STUDIA MATHEMATICA 182 (3) (2007)

Weak amenability of the second dual of a Banach algebra

by

M. Eshaghi Gordji (Semnan) and M. Filali (Oulu)

Abstract. It is known that a Banach algebra $\mathcal{A}$ inherits amenability from its second Banach dual $\mathcal{A}^{**}$. No example is yet known whether this fails if one considers the weak amenability instead, but the property is known to hold for the group algebra $L^1(G)$, the Fourier algebra $A(G)$ when G is amenable, the Banach algebras $\mathcal{A}$ which are left ideals in $\mathcal{A}^{**}$, the dual Banach algebras, and the Banach algebras $\mathcal{A}$ which are Arens regular and have every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ weakly compact. In this paper, we extend this class of algebras to the Banach algebras for which the second adjoint of each derivation $D : \mathcal{A} \to \mathcal{A}^*$ satisfies $D''(\mathcal{A}^{**}) \subseteq \mathrm{WAP}(\mathcal{A})$, the Banach algebras $\mathcal{A}$ which are right ideals in $\mathcal{A}^{**}$ and satisfy $\mathcal{A}^{**}\mathcal{A} = \mathcal{A}^{**}$, and to the Figà-Talamanca–Herz algebra $A_p(G)$ for G amenable. We also provide a short proof of the interesting recent criterion on when the second adjoint of a derivation is again a derivation.

1. Introduction. Amenable Banach algebras were introduced by Johnson in [23]. A *derivation* from a Banach algebra $\mathcal{A}$ to a Banach $\mathcal{A}$-bimodule X is a bounded linear mapping $D : \mathcal{A} \to X$ such that

$$D(ab) = aD(b) + D(a)b \quad \text{for all } a, b \in \mathcal{A}.$$

Easy examples of derivations are the *inner derivations*, which are given for each $x \in X$ by

$$D_x(a) = ax - xa \quad \text{for all } a \in \mathcal{A}.$$

The Banach algebra $\mathcal{A}$ is said to be *amenable* when for every Banach $\mathcal{A}$-bimodule X, the inner derivations are the only derivations existing from $\mathcal{A}$ to X^* (note that X^* is also a Banach $\mathcal{A}$-bimodule).

A Banach algebra $\mathcal{A}$ is *weakly amenable* if every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner ([3] and [24]).

It is clear that an amenable algebra is weakly amenable, but the converse is not true as can be checked with the algebra ℓ_p $(1 \le p < \infty)$ with pointwise multiplication. This algebra is weakly amenable but not amenable since it does not have a bounded approximate identity ([24]). Another less trivial but

2000 *Mathematics Subject Classification*: Primary 46H20; Secondary 43A20.
Key words and phrases: derivation, weak amenability, Arens product.

much more interesting example is the group algebra $L^1(G)$. It is amenable if and only if G amenable ([23]), but it is always weakly amenable ([25] or [10]).

Consider now the second dual $\mathcal{A}^{**}$ of our Banach algebra $\mathcal{A}$. Then $\mathcal{A}^{**}$ is also a Banach algebra with an Arens product, and so one may consider the question of how the amenability and weak amenability of $\mathcal{A}^{**}$ and $\mathcal{A}$ relate. It is known that neither the amenability of $\mathcal{A}$ implies that of $\mathcal{A}^{**}$, nor the weak amenability of $\mathcal{A}$ implies that of $\mathcal{A}^{**}$; for example $L^1(\mathbb{R})$ is amenable (and hence weakly amenable) but its second dual is not weakly amenable (and so not amenable); see [19] or the end of this paper. The amenability of $\mathcal{A}^{**}$, however, implies the amenability of $\mathcal{A}$ (see [20] or [19]). The unavoidable question is therefore whether weak amenability passes from $\mathcal{A}^{**}$ to $\mathcal{A}$. This problem was considered by a few authors and a positive answer has been given in each of the following cases:

- $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$, [19].
- $\mathcal{A}$ is a dual Banach algebra, [17].
- $\mathcal{A}$ is Arens regular and every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is weakly compact, [8].

In this paper, we go on being positive and show that the implication holds in the following instances:

- The second adjoint of each derivation $D : \mathcal{A} \to \mathcal{A}^*$ satisfies $D''(\mathcal{A}^{**}) \subseteq \mathrm{WAP}(\mathcal{A})$; this includes, in particular, the result proved in [8, Corollary 7.5] stated above.
- $\mathcal{A}$ is a right ideal in $\mathcal{A}^{**}$ and $\mathcal{A}^{**}\mathcal{A} = \mathcal{A}^{**}$.

The result in [8, Corollary 7.5] is proved using the main theorem [8, Theorem 7.1] of that paper which gives a criterion on when the second conjugate of D is a derivation. A by-product of our arguments is a very short and straightforward proof of that theorem. Another by-product is that if the topological centre of $\mathcal{A}^{**}$ is weakly amenable and each derivation $D : \mathcal{A} \to \mathcal{A}^*$ is weakly compact, then $\mathcal{A}$ is weakly amenable. That the amenability of the topological centre of $\mathcal{A}^{**}$ implies that of $\mathcal{A}$ without any extra condition is proved in [17].

At the end of the paper we consider the Figà-Talamanca–Herz algebra $A_p(G)$ $(1 < p < \infty)$ and prove in particular that the weak amenability of $A_p(G)^{**}$ implies that of $A_p(G)$ when G is amenable. This was proved in [28, Proposition 6.3] for the Fourier algebra $A(G)$, i.e., when $p = 2$.

The *first Arens product* on $\mathcal{A}^{**}$ is defined in three stages as follows (see [2]). For every $a'', b'' \in \mathcal{A}^{**}$, $a' \in \mathcal{A}^*$ and $a \in \mathcal{A}$, we define $a'a \in \mathcal{A}^*$, $b''a' \in \mathcal{A}^*$ and $a''b'' \in \mathcal{A}^{**}$, respectively, by

$$\langle a'a, b\rangle = \langle a', ab\rangle, \qquad b \in \mathcal{A},$$
$$\langle a''a', a\rangle = \langle a'', a'a\rangle, \qquad a \in \mathcal{A},$$
$$\langle a''b'', a'\rangle = \langle a'', b''a'\rangle, \qquad a' \in \mathcal{A}^*.$$

When $\mathcal{A}^{**}$ is given the weak*-topology, we see that the mapping

$$a'' \mapsto a''b'' : \mathcal{A}^{**} \to \mathcal{A}^{**}$$

is continuous for each fixed $b'' \in A^{**}$. It is not difficult to verify that the mapping

$$b'' \mapsto ab'' : \mathcal{A}^{**} \to \mathcal{A}^{**}$$

is also continuous for each fixed $a \in \mathcal{A} \subseteq \mathcal{A}^{**}$. But, in general, this mapping is not continuous. The *topological centre* of $\mathcal{A}^{**}$ is defined by

$$Z(\mathcal{A}^{**}) = \{a'' \in \mathcal{A}^{**} : \text{the map } b'' \mapsto a''b'' : \mathcal{A}^{**} \to \mathcal{A}^{**} \\ \text{is weak}^*\text{-weak}^*\text{-continuous}\}.$$

As already noted, $\mathcal{A}$ is a subalgebra of $Z(\mathcal{A}^{**})$. The algebra $\mathcal{A}$ is said to be *Arens regular* when $Z(\mathcal{A}^{**}) = \mathcal{A}^{**}$. We may recall that any C^*-algebra is Arens regular, and that the group algebra $L^1(G)$ of a locally compact group G is strongly Arens irregular, i.e., $Z(L^1(G)^{**}) = L^1(G)$ (see [27], or [29] and [14] for different proofs). For more details, the reader is directed for example to [15], [5] or [7].

As already done and throughout the rest of the paper, we shall identify every Banach space with its canonical image in its second dual.

2. Weak amenability of the second dual. Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}^{**}$ is weakly amenable. If $D : \mathcal{A} \to \mathcal{A}^*$ is a derivation, then we wish to extend D to a (bounded) derivation from $\mathcal{A}^{**}$ to $\mathcal{A}^{***}$. The natural extension of D as a bounded linear map is the second conjugate D'' of D. But in general, nothing guarantees that D'' is a derivation, i.e.,

$$D''(a''b'') = a''D(b'') + D(b'')a'',$$

where the module actions of $\mathcal{A}$ on $\mathcal{A}^*$ are given by

$$\langle a'a, b\rangle = \langle a', ab\rangle, \quad \langle aa', b\rangle = \langle a', ba\rangle, \qquad a' \in \mathcal{A}^*,\ a, b \in \mathcal{A},$$

and the module actions of $\mathcal{A}^{**}$ on $\mathcal{A}^{***}$ are given by

$$\langle a''a''', b''\rangle = \langle a''', b''a''\rangle,$$
$$\langle a'''a'', b''\rangle = \langle a''', a''b''\rangle, \qquad a''' \in \mathcal{A}^{***},\ a'', b'' \in \mathcal{A}^{**}.$$

To see this, let $a'', b'' \in \mathcal{A}^{**}$ and take nets (a_α) and (b_β) in $\mathcal{A}$ which converge, respectively, to a'' and b'' in the weak*-topology of $\mathcal{A}^{**}$. Using the weak*-weak*-continuity of D'' , we obtain

$$D''(a''b'') = \lim_\alpha \lim_\beta D(a_\alpha b_\beta) = \lim_\alpha \lim_\beta D(a_\alpha) b_\beta + \lim_\alpha \lim_\beta a_\alpha D(b_\beta).$$

The first limit term is easily computed since $b'' \mapsto b''c'' : \mathcal{A}^{**} \to \mathcal{A}^{**}$ is weak*-weak*-continuous. For every $c'' \in \mathcal{A}^{**}$, we obtain

$$(1)\quad \lim_\alpha \lim_\beta \langle D(a_\alpha)b_\beta, c''\rangle = \lim_\alpha \lim_\beta \langle D(a_\alpha), b_\beta c''\rangle = \lim_\alpha \lim_\beta \langle b_\beta c'', D(a_\alpha)\rangle$$
$$= \lim_\alpha \langle b''c'', D(a_\alpha)\rangle = \lim_\alpha \langle D(a_\alpha), b''c''\rangle$$
$$= \langle D''(a''), b''c''\rangle = \langle D''(a'')b'', c''\rangle.$$

To compute the second limit, we can go as far as

$$(2)\quad \lim_\alpha \lim_\beta \langle a_\alpha D(b_\beta), c''\rangle = \lim_\alpha \lim_\beta \langle D(b_\beta), c''a_\alpha\rangle = \lim_\alpha \langle D''(b''), c''a_\alpha\rangle.$$

But this cannot be pushed any further due first to the fact that the mapping

$$a'' \mapsto c''a'' : \mathcal{A}^{**} \to \mathcal{A}^{**}$$

is not weak*-weak*-continuous for every $c'' \in \mathcal{A}^{**}$ unless $\mathcal{A}$ is Arens regular; and secondly, even if $\mathcal{A}$ were Arens regular, $D''(a'')$ is in $\mathcal{A}^{***}$ and may very likely be outside of $\mathcal{A}^*$. Therefore additional conditions must be assumed if we want D'' to be a derivation. This is done in our first theorem. Recall that $a' \in \mathcal{A}^*$ is *weakly almost periodic* if the set

$$\{a'a : a \in \mathcal{A},\ \|a\| \le 1\}$$

is relatively weakly compact. Let $\mathrm{WAP}(\mathcal{A})$ be the subspace of all weakly almost periodic functionals in $\mathcal{A}^*$. Recall also the known characterization of $\mathrm{WAP}(\mathcal{A})$ ([30]),

$$\mathrm{WAP}(\mathcal{A}) = \{a' \in \mathcal{A}^* : a'' \mapsto \langle b''a'', a'\rangle \text{ is continuous on } \mathcal{A}^{**} \text{ for every } b'' \in \mathcal{A}^{**}\}.$$

THEOREM 2.1. *If $\mathcal{A}^{**}$ is weakly amenable and if every derivation $D : \mathcal{A} \to \mathcal{A}^*$ satisfies $D''(\mathcal{A}^{**}) \subseteq \mathrm{WAP}(\mathcal{A})$, then $\mathcal{A}$ is also weakly amenable.*

Proof. We continue the argument started in (2). Since $D''(b'') \in \mathrm{WAP}(\mathcal{A})$, for each $c'' \in \mathcal{A}^{**}$ we have

$$\lim_\alpha \langle a_\alpha D''(b''), c''\rangle = \lim_\alpha \langle c''a_\alpha, D''(b'')\rangle = \langle c''a'', D''(b'')\rangle$$
$$= \langle a''D''(b''), c''\rangle.$$

Therefore

$$D''(a''b'') = D''(a'')b'' + a''D''(b''),$$

and so D'' is a derivation. Since $\mathcal{A}''$ is weakly amenable, D'' is inner, and so

$$D''(a'') = a''a''' - a'''a'' \quad \text{for some } a''' \in A^{***} \text{ and for all } a'' \in A^{**}.$$

In particular, $D(a) = aa''' - a'''a$, and regarding $D(a)$ in $\mathcal{A}^*$, we obtain $D(a) = aa' - a'a$, where $a' = a'''|_{\mathcal{A}}$. Thus $\mathcal{A}$ is weakly amenable. ∎

The following corollary was obtained in [8, Corollary 7.5].

Corollary 2.1. *Let $\mathcal{A}$ be Arens regular and suppose that every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is weakly compact. If $\mathcal{A}^{**}$ is weakly amenable, then so is $\mathcal{A}$.*

Proof. Since $\mathcal{A}$ is Arens regular, $\mathrm{WAP}(\mathcal{A}) = \mathcal{A}^*$. Let $D : \mathcal{A} \to \mathcal{A}^*$ be a derivation. Since D is weakly compact, $D''(\mathcal{A}^{**}) \subseteq \mathcal{A}^* = \mathrm{WAP}(\mathcal{A})$. So, by Theorem 2.1, the weak amenability of $\mathcal{A}^{**}$ implies that of $\mathcal{A}$. ■

In [8], Corollary 2.1 was deduced from the main theorem of that paper which provided a criterion for D'' to be a derivation. Next we show that this criterion may also be deduced with the same type of argument as above. Putting (1) and (2) together, we obtain

$$\langle D''(a''b''), c''\rangle = \langle D''(a'')b'', c''\rangle + \lim_{\alpha} \langle D''(b''), c''a_\alpha\rangle$$

for every $a'', b'', c'' \in \mathcal{A}^{**}$ and whenever (a_α) is a net in $\mathcal{A}$ weak*-converging to a'' in $\mathcal{A}^{**}$. It follows that D'' is a derivation if and only if

$$\lim_{\alpha} a_\alpha D''(b'') = a''D''(b'').$$

This is true if and only if for every $c'' \in \mathcal{A}^{**}$,

$$\lim_{\alpha} \langle a_\alpha D''(b''), c''\rangle = \langle a''D''(b''), c''\rangle,$$

which in turn holds if and only if

$$\lim_{\alpha} \langle D''(b'')c'', a_\alpha\rangle = \langle D''(b'')c'', a''\rangle.$$

This means that $D''(b'')c'' : \mathcal{A}^{**} \to \mathbb{C}$ is weak*-weak*-continuous. Hence $D''(b'')c'' \in \mathcal{A}^*$. Thus we obtain Theorem 7.1 of [8].

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra and let $D : \mathcal{A} \to \mathcal{A}^*$ be a derivation. Then D'' is a derivation if and only if $D''(\mathcal{A}^{**})\mathcal{A}^{**} \subseteq \mathcal{A}^*$.*

Theorem 2.3. *Suppose that every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is weakly compact and let B be a closed subalgebra of $\mathcal{A}^{**}$ containing $\mathcal{A}$ such that for every $b'' \in B$, $a'' \mapsto b''a'' : B \to B$ is weak*-weak*-continuous. If B is weakly amenable, then $\mathcal{A}$ is weakly amenable.*

Proof. Let $D : \mathcal{A} \to \mathcal{A}^*$ be a derivation, and let $R : \mathcal{A}^{***} \to B^*$ be the restriction map, defined by

$$R(a''') = a'''_{|B} \quad \text{for every } a''' \in \mathcal{A}^{***}.$$

Then R is a B-module homomorphism. We show that $\overline{D} := R \circ D''_{|B} : B \to B^*$ is a derivation. Obviously, $\overline{D}$ is an extension of D. Let $a'', b'' \in B$, and as before, pick nets (a_α) and (b_β) in $\mathcal{A}$ converging, respectively, to a'' and b'' in the weak*-topology of $\mathcal{A}^{**}$. Then for every $c'' \in B$, we have $\lim_\alpha c''a_\alpha = c''a''$, and so

$$\lim_{\alpha} \langle a_\alpha D''(b''), c''\rangle = \langle c''a'', D''(b'')\rangle = \langle a''D''(b''), c''\rangle.$$

Therefore, with (1) in mind,

$$\begin{aligned}\overline{D}(a''b'') &= R(\lim_\alpha \lim_\beta D(a_\alpha b_\beta)) = R(\lim_\alpha \lim_\beta D(a_\alpha)b_\beta + \lim_\alpha \lim_\beta a_\alpha D(b_\beta)) \\ &= R(D''(a'')b'') + R(a''D''(b'')) \\ &= (R\circ D'')(a'')b'' + a''(R\circ D'')(b'') = \overline{D}(a'')b'' + a''\overline{D}(b'').\end{aligned}$$

Since B is weakly amenable, $\overline{D} = D_{b'}$ for some $b' \in B^*$. As before, letting $a' = b'_{|\mathcal{A}}$, we see that $D = D_{a'}$. Thus, $\mathcal{A}$ is weakly amenable. ■

COROLLARY 2.2. *If the topological centre $Z(\mathcal{A}^{**})$ is weakly amenable, then $\mathcal{A}$ is weakly amenable in each of the following cases.*

(1) *Every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is weakly compact.*
(2) *$\mathcal{A}^*$ is weakly sequentially complete.*

Proof. It is well known that if $\mathcal{A}^*$ is weakly sequentially complete, then every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is weakly compact (see for example [1]). So (1) follows from Theorem 2.2 by letting $B = Z(\mathcal{A}^{**})$, and (2) follows from (1). ■

The following result may puzzle the unprepared reader since in [19] it is proved that $\mathcal{A}$ is weakly amenable if $\mathcal{A}^{**}$ is weakly amenable and $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$. Passing from $\mathcal{A}$ to the Banach algebra with reversed multiplication would thus yield Theorem 2.4 without even the additional hypothesis that $\mathcal{A}^{**}\mathcal{A} = \mathcal{A}^{**}$. Unfortunately (or fortunately), reversing the multiplication in $\mathcal{A}$ leads to the second Arens product in $\mathcal{A}^{**}$. So the right ideal in $\mathcal{A}^{**}$ becomes a left ideal in $\mathcal{A}^{**}$ (as one wishes) but with $\mathcal{A}^{**}$ now having the second Arens product, and so [19] cannot be applied.

THEOREM 2.4. *Let $\mathcal{A}$ be a right ideal in $\mathcal{A}^{**}$ and suppose $\mathcal{A}^{**}\mathcal{A} = \mathcal{A}^{**}$. If $\mathcal{A}^{**}$ is weakly amenable, then $\mathcal{A}$ is weakly amenable.*

Proof. Let $D : \mathcal{A} \to \mathcal{A}^*$ be a derivation. To show that $D'' : \mathcal{A}^{**} \to \mathcal{A}^{***}$ is a derivation, we apply Theorem 2.2 and verify that $D''(\mathcal{A}^{**})\mathcal{A}^{**} \subseteq \mathcal{A}^*$. For this, we prove that every element in $D''(\mathcal{A}^{**})\mathcal{A}^{**}$ is in fact weak*-continuous. So let $(c''_\alpha)_\alpha$ be a net in $\mathcal{A}^{**}$ with $\lim_\alpha c''_\alpha = c''$ in $\mathcal{A}^{**}$, let $a'', b'' \in \mathcal{A}^{**}$ and write $b'' = d''a$ with $d'' \in \mathcal{A}^{**}$ and $a \in \mathcal{A}$. Then

$$\langle D''(a'')b'', c''_\alpha\rangle = \langle D''(a'')d''a, c''_\alpha\rangle = \langle D''(a'')d'', ac''_\alpha\rangle.$$

Since $a \in \mathcal{A} \subseteq Z(\mathcal{A}^{**})$, we have $\lim_\alpha ac_\alpha = ac''$ in $\mathcal{A}^{**}$ with respect to the weak*-topology. But since $\mathcal{A}$ is a right ideal in $\mathcal{A}^{**}$, for every α, ac_α and ac'' are members of $\mathcal{A}$. So $\lim_\alpha ac_\alpha = ac''$ weakly in $\mathcal{A}$.

Now the linear map $D''(a'')d''_{|\mathcal{A}} : \mathcal{A} \to \mathbb{C}$ is norm-norm-continuous, and so it is weak-weak-continuous (see for example [11, Theorem V.3.15]). Therefore,

$$\lim_\alpha \langle D''(a'')d'', ac''_\alpha\rangle = \langle D''(a'')d'', ac''\rangle = \langle D''(a'')b'', c''\rangle.$$

Therefore $D''(a'')b'' : \mathcal{A}^{**} \to \mathbb{C}$ is weak*-continuous, and so $D''(a'')b'' \in \mathcal{A}^*$ and the proof is complete.

Remark. We can also prove Theorem 2.4 by applying Corollary 2.1. In fact, the conditions imposed on $\mathcal{A}$ imply that $\mathcal{A}$ is Arens regular (see [12, Theorem 4.3]). Furthermore, it is not difficult to check in this case that every derivation $D : \mathcal{A} \to \mathcal{A}^*$ satisfies $D''(\mathcal{A}^{**}) \subseteq \mathcal{A}^*$.

As promised, we finish the exposition with the Figà-Talamanca–Herz algebra. Recall first that when $\mathcal{A} = L^1(G)$, $\mathcal{A}^*\mathcal{A}$ is the space $\mathrm{LUC}(G)$ of all bounded left uniformly continuous functions. By using the known decompositions $L^1(G)^{**} = \mathrm{LUC}(G)^* \oplus \mathrm{LUC}(G)^\perp$ and $\mathrm{LUC}(G)^* = M(G) \oplus C_0(G)^\perp$ (see [13] or [18]), the weak amenability passes from $L^1(G)^{**}$ to $\mathrm{LUC}(G)^*$ and then to $M(G)$. This is done in [19, Theorem 2.1]. By [6], G is therefore discrete and so $L^1(G)$ is weakly amenable.

If we consider $\mathcal{A}$ as the Fourier algebra $A(G)$, we take $B_\varrho(G)$ which is the dual of the reduced C^*-algebra $C^*_\varrho(G)$. The latter is the closure of $L^1(G)$ with respect to the norm in $\mathcal{B}(L^2(G))$ when $L^1(G)$ is regarded as convolution operators on $L^2(G)$. Then we apply the decomposition theorem proved in [26] to see that $B_\varrho(G)$ inherits the weak amenability from $\mathrm{UC}(\widehat{G})^*$. As noted in the proof of [16, Proposition 3.6], this in turn implies that $A(G)$ is weakly amenable by [21, Proposition 2.2]. If G is amenable, then $A(G)$ has a bounded approximate identity and so the decomposition $A(G)^{**} = \mathrm{UC}(\widehat{G})^* \oplus \mathrm{UC}(\widehat{G})^\perp$ together with the previous one enables the weak amenability to be inherited successively from $A(G)^{**}$ to $\mathrm{UC}(\widehat{G})^*$, to the Fourier–Stieltjes algebra $B(G)$, and then to $A(G)$. This is proved in [28, Proposition 6.3].

We prove this result for $A_p(G)$. Recall that $A_p(G)$ is the space of all complex-valued functions f on G which can be represented as

$$f = \sum_{n=1}^{\infty} u_n * \widetilde{v}_n,$$

where $u_n \in L^p(G)$, $v_n \in L^q(G)$, $1/p + 1/q = 1$, and $\sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q < \infty$. The norm of f is then defined by

$$\|f\| = \inf\Big\{\sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q : f = \sum_{n=1}^{\infty} u_n * \widetilde{v}_n,\ u_n \in L^p(G),\ v_n \in L^q(G)\Big\},$$

making $A_p(G)$ a commutative Banach algebra [22]. We consider $L^1(G)$ as an algebra of convolution operators on $L_p(G)$, and define $\mathrm{PF}_p(G)$ and $\mathrm{PM}_p(G)$ as the closure of $L^1(G)$ in $\mathcal{B}(L^p(G))$ with respect to the norm operator topology and the weak operator topology, respectively. It is known that $\mathrm{PM}_p(G) = A_p(G)^*$. We denote $\mathrm{PF}_p(G)^*$ by $W_p(G)$. Then $A_p(G) \subseteq W_p(G) \subseteq$

$M(A_p(G))$, the latter is the multiplier algebra of $A_p(G)$, and $W_p(G) = M(A_p(G))$ when G is amenable (see [4]). When $p = 2$, $A_p(G) = A(G)$, $\mathrm{PM}_p(G)$ is the von Neumann algebra $\mathrm{VN}(G)$, and $\mathrm{PF}_p(G)$ is the reduced C^*-algebra $C^*_\varrho(G)$ of G.

Let $\mathrm{UC}_p(\widehat{G})$ be the norm closed linear span of $A_p(G)\,\mathrm{PM}_p(G)$.

THEOREM 2.5. *Let G be a locally compact group.*

(1) *The weak amenability of* $\mathrm{UC}_p(\widehat{G})^*$ *implies that of* $W_p(G)$.
(2) *If G is amenable, then the weak amenability of* $A_p(G)^{**}$ *implies that of* $\mathrm{UC}_p(\widehat{G})$, $B_p(G)$ *and* $A_p(G)$

Proof. To prove (1) we use the decomposition $\mathrm{UC}_p(\widehat{G})^* = W_p(G) \oplus F_p(G)^\perp$ proved in [9, Corollary 3.5], then apply [28, Lemma 2.3].

To prove (2), as for $p = 2$, when G is amenable, $\mathrm{UC}_p(\widehat{G}) = A_p(G)\,\mathrm{PM}_p(G)$ by the Cohen factorisation theorem. We use first the decomposition $A_p(G)^{**} = \mathrm{UC}_p(\widehat{G})^* \oplus \mathrm{UC}_p(\widehat{G})^\perp$ to deduce the weak amenability of $\mathrm{UC}_p(\widehat{G})^*$. Then as in (1), we deduce the weak amenability of $W_p(G)$. Since $A_p(G)$ has a bounded approximate identity, we see that $\overline{A_p(G)^2} = A_p(G)$. It is clear that $A_p(G)$ is also an ideal in $W_p(G)$. Consequently, $A_p(G)$ is also weakly amenable by [21, Proposition 2.2]. ∎

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Department of Mathematics
University of Semnan
Semnan, Iran
E-mail: maj_ess@yahoo.com

Department of Mathematical Sciences
University of Oulu
Oulu 90014, Finland
E-mail: mfilali@cc.oulu.fi

Received November 9, 2006
Revised version July 18, 2007 (6038)