n$ then we say that T *satisfies the hypercyclicity criterion.*

Corollary 5.2. *Fix a sequence $\{\lambda_n\}$ of positive numbers such that*

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1.$$

Let T be a bounded continuous operator acting on a Fréchet space X. If T satisfies the hypercyclicity criterion and the sequence $\{\lambda_n T^n\}$ is hypercyclic, then for every λ with $|\lambda| = 1$ the sequence $\{\lambda_n(\lambda T)^n\}$ is hypercyclic and

$$\mathrm{HC}(\{\lambda_n T^n\}) = \mathrm{HC}(\{\lambda_n(\lambda T)^n\}).$$

Proof. Fix any λ with $|\lambda| = 1$. Then it is obvious that λT satisfies the hypercyclicity criterion and Theorem 5.1 yields the conclusion. ∎

Corollary 5.3. *Let X be a complex Banach space. Fix a sequence $\{\lambda_n\}$ of positive numbers such that* $\lim_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$ *and let T be a continuous linear operator on X. If T is hypercyclic and the sequence $\{\lambda_n T^n\}$ is hypercyclic then for every λ with $|\lambda| = 1$ the sequence $\{\lambda_n(\lambda T)^n\}$ is hypercyclic and*

$$\mathrm{HC}(\{\lambda_n T^n\}) = \mathrm{HC}(\{\lambda_n(\lambda T)^n\}).$$

Proof. This is an immediate consequence of Theorem 5.1 and the theorem of León-Saavedra and Müller. ∎

Corollary 5.4.

(i) *Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be a Cesàro hypercyclic unilateral weighted backward shift. Then for every λ with $|\lambda| = 1$, λT is Cesàro hypercyclic and*

$$\mathrm{HC}\left(\left\{\frac{T^n}{n}\right\}\right) = \mathrm{HC}\left(\left\{\frac{(\lambda T)^n}{n}\right\}\right).$$

(ii) *Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be a unilateral weighted backward shift. Then for every λ with $|\lambda| = 1$ and every r positive integer the sequence $\{\lambda^n(I+T)^n/n^r\}$ is hypercyclic and*

$$\mathrm{HC}\left(\left\{\frac{\lambda^n(I+T)^n}{n^r}\right\}\right) = \mathrm{HC}\left(\left\{\frac{(I+T)^n}{n^r}\right\}\right).$$

Proof. (i) Using Proposition 3.1 in [22] we conclude that for every λ with $|\lambda| = 1$, λT is Cesàro hypercyclic. On the other hand, Corollary 3.3 in [22] implies that T is also hypercyclic. Hence the requirements of Theorem 5.1 are satisfied and the assertion follows.

(ii) Combine Theorem 4.1 in [22] and Corollary 5.2 above. ■

We shall give one more application of Theorem 5.1. Recall that $H(\mathbb{C})$ is the Fréchet space of entire functions endowed with the topology of uniform convergence on compact subsets of $\mathbb{C}$ and $T_1, D : H(\mathbb{C}) \to H(\mathbb{C})$ are the translation and differentiation operators respectively, defined in Section 2. We establish the following.

COROLLARY 5.5. *Fix a sequence $\{\lambda_n\}$ of positive numbers such that*

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1.$$

Then for every λ with $|\lambda| = 1$ the following hold.

(i) $\mathrm{HC}(\{\lambda_n D^n\}) = \mathrm{HC}(\{\lambda_n (\lambda D)^n\})$.
(ii) $\mathrm{HC}(\{\lambda_n T_1^n\}) = \mathrm{HC}(\{\lambda_n (\lambda T_1)^n\})$.

Proof. It is known and easy to see—using the hypercyclicity criterion—that $\{\lambda_n D^n\}$ and $\{\lambda_n T_1^n\}$ are hypercyclic (see for example [3] and also [10], [5] for further results). Since both D and T_1 satisfy the hypercyclicity criterion, Corollary 5.2 gives the desired result. ■

6. Concluding remarks and open problems. In this section we collect some open problems that arise naturally after the present investigation. Assume that X is a topological vector space and let T be a continuous linear operator on X.

PROBLEM 1. *If for some $x \in X$ the set $\mathrm{Orb}_{\mathrm{C}}(T, x)$ is somewhere dense, is it true that it is everywhere dense?*

Observe that a positive answer to the above question would imply versions of Ansari's theorem [1] and Herrero's conjecture [11], [29] for Cesàro hypercyclic operators. For details on this simple implication we refer to [12]. We also note that in a forthcoming paper [13] we prove that the assumption that the sequence $\{T^n/n\}$ satisfies the hypercyclicity criterion implies a version of Ansari's theorem for Cesàro hypercyclic operators.

PROBLEM 2. *Suppose that T is Cesàro hypercyclic. Is it true that for every λ with $|\lambda| = 1$ the operator λT is Cesàro hypercyclic and* $\mathrm{HC}(\{T^n/n\}) = \mathrm{HC}(\{\lambda T^n/n\})$*?*

Finally, we raise a question in connection with the following impressive result of Read [30]: there exists a continuous linear operator S acting on the space $l^1(\mathbb{N})$ of absolutely summable sequences such that every non-zero vector is hypercyclic for S.

PROBLEM 3. *Does there exist a continuous linear operator* $S : l^1(\mathbb{N}) \to l^1(\mathbb{N})$ *so that the set* $\mathrm{Orb}_{\mathrm{C}}(S,x) = \{n^{-1}S^n x : n = 1,2,\ldots\}$ *is dense in* $l^1(\mathbb{N})$ *for every* $x \in l^1(\mathbb{N}) \setminus \{0\}$*?*

Let us mention that in the above problems one may replace the sequence of weights $\{n^{-1}\}$ with the sequence $\{n^{\alpha}\}$ for any real α and ask similar questions. As a stronger version of Problem 3 we ask the following.

PROBLEM 4. *Does there exist a continuous linear operator* $S : l^1(\mathbb{N}) \to l^1(\mathbb{N})$ *so that the set* $\{n^{\alpha}S^n x : n = 1,2,\ldots\}$ *is dense in* $l^1(\mathbb{N})$ *for every* $x \in l^1(\mathbb{N}) \setminus \{0\}$ *and every* $\alpha \in \mathbb{R}$*?*

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