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When a unital F-algebra has all maximal left (right) ideals closed?

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Abstract. We prove that a real or complex unital F-algebra has all maximal left ideals closed if and only if the set of all its invertible elements is open. Consequently, such an algebra also automatically has all maximal right ideals closed.

A *topological algebra* is a real or complex algebra A which is a topological vector space (t.v.s.) such that the multiplication $(x, y) \mapsto xy$ is a jointly continuous map from $A \times A$ to A.

An *F-algebra* (an algebra of type F) is a topological algebra which is an F-space, i.e. a complete metrizable t.v.s. The topology of an F-space X can be given by means of an *F-norm*, i.e. a map $x \mapsto \|x\|$ from X to non-negative real numbers such that

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,
(ii) $\|x + y\| \leq \|x\| + \|y\|$,
(iii) the map $(\lambda, x) \mapsto \|\lambda x\|$ from $\mathbb{K}\times X$ to X is jointly continuous ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

The metric (distance) in an F-space X is given by means of $\|x - y\|$ $(x, y \in X)$; we shall also write $x_n \to x_0$ if $\lim_n \|x_n - x_0\| = 0$.

A *B_0-algebra* is a locally convex F-algebra; it is called *m-convex* if its topology can be given by means of a sequence of submultiplicative seminorms.

For further information on F-spaces the reader is referred to [2] and [5], and for more information on F-algebras and B_0-algebras, to [3], [4] and [6].

Ideal theory is one of the main chapters of the theory of topological algebras, and there is a natural question under which conditions all maximal ideals (left, right or two-sided) are closed. In this paper we deal with this question for maximal left (or right) ideals in unital F-algebras (generally speaking, such an algebra may have dense maximal ideals). The problem

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of characterization of such algebras was explicitly posed in [9, Problem 7]. The exact wording of this problem is as follows: Let A be an F-algebra with all maximal left ideals closed. Does it follow that A is a Q-algebra? Here we solve this problem positively. Recall that a unital topological algebra A is called a *Q-algebra* if the set (group) $G(A)$ of its invertible elements is open (it is well known that this happens if and only if $G(A)$ contains a neighbourhood of the identity of A, or equivalently, if and only if the interior of $G(A)$ is non-void).

Some related problems have already been considered. In [7], among other results, we have shown that a commutative complex m-convex B_0-algebra A has all maximal ideals closed if and only if it is a Q-algebra, and this holds if and only if all its maximal ideals are of codimension one. M. Akkar and C. Nacir ([1, Proposition 17]) proved that a commutative unital F-algebra has all its maximal ideals closed if and only if it is a Q-algebra, Further we have shown ([10]) that a unital F-algebra has all maximal one-sided ideals closed if and only if it is a Q-algebra. Our present theorem improves this result in the sense that we can replace one-sided ideals by left (or right) ideals. The result seems to be new even in the case of an m-convex B_0-algebra.

Let A be a unital topological algebra. We denote by $G_l(A)$ (resp. $G_r(A)$) the set of all left (resp. right) invertible elements in A. For convenience of the reader we shall prove the following (folklore) lemma, which will be used in the proof of our theorem (we shall improve it in Corollary 1).

LEMMA 1. *Let A be an F-algebra with unity e. Then the following are equivalent.*

(i) *The set $G_l(A)$ (resp. $G_r(A)$) contains a neighbourhood of the unity.*
(ii) *The set $G_l(A)$ (resp. $G_r(A)$) is open.*

Proof. We have to prove only the implication (i)⇒(ii). We give the proof for $G_l(A)$ since for $G_r(A)$ the argument is analogous. Suppose that there is an open neighbourhood U of e consisting of left invertible elements. Let a be an arbitrary left invertible element in A. Since the map $x \mapsto xa$ maps A onto itself, the open mapping theorem for F-spaces (see e.g. [5, Theorem 2.3.1]) implies that the set Ua is open. Since it contains a and consists of left invertible elements, every element of $G_l(A)$ is an interior point of it, and so the conclusion follows.

In the proof of the theorem we shall also need the concept of a *topologically right invertible element*. Let A be a unital F-algebra with unity e. An element x in A is called *topologically right invertible* if there is a sequence $(z_i) \subset A$ such that $\lim_i xz_i = e$.

We shall need the following lemma (it follows from Lemmas 1 and 5 in [11]).

LEMMA 2. *Let A be a unital F-algebra with unity e and let $(x_i) \subset A$ be a sequence tending to e. Then there is a subsequence (a_i) of (x_i) such that for all $k \in \mathbb{N}$ the infinite products*

$$u_k = \lim_i a_k a_{k+1} \cdots a_{k+i} \tag{1}$$

are convergent and moreover

$$\lim_k u_k = e. \tag{2}$$

If all x_i are right invertible, then all the products u_k are topologically right invertible.

Note that the formula (1) implies that

$$u_k = a_k u_{k+1} \tag{3}$$

for all k.

Our main result reads as follows.

THEOREM. *Let A be a unital F-algebra. Then the following statements are equivalent*:

(a) *all maximal left ideals in A are closed*;
(b) *the set $G_l(A)$ of all left invertible elements is open*;
(c) *the set $G(A)$ of all invertible elements is open*, *i.e. A is a Q-algebra*;
(d) *the set $G_r(A)$ of all right invertible elements is open*;
(e) *all maximal right ideals in A are closed.*

Proof. (a)⇒(b). Assume towards a contradiction that $G_l(A)$ is not open. We shall be done if we show that A has a dense left ideal I, for then every maximal left ideal containing I will be dense and so non-closed. By Lemma 1, there is a sequence $(x_i) \subset A$ of elements in $A \setminus G_l(A)$ tending to the unity e.

Consider the elements a_i and u_k obtained from the sequence (x_i) by the formula (1) of Lemma 2. We shall be looking for an ideal I of the form

$$I = \bigcup_{k=1}^{\infty} A u_k. \tag{4}$$

If all the u_k are non-left invertible, then the left ideals Au_k are proper. In this case, by (3), we have $Au_k = Aa_k u_{k+1} \subset Au_{k+1}$. Thus the ideal I is a proper left ideal in A as the union of an increasing sequence of proper ideals. We also have $u_k \in I$ for all k. Let x be an arbitrary element in A. Then $xu_k \in I$ for all k, and, by (2), $xu_k \to x$. Since x was chosen arbitrarily, I is dense in A, and the conclusion follows. Thus we shall be done if we show that all elements u_k are non-left invertible.

To prove this fact, we consider first the case where our sequence (x_i) of non-left invertible elements can be chosen so that it consists of right invertible elements. We claim that in this case the elements u_k cannot belong to $G_l(A)$ for all k. Otherwise

$$bu_{k_0} = e \tag{5}$$

for some natural k_0 and some b in A. By Lemma 2, u_{k_0} is topologically right invertible, and so there are $z_i \in A$ with $u_{k_0} z_i \to e$. Multiplying (5) from the right by z_i and passing to the limit, we obtain

$$z_i = bu_{k_0} z_i \to b.$$

Thus

$$u_{k_0} b = \lim_i u_{k_0} z_i = e = bu_{k_0},$$

and so u_{k_0} is invertible with inverse b. We have $e = bu_{k_0} = ba_{k_0} u_{k_0+1}$, so that u_{k_0+1} is left invertible as well. The above reasoning shows that u_{k_0+1} is also invertible, which implies the invertibility of $a_{k_0} = u_{k_0}(u_{k_0+1})^{-1}$. This is a contradiction, since our hypothesis is that no x_i is left invertible. Thus in the case considered our claim follows.

In the opposite case there is no sequence (x_i) of non-invertible elements with $x_i \in G_r(A)$ for all i. In this case there exists a neighbourhood U of the identity which contains no right invertible elements except the invertible ones. Take now a sequence (x_i) so that not only all its terms, but also all elements u_k, obtained by Lemma 2, are in U. This can be done by formula (2). Our conclusion will be obtained if we show that the u_k cannot be left invertible, because then (4) would give a dense left ideal. Suppose for contradiction that u_{k_0} is left invertible for some k_0. By (1), we have

$$e = bu_{k_0} = \lim_i ba_{k_0} \cdots a_{k_0+i},$$

and so for sufficiently large i, say $i \geq i_0$, the elements $ba_{k_0} \cdots a_{k_0+i}$ are in U. By (3) we have

$$bu_{k_0} = bua_{k_0} \cdots a_{k_0+i_0+i} u_{k_0+i_0+i+1} = e,$$

and thus all elements $ba_{k_0} \ldots a_{k_0+i_0+i}$, $i \in \mathbb{N}$, are invertible, being right invertible elements of U. Consequently,

$$a_{k_0+i_0+1} = (ba_{k_0} \cdots a_{k_0+i_0})^{-1} ba_{k_0} \cdots a_{k_0+i_0+1},$$

and so the left-hand element is invertible. This contradiction completes the proof of the implication (a)⇒(b).

(b)⇒(c). This proof is similar to the last part of the previous one. Assume for contradiction that $G_l(A)$ is open whilst $G(A)$ is not. Denote by U an open neighbourhood of e which consists of left invertible elements. By our assumption, there is in U a sequence $x_i \to e$ consisting of non-invertible elements. Using Lemma 2 we obtain elements a_i and u_i, and, by (2), there is an index k_0 such that $u_k \in U$ for all $k \geq k_0$. In particular, by (3), there is an element $b \in A$ such that

$$e = bu_{k_0} = \lim_i ba_{k_0} \cdots a_{k_0+i}. \tag{6}$$

Hence $ba_{k_0} \cdots a_{k_0+i} \in U$ for sufficiently large i, say $i \geq i_0$. We have

$$bu_{k_0} = ba_{k_0} \dots a_{k_0+i_0} u_{k_0+i_0+1},$$

so that, by (6), $ba_{k_0} \dots a_{k_0+i_0}$ is right invertible, and so invertible. Similarly, $ba_{k_0} \dots a_{k_0+i_0+1}$ is also invertible. Thus

$$a_{k_0+i_0+1} = (ba_{k_0} \dots a_{k_0+i_0})^{-1} ba_{k_0} \dots a_{k_0+i_0+1}$$

is invertible, which is a contradiction. Our implication follows.

(c)⇒(a). We have $G(A)$ open. If M is a maximal left ideal in A, then $A \cap G(A) = \emptyset$, and so M is not dense in A. Since a maximal left ideal is either dense or closed, M is closed, and the implication follows.

Thus we have (a)⇔(b)⇔(c). The equivalence (c)⇔(d)⇔(e) is just the "right" version of the above one, and so the conclusion follows.

We also have the following corollaries.

COROLLARY 1. *Let A be as above. Then the set $G_l(A)$ (and $G_r(A)$) is open if and only if its interior is non-void.*

For the proof observe that any non-void open subset of $G_l(A)$ is disjoint from any proper left ideal in A. Thus all maximal left ideals in A are closed, and so A is a Q-algebra. Lemma 1 implies the desired result.

Consequently, we can add to Lemma 1 the following equivalent condition:

(iii) the interior of $G_l(A)$ (resp. $G_r(A)$) is non-void.

The above means, in particular, that in a Q-algebra A of type F the sets $G_l(A)$ and $G_r(A)$ are open. Call an element x of A *properly left* (resp. *right*) *invertible* if it is left (right) invertible but non-invertible.

COROLLARY 2. *Let A be a Q-algebra of type F. Then every connected component of the open set*

$$G_l(A) \cup G_r(A)$$

consists entirely either of invertible elements, or of properly left-invertible elements, or of properly right-invertible elements.

The above statement follows from the fact that $G(A)$ is both closed and open in $G_l(A)$ (and in $G_r(A)$). To prove its closedness observe that, if U is a neighbourhood of e consisting of invertible elements, and an element x in A is properly left (resp. right) invertible, then the set Ux (resp. xU) is open, and consists entirely of left (resp. right) properly invertible elements.

Final remarks. It was a natural conjecture that in a unital F-algebra A all maximal left ideals are closed if and only if the set $G_l(A)$ is open (such an algebra could be called a left Q-algebra, or a Q_l-algebra). The first version of our theorem was obtained in such a form. However, it was not clear whether a proper Q_l-algebra of type F exists (by that we mean a Q_l-algebra which is not a Q-algebra). Our result says that such algebras do not exist. The author does not know whether there exists a unital topological algebra which is not a Q-algebra, but is a Q_l-algebra.

Finally, let us remark that our theorem does not extend to general topological algebras. In [8, Example 12] we gave an example of a complete commutative locally convex unital algebra with all ideals closed, but which is not a Q-algebra.

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