STUDIA MATHEMATICA 182 (3) (2007)

On L^p integrability and convergence of trigonometric series

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Abstract. We first give a necessary and sufficient condition for $x^{-\gamma}\phi(x) \in L^p$, $1 < p < \infty$, $1/p - 1 < \gamma < 1/p$, where $\phi(x)$ is the sum of either $\sum_{k=1}^{\infty} a_k \cos kx$ or $\sum_{k=1}^{\infty} b_k \sin kx$, under the condition that $\{\lambda_n\}$ (where λ_n is a_n or b_n respectively) belongs to the class of so called Mean Value Bounded Variation Sequences (MVBVS). Then we discuss the relations among the Fourier coefficients λ_n and the sum function $\phi(x)$ under the condition that $\{\lambda_n\} \in$ MVBVS, and deduce a sharp estimate for the weighted modulus of continuity of $\phi(x)$ in L^p norm.

1. Introduction. Let L^p, $1 < p < \infty$, be the space of all p-power integrable functions of period 2π equipped with the norm

$$\|f\|_p = \Big(\int_{-\pi}^{\pi} |f(x)|^p \, dx\Big)^{1/p}.$$

Write

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx, \qquad g(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

for those x where the series converge. Denote by $\phi(x)$ either $f(x)$ or $g(x)$ and let λ_n be the associated Fourier coefficients, i.e., λ_n is either a_n or b_n. In this paper, we first consider necessary and sufficient conditions for $x^{-\gamma}\phi(x) \in L^p$, $1/p-1 < \gamma < 1/p$, and give an answer to Boas' [2] Question 6.12, "What condition is necessary and sufficient for $x^{-\gamma}\phi(x) \in L^p$, $1/p - 1 < \gamma < 1/p$, when $\lambda_n \geq 0$?", under some weak condition on the sequence $\{\lambda_n\}$.

2000 *Mathematics Subject Classification*: 42A20, 42A32.

Key words and phrases: Fourier series, L^p integrability, modulus of continuity, mean value bounded variation sequences.

Research of D. S. Yu supported by NSERC RCD grant and AARMS of Canada.

Research of P. Zhou supported by NSERC of Canada.

S. P. Zhou is W. F. James Professor of St. Francis Xavier University. His research was also supported in part by Natural Science Foundation of China under grant number 10471130.

The definition below introduces a new class of sequences called Mean Value Bounded Variation Sequences (MVBVS) which was first defined in [10]:

DEFINITION. A nonnegative sequence $\mathbf{A} = \{a_n\}_{n=0}^{\infty}$ is said to be a *mean value bounded variation sequence* ($\{a_n\} \in$ MVBVS) if there is a $\lambda \geq 2$ such that

$$\sum_{k=n}^{2n} |a_k - a_{k+1}| \leq \frac{C(\mathbf{A})}{n} \sum_{k=[\lambda^{-1}n]}^{[\lambda n]} a_k$$

for all $n = 1, 2, \ldots$ and some constant $C(\mathbf{A})$ depending only upon the sequence $\mathbf{A}$.

Our first main result is:

THEOREM 1. *Let* $1 < p < \infty$. *If* $\{\lambda_n\} \in$ MVBVS, *then* $x^{-\gamma}\phi(x) \in L^p$, $1/p - 1 < \gamma < 1/p$, *if and only if*

$$\sum_{n=1}^{\infty} n^{p+p\gamma-2}\lambda_n^p < \infty. \tag{1}$$

Theorem 1 answers Boas' Question 6.12 under the condition that $\{\lambda_n\} \in$ MVBVS. Answers under stronger conditions on $\{\lambda_n\}$ were given earlier by Chen ([3], [4]) for monotonic $\{\lambda_n\}$, Leindler ([6]) for "rest bounded variation" $\{\lambda_n\}$, Zhou–Le ([9]) for "group bounded variation" $\{\lambda_n\}$, and Yu–Zhou ([8]) for "non-onesided bounded variation" $\{\lambda_n\}$.

Throughout this paper, we use C to denote a positive constant independent of the integer n; C may depend on the parameters such as p, γ, and λ, and it may have different values in different occurrences.

The next aim of this paper is to discuss the relations between the Fourier coefficients λ_n and the sum function $\phi(x)$, under the condition that $\{\lambda_n\} \in$ MVBVS. Let $f \in L^p$, $1 < p < \infty$ and $1/p - 1 < \gamma < 1/p$. Define the weighted modulus of continuity of f in L^p norm as follows:

$$\omega(f, h)_{p,x^{-\gamma}} := \omega(f, h)_{p,\gamma} := \sup_{|t| \leq h} \|x^{-\gamma}(f(x+t) - f(x))\|_p.$$

Our second main result is:

THEOREM 2. *Let* $1 < p < \infty$. *If* $\{\lambda_n\} \in$ MVBVS *satisfies* (1), *then for* $1/p - 1 < \gamma < 1/p$, *we have*

$$\omega(\phi, 1/n)_{p,\gamma} \leq Cn^{-1}\Big(\sum_{k=1}^{n-1} k^{2p+p\gamma-2}\lambda_k^p\Big)^{1/p} + C\Big(\sum_{k=n}^{\infty} k^{p+p\gamma-2}\lambda_k^p\Big)^{1/p}.$$

The special case of this result, when $\gamma = 0$ (non-weighted case) and $\{\lambda_n\}$ is monotonic, was first given by Aljančić [1]. Then the monotonicity condition on the sequence $\{\lambda_n\}$ was weakened by Leindler [7] to "rest bounded variation", by Zhou–Le [9] to "group bounded variation", and by Yu–Zhou [8] to "non-onesided bounded variation". Our Theorem 2 above is the first result in the case of weighted modulus of continuity in L^p norm. We also weaken the condition on the sequence $\{\lambda_n\}$ to the weakest condition so far that $\{\lambda_n\} \in$ MVBVS. See Zhou–Zhou–Yu [10] for details on the relations between the classes of sequences mentioned above.

2. Proof of Theorem 1. Throughout this paper, we set $\lambda_0 = 0$. We need the following lemmas:

LEMMA 1 (Boas [2]). *Let $1 < p < \infty$. If $\lambda_n \ge 0$ and $1/p - 1 < \gamma < 1/p$, then a sufficient condition for $x^{-\gamma}\phi(x) \in L^p$ is*

$$\sum_{n=1}^{\infty} n^{p+p\gamma-2}\Big(\sum_{k=n}^{\infty} |\Delta\lambda_k|\Big)^p < \infty, \tag{2}$$

and a necessary condition is

$$\sum_{n=1}^{\infty} n^{p+p\gamma-2}\Big(\sum_{k=n}^{\infty} k^{-1}\lambda_k\Big)^p < \infty. \tag{3}$$

LEMMA 2 (Leindler [5]). *Let $p \ge 1$, $\alpha_n \ge 0$, and $\beta_n > 0$. Then*

$$\sum_{n=1}^{\infty} \beta_n\Big(\sum_{k=1}^{n} \alpha_k\Big)^p \le p^p \sum_{n=1}^{\infty} \beta_n^{1-p}\Big(\sum_{k=n}^{\infty} \beta_k\Big)^p \alpha_n^p, \tag{4}$$

$$\sum_{n=1}^{\infty} \beta_n\Big(\sum_{k=n}^{\infty} \alpha_k\Big)^p \le p^p \sum_{n=1}^{\infty} \beta_n^{1-p}\Big(\sum_{k=1}^{n} \beta_k\Big)^p \alpha_n^p. \tag{5}$$

Proof of Theorem 1. Sufficiency. Suppose that (1) holds. For $\{\lambda_n\} \in$ MVBVS and sufficiently large n, there exists a $\lambda \ge 2$ such that

$$\begin{aligned} \sum_{k=n}^{\infty} |\Delta\lambda_k| &\le \sum_{j=0}^{\infty} \sum_{k=2^j n}^{2^{j+1}n} |\Delta\lambda_k| \le C \sum_{j=0}^{\infty} \frac{1}{2^j n} \sum_{k=[\lambda^{-1}2^j n]}^{[\lambda 2^j n]} \lambda_k \\ &\le C \sum_{k=[\lambda^{-1}n]}^{\infty} \frac{\lambda_k}{k+1}. \end{aligned} \tag{6}$$

Thus,

$$(7)\quad \sum_{n=[\lambda]+1}^{\infty} n^{p+p\gamma-2}\Big(\sum_{k=n}^{\infty}|\Delta\lambda_k|\Big)^p$$
$$\le C\sum_{n=[\lambda]+1}^{\infty} n^{p+p\gamma-2}\bigg(\sum_{k=[\lambda^{-1}n]}^{\infty}\frac{\lambda_k}{k+1}\bigg)^p \le C\sum_{n=[\lambda]+1}^{\infty} n^{p+p\gamma-2}\bigg(\sum_{k=[\lambda^{-1}n]}^{\infty}\frac{\lambda_k}{k}\bigg)^p$$
$$\le C\sum_{n=[\lambda]+1}^{\infty}([\lambda^{-1}n])^{p+p\gamma-2}\bigg(\sum_{k=[\lambda^{-1}n]}^{\infty}\frac{\lambda_k}{k}\bigg)^p \le C\sum_{n=1}^{\infty} n^{p+p\gamma-2}\bigg(\sum_{k=n}^{\infty}\frac{\lambda_k}{k}\bigg)^p$$
$$\le C\sum_{n=1}^{\infty} n^{(p+p\gamma-2)(1-p)}\Big(\sum_{k=1}^{n}k^{p+p\gamma-2}\Big)^p\Big(\frac{\lambda_n}{n}\Big)^p \quad \text{(by (5))}$$
$$\le C\sum_{n=1}^{\infty} n^{p+p\gamma-2}\lambda_n^p.$$

Combining this with (2), we obtain the sufficiency.

Necessity. If $x^{-\gamma}\phi(x)\in L^p$, then (3) holds. For $\{\lambda_n\}\in$ MVBVS, we have, for all $[n/2]+1\le k\le n$,

$$\lambda_n \le \sum_{i=k}^{n-1}|\Delta\lambda_i| + \lambda_k \le \sum_{i=k}^{2k}|\Delta\lambda_i| + \lambda_k$$
$$\le C\,\frac{1}{k}\sum_{i=[\lambda^{-1}k]}^{[\lambda k]}\lambda_i + \lambda_k \le C\sum_{i=[n/(2\lambda)]}^{[\lambda n]}\frac{\lambda_i}{i+1} + \lambda_k,$$

and so

$$(8)\qquad \lambda_n \le C\sum_{k=[n/(2\lambda)]}^{[\lambda n]}\frac{\lambda_k}{k+1} + \frac{2}{n}\sum_{k=[n/2]+1}^{n}\lambda_k \le C\sum_{k=[n/(2\lambda)]}^{[\lambda n]}\frac{\lambda_k}{k+1}.$$

Therefore,

$$\sum_{n=[2\lambda]+1}^{\infty} n^{p+p\gamma-2}\lambda_n^p \le C\sum_{n=[2\lambda]+1}^{\infty} n^{p+p\gamma-2}\bigg(\sum_{k=[n/(2\lambda)]}^{[\lambda n]}\frac{\lambda_k}{k}\bigg)^p$$
$$\le C\sum_{n=[2\lambda]+1}^{\infty}([n/(2\lambda)])^{p+p\gamma-2}\bigg(\sum_{k=[n/(2\lambda)]}^{\infty}\frac{\lambda_k}{k}\bigg)^p$$
$$\le C\sum_{n=1}^{\infty} n^{p+p\gamma-2}\bigg(\sum_{k=n}^{\infty}\frac{\lambda_k}{k}\bigg)^p < \infty,$$

by (3). This completes the proof of Theorem 1.

3. Proof of Theorem 2. Now we prove Theorem 2. First, we prove two lemmas.

LEMMA 3. *Let* $1 < p < \infty$ *and* $\{a_n\} \in$ MVBVS*. Then for* $1/p - 1 < \gamma < 1/p$,

$$n^{-p}\sum_{m=1}^{n-1} m^{p\gamma-2}\Big(\sum_{\nu=1}^{m}\nu^2|\Delta a_\nu|\Big)^p \le Cn^{-p}\sum_{\nu=1}^{n-1}\nu^{2p+p\gamma-2}a_\nu^p + C\sum_{\nu=n}^{\infty}\nu^{p+p\gamma-2}a_\nu^p.$$

Proof. Let

$$m^* := \begin{cases} m, & m \text{ is even}, \\ m-1, & m \text{ is odd}. \end{cases}$$

For $\{a_n\} \in$ MVBVS, we have

$$\text{(9)}\qquad \sum_{\nu=1}^{m}\nu^2|\Delta a_\nu| \le \sum_{j=1}^{[\log m/\log 2]}\sum_{\nu=2^{j-1}}^{2^j}\nu^2|\Delta a_\nu| + \sum_{\nu=[m/2]}^{m}\nu^2|\Delta a_\nu|$$
$$\le C\sum_{j=1}^{[\log m/\log 2]}\sum_{\nu=[\lambda^{-1}2^{j-1}]}^{[\lambda 2^{j-1}]}\nu a_\nu$$
$$+ C\sum_{\nu=[m^*/(2\lambda)]}^{[\lambda m^*/2]}\nu a_\nu + m^2(a_m + a_{m+1})$$
$$\le C\sum_{\nu=1}^{[\lambda m/2]}\nu a_\nu + m^2(a_m + a_{m+1}).$$

By applying (4) with $\beta_m = m^{p\gamma-2}$, $\alpha_\nu = \nu a_\nu$ for $\nu < n$, and $\alpha_\nu = 0$ for $\nu \ge n$, we obtain (note that $p\gamma - 2 < -1$)

$$\text{(10)}\quad \sum_{m=1}^{n-1} m^{p\gamma-2}\Big(\sum_{\nu=1}^{m}\nu a_\nu\Big)^p \le \sum_{m=1}^{\infty}\beta_m\Big(\sum_{\nu=1}^{m}\alpha_\nu\Big)^p$$
$$\le p^p\sum_{m=1}^{\infty}\beta_m^{1-p}\Big(\sum_{\nu=m}^{\infty}\beta_\nu\Big)^p\alpha_m^p \le C\sum_{\nu=1}^{n-1}\nu^{2p+p\gamma-2}a_\nu^p.$$

By applying (5) with $\beta_m = m^{p+p\gamma-2}$, $\alpha_\nu = a_\nu$ for $\nu < n$, and $\alpha_\nu = 0$ for $\nu \ge n$, we deduce that (note that $p + p\gamma - 2 > -1$)

$$\text{(11)}\qquad \sum_{\lambda m/2 \le n-1} m^{p\gamma-2}\Big(\sum_{\nu=m+1}^{[\lambda m/2]}\nu a_\nu\Big)^p \le C\sum_{\lambda m/2 \le n-1} m^{p+p\gamma-2}\Big(\sum_{\nu=m}^{[\lambda m/2]}a_\nu\Big)^p$$
$$\le C\sum_{m=1}^{n-1} m^{p+p\gamma-2}\Big(\sum_{\nu=m}^{n-1}a_\nu\Big)^p \le C\sum_{\nu=1}^{n-1}\nu^{2p+p\gamma-2}a_\nu^p.$$

Similarly,

$$\sum_{n<\lambda m/2\le\lambda(n-1)/2} m^{p\gamma-2}\Big(\sum_{\nu=m+1}^{n-1}\nu a_\nu\Big)^p \le C\sum_{n<\lambda m/2\le\lambda(n-1)/2} m^{p+p\gamma-2}\Big(\sum_{\nu=m}^{n-1}a_\nu\Big)^p\le C\sum_{\nu=1}^{n-1}\nu^{2p+p\gamma-2}a_\nu^p. \tag{12}$$

Also, by using Hölder's inequality, we get

$$n^{-p}\sum_{n<\lambda m/2\le\lambda(n-1)/2} m^{p+p\gamma-2}\Big(\sum_{\nu=n}^{[\lambda m/2]}a_\nu\Big)^p\le Cn^{p\gamma-1}\Big(\sum_{\nu=n}^{[\lambda n/2]}a_\nu\Big)^p \le Cn^{p+p\gamma-2}\sum_{\nu=n}^{[\lambda n/2]}a_\nu^p\le C\sum_{\nu=n}^{\infty}\nu^{p+p\gamma-2}a_\nu^p. \tag{13}$$

Combining (10)–(13) gives

$$\begin{aligned} n^{-p}\sum_{m=1}^{n-1}m^{p\gamma-2}\Big(\sum_{\nu=1}^{[\lambda m/2]}\nu a_\nu\Big)^p &\le n^{-p}\sum_{m=1}^{n-1}m^{p\gamma-2}\Big(\sum_{\nu=1}^{m}\nu a_\nu\Big)^p \\ &\quad + n^{-p}\sum_{\lambda m/2\le n-1}m^{p\gamma-2}\Big(\sum_{\nu=m+1}^{[\lambda m/2]}\nu a_\nu\Big)^p \\ &\quad + n^{-p}\sum_{n<\lambda m/2\le\lambda(n-1)/2}m^{p\gamma-2}\Big(\sum_{\nu=m+1}^{n-1}\nu a_\nu\Big)^p \\ &\quad + n^{-p}\sum_{n<\lambda m/2\le\lambda(n-1)/2}m^{p+p\gamma-2}\Big(\sum_{\nu=n}^{[\lambda m/2]}a_\nu\Big)^p \\ &\le Cn^{-p}\sum_{\nu=1}^{n-1}\nu^{2p+p\gamma-2}a_\nu^p+C\sum_{\nu=n}^{\infty}\nu^{p+p\gamma-2}a_\nu^p. \end{aligned} \tag{14}$$

Finally, we estimate

$$n^{-p}\sum_{m=1}^{n-1}m^{p\gamma-2}(m^2(a_m+a_{m+1}))^p \tag{15}$$

$$\le 2n^{-p}\sum_{m=1}^{n-1}m^{2p+p\gamma-2}a_m^p+n^{p+p\gamma-2}a_n^p\le 2n^{-p}\sum_{\nu=1}^{n-1}\nu^{2p+p\gamma-2}a_\nu^p+\sum_{\nu=n}^{\infty}\nu^{p+p\gamma-2}a_\nu^p.$$

We complete the proof of Lemma 3 by combining (9), (14), and (15).

LEMMA 4. *Let* $1 < p < \infty$ *and* $\{a_n\} \in$ MVBVS. *Then for* $1/p - 1 < \gamma < 1/p$,

$$n^{-p}\sum_{m=1}^{n-1} m^{p+p\gamma-2}\Big(\sum_{\nu=m+1}^{n}\nu|\Delta a_\nu|\Big)^p \le Cn^{-p}\sum_{\nu=1}^{n-1}\nu^{2p+p\gamma-2}a_\nu^p + C\sum_{\nu=n}^{\infty}\nu^{p+p\gamma-2}a_\nu^p. \tag{16}$$

Proof. In a similar way to the proof of (8), we have for $\{a_n\} \in$ MVBVS,

$$a_n + a_{n+1} \le C\sum_{\nu=[n/(2\lambda)]}^{[\lambda(n+1)]}\frac{a_\nu}{\nu+1} \le Cn^{-1}\sum_{\nu=[n/(2\lambda)]}^{[\lambda(n+1)]}a_\nu.$$

Similar to the proof of Lemma 3, we obtain

$$\begin{aligned}\sum_{\nu=m+1}^{n}\nu|\Delta a_\nu| &\le \sum_{j=[\log(m+1)/\log 2]}^{[\log n/\log 2]-1} 2^j\sum_{\nu=2^j}^{2^{j+1}}|\Delta a_\nu| + \sum_{\nu=[n^*/2]}^{n^*}\nu|\Delta a_\nu| + n(a_n + a_{n+1})\\ &\le C\sum_{\nu=[(m+1)/(2\lambda)]}^{[\lambda n/2]}a_\nu + C\sum_{\nu=[(n-1)/(2\lambda)]}^{[\lambda(n+1)]}a_\nu\\ &\le C\sum_{\nu=[m/(2\lambda)]}^{[\lambda n/2]}a_\nu + C\sum_{\nu=[\lambda n/2]+1}^{[\lambda(n+1)]}a_\nu \le C\sum_{\nu=[m/(2\lambda)]}^{[\lambda(n+1)]}a_\nu.\end{aligned}$$

So we can split the left-hand side of the inequality in (16) into

$$\begin{aligned}&n^{-p}\sum_{m=1}^{n-1} m^{p+p\gamma-2}\Big(\sum_{\nu=m+1}^{n}\nu|\Delta a_\nu|\Big)^p\\ &\le Cn^{-p}\sum_{m=1}^{n-1} m^{p+p\gamma-2}\Big(\sum_{\nu=[m/(2\lambda)]}^{m}a_\nu\Big)^p + Cn^{-p}\sum_{m=1}^{n-1} m^{p+p\gamma-2}\Big(\sum_{\nu=m+1}^{n-1}a_\nu\Big)^p\\ &\quad + Cn^{-p}\sum_{m=1}^{n-1} m^{p+p\gamma-2}\Big(\sum_{\nu=n}^{[\lambda(n+1)]}a_\nu\Big)^p =: I_1 + I_2 + I_3.\end{aligned} \tag{17}$$

Evidently,

$$I_3 \le Cn^{p+p\gamma-2}\sum_{\nu=n}^{[\lambda(n+1)]}a_\nu^p \le C\sum_{\nu=n}^{\infty}\nu^{p+p\gamma-2}a_\nu^p. \tag{18}$$

From the proof of (11) of Lemma 3,

$$I_2 \le Cn^{-p} \sum_{\nu=1}^{n-1} \nu^{2p+p\gamma-2} a_\nu^p. \tag{19}$$

For I_1, we have

$$\begin{aligned} I_1 &\le Cn^{-p} \sum_{m=1}^{[2\lambda]} m^{p+p\gamma-2} \Big(\sum_{\nu=[m/(2\lambda)]}^{m} a_\nu \Big)^p \\ &\quad + Cn^{-p} \sum_{m=[2\lambda]+1}^{n-1} ([m/(2\lambda)])^{p+p\gamma-2} \Big(\sum_{\nu=[m/(2\lambda)]}^{n-1} a_\nu \Big)^p \\ &\le Cn^{-p} \sum_{\nu=1}^{n-1} \nu^{2p+p\gamma-2} a_\nu^p + CI_2 \le C \sum_{\nu=n}^{\infty} \nu^{p+p\gamma-2} a_\nu^p. \end{aligned} \tag{20}$$

Combining (17)–(20), we obtain (16).

Now we are ready to prove Theorem 2.

Proof of Theorem 2. We prove the theorem for the case when $\phi(x) = f(x)$. The case when $\phi(x) = g(x)$ can be proved similarly. Let $h = \pi/(2n)$. Since f is an even function, it is clear that

$$\begin{aligned} \omega(f,h)_{p,\gamma} &\le C \sup_{0<t\le h} \Big(\Big\{ \int_0^{\pi/n} x^{-p\gamma} |f(x\pm t) - f(x)|^p \, dx \Big\}^{1/p} \\ &\quad + \Big\{ \int_{\pi/n}^{\pi} x^{-p\gamma} |f(x\pm t) - f(x)|^p \, dx \Big\}^{1/p} \Big) =: C \sup_{0<t\le h} (J_1 + J_2). \end{aligned}$$

By Minkowski's inequality,

$$\begin{aligned} \frac{1}{2} J_1 \le{}& \Big(\int_0^{\pi/n} x^{-p\gamma} \Big| \sum_{\nu=1}^{n-1} a_\nu \sin \frac{1}{2}\nu t \sin\Big(x \pm \frac{1}{2}t\Big) \Big|^p \, dx \Big)^{1/p} \\ &+ \Big(\int_0^{\pi/n} x^{-p\gamma} \Big| \sum_{\nu=n}^{\infty} a_\nu [\cos \nu(x\pm t) - \cos \nu x] \Big|^p \, dx \Big)^{1/p} \\ \le{}& t \Big\{ \int_0^{\pi/n} x^{-p\gamma} \Big(\sum_{\nu=1}^{n-1} \nu a_\nu \Big)^p \, dx \Big\}^{1/p} \\ &+ C \Big\{ \sum_{m=n}^{\infty} \int_{3\pi/(2(m+1))}^{3\pi/(2m)} x^{-p\gamma} \Big| \sum_{\nu=n}^{\infty} a_\nu \cos \nu x \Big|^p \, dx \Big\}^{1/p} \\ =:{}& J_{11} + J_{12}. \end{aligned}$$

By Hölder's inequality, we have (note that $2p + p\gamma - 2 > p - 1 > 0$)

$$(21)\quad \begin{aligned} J_{11} &\le Cn^{-1-1/p+\gamma} \sum_{\nu=1}^{n-1} \nu a_\nu \\ &\le Cn^{-1-1/p+\gamma} \Big(\sum_{\nu=1}^{n-1} \nu^{2p+p\gamma-2} a_\nu^p \Big)^{1/p} \Big(\sum_{\nu=1}^{n-1} \nu^{\frac{(-1-\gamma+2/p)p}{p-1}} \Big)^{(p-1)/p} \\ &\le Cn^{-1} \Big(\sum_{\nu=1}^{n-1} \nu^{2p+p\gamma-2} a_\nu^p \Big)^{1/p}. \end{aligned}$$

The condition that $\{\lambda_n\}$ satisfies (1), i.e., $\{a_\nu\}$ satisfies (1), implies that $\lim_{\nu\to\infty} a_\nu = 0$. By Abel's transformation, with the same argument as in the proof of (6), we obtain

$$\Big| \sum_{\nu=n}^{\infty} a_\nu \cos \nu x \Big| \le \sum_{\nu=n}^{m} a_\nu + (m+1) \sum_{\nu=m+1}^{\infty} |\Delta a_\nu| \le \sum_{\nu=n}^{m} a_\nu + m \sum_{\nu=[\lambda^{-1}(m+1)]}^{\infty} \frac{a_\nu}{\nu+1}.$$

Set $\beta_m = m^{p\gamma-2}$, $\alpha_\nu = 0$ for $\nu < n$, and $\alpha_\nu = a_\nu$ for $\nu \ge n$. Then by (4),

$$(22)\qquad \sum_{m=n}^{\infty} m^{p\gamma-2} \Big(\sum_{\nu=n}^{m} a_\nu \Big)^p \le C \sum_{m=n}^{\infty} m^{p+p\gamma-2} a_m^p.$$

Note that

$$(23)\quad \begin{aligned} &\sum_{m=n}^{\infty} m^{p+p\gamma-2} \Big(\sum_{\nu=[\lambda^{-1}(m+1)]}^{\infty} \frac{a_\nu}{\nu} \Big)^p \\ &\le \sum_{n\le m\le \lambda n-1} m^{p+p\gamma-2} \Big(\sum_{\nu=[\lambda^{-1}(m+1)]}^{m} \frac{a_\nu}{\nu} \Big)^p \\ &\quad + \sum_{n\le m\le \lambda n-1} m^{p+p\gamma-2} \Big(\sum_{\nu=m}^{\infty} \frac{a_\nu}{\nu} \Big)^p + \sum_{m\ge \lambda n} m^{p+p\gamma-2} \Big(\sum_{\nu=[\lambda^{-1}(m+1)]}^{\infty} \frac{a_\nu}{\nu} \Big)^p \\ &\le Cn^{p\gamma-1} \Big(\sum_{\nu=[\lambda^{-1}(n+1)]}^{[\lambda n]} a_\nu \Big)^p + C \sum_{m=n}^{\infty} m^{p+p\gamma-2} \Big(\sum_{\nu=m}^{\infty} \frac{a_\nu}{\nu} \Big)^p \\ &\le Cn^{-p} \sum_{\nu=[\lambda^{-1}(n+1)]}^{[\lambda n]} \nu^{2p+p\gamma-2} a_\nu^p + C \sum_{m=n}^{\infty} m^{p+p\gamma-2} \Big(\sum_{\nu=m}^{\infty} \frac{a_\nu}{\nu} \Big)^p \end{aligned}$$

(by Hölder's inequality)

$$\leq Cn^{-p}\sum_{\nu=1}^{n-1}\nu^{2p+p\gamma-2}a_\nu^p + C\sum_{\nu=n}^{\infty}\nu^{p+p\gamma-2}a_\nu^p + C\sum_{m=n}^{\infty}m^{p+p\gamma-2}\Big(\sum_{\nu=m}^{\infty}\frac{a_\nu}{\nu}\Big)^p$$
$$\leq Cn^{-p}\sum_{\nu=1}^{n-1}\nu^{2p+p\gamma-2}a_\nu^p + C\sum_{\nu=n}^{\infty}\nu^{p+p\gamma-2}a_\nu^p,$$

where in the last inequality, we have used the following inequality:

$$\sum_{m=n}^{\infty}m^{p+p\gamma-2}\Big(\sum_{\nu=m}^{\infty}\frac{a_\nu}{\nu}\Big)^p \leq \sum_{m=1}^{\infty}\beta_m\Big(\sum_{\nu=m}^{\infty}\alpha_\nu\Big)^p \leq C\sum_{\nu=n}^{\infty}\nu^{p+p\gamma-2}a_\nu^p,$$

which can be deduced from (5) by taking $\beta_m = m^{p+p\gamma-2}$, $\alpha_\nu = 0$ for $\nu < n$, $\alpha_\nu = a_\nu/\nu$ for $\nu \geq n$. Thus, it follows from (22) and (23) that

$$J_{12} \leq Cn^{-1}\Big(\sum_{k=1}^{n-1}\nu^{2p+p\gamma-2}a_\nu^p\Big)^{1/p} + C\Big(\sum_{k=n}^{\infty}\nu^{p+p\gamma-2}a_\nu^p\Big)^{1/p}. \tag{24}$$

Denote by $D_k(x)$ the Dirichlet kernel of order k. Following Leindler [6], we have

$$\begin{aligned} J_2 \leq\ & \Big\{\int_{\pi/n}^{\pi} x^{-p\gamma}\Big|\sum_{\nu=1}^{n}\Delta a_\nu(D_\nu(x\pm t)-D_\nu(x))\Big|^p dx\Big\}^{1/p} \\ & + \Big\{\int_{\pi/n}^{\pi} x^{-p\gamma}\Big|\sum_{\nu=n+1}^{\infty}\Delta a_\nu(D_\nu(x\pm t)-D_\nu(x))\Big|^p dx\Big\}^{1/p} \\ =:\ & J_{21}+J_{22}. \end{aligned}$$

Now

$$\begin{aligned} (J_{21})^p &\leq C\sum_{m=1}^{n-1}\int_{\pi/(m+1)}^{\pi/m} x^{-p\gamma}\sum_{\nu=1}^{n}|\Delta a_\nu(D_\nu(x\pm t)-D_\nu(x))|^p\,dx \\ &\leq Ct^p\Big\{\sum_{m=1}^{n-1}m^{p\gamma-2}\Big(\sum_{\nu=1}^{m}\nu^2|\Delta a_\nu|\Big)^p + \sum_{m=1}^{n-1}m^{p\gamma+p-2}\Big(\sum_{\nu=m+1}^{n}\nu|\Delta a_\nu|\Big)^p\Big\}. \end{aligned}$$

Hence, it follows from Lemmas 3 and 4 that

$$J_{21} \leq Cn^{-1}\Big(\sum_{k=1}^{n-1}\nu^{2p+p\gamma-2}a_\nu^p\Big)^{1/p} + C\Big(\sum_{k=n}^{\infty}\nu^{p+p\gamma-2}a_\nu^p\Big)^{1/p}. \tag{25}$$

For J_{22}, we have

$$J_{22} \le \Big\{ \int_{\pi/(2n)}^{\pi+\pi/(2n)} \Big| \sum_{\nu=n+1}^{\infty} |\Delta a_\nu|\,|D_\nu(x)| \Big|^p dx \Big\}^{1/p}$$
$$\le C \Big| \sum_{\nu=n+1}^{\infty} |\Delta a_\nu| \Big|^p \Big\{ \int_{\pi/(2n)}^{\infty} x^{-p-p\gamma}\, dx \Big\}^{1/p}$$
$$\le C n^{1+\gamma-1/p} \sum_{k=[\lambda^{-1}(n+1)]}^{\infty} \frac{a_k}{k} \quad \text{(by (6))}$$
$$\le C n^{1+\gamma-1/p} \sum_{k=[\lambda^{-1}(n+1)]}^{n-1} \frac{a_k}{k} + C n^{1+\gamma-1/p} \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

By Hölder's inequality,

$$n^{1+\gamma-1/p} \sum_{k=[\lambda^{-1}(n+1)]}^{n-1} \frac{a_k}{k} \le n^{1+\gamma-1/p} \Big(\sum_{k=[\lambda^{-1}(n+1)]}^{n-1} k^{2p+p\gamma-2} a_k^p \Big)^{1/p}$$
$$\times \Big(\sum_{k=[\lambda^{-1}(n+1)]}^{n-1} k^{\frac{(-3-\gamma+2/p)p}{p-1}} \Big)^{(p-1)/p}$$
$$\le C n^{-1} \Big(\sum_{\nu=1}^{n-1} \nu^{2p+p\gamma-2} a_\nu^p \Big)^{1/p},$$

and

$$n^{1+\gamma-1/p} \sum_{k=n}^{\infty} \frac{a_k}{k} \le n^{1+\gamma-1/p} \Big(\sum_{k=n}^{\infty} k^{p+p\gamma-2} a_k^p \Big)^{1/p} \Big(\sum_{k=n}^{\infty} k^{\frac{(-2-\gamma+2/p)p}{p-1}} \Big)^{(p-1)/p}$$
$$\le C \Big(\sum_{\nu=n}^{\infty} \nu^{p+p\gamma-2} a_\nu^p \Big)^{1/p}.$$

Therefore,

$$J_{22} \le C n^{-1} \Big(\sum_{\nu=1}^{n-1} \nu^{2p+p\gamma-2} a_\nu^p \Big)^{1/p} + C \Big(\sum_{\nu=n}^{\infty} \nu^{p+p\gamma-2} a_\nu^p \Big)^{1/p}. \tag{26}$$

Combining (21)–(26), we complete the proof of Theorem 2.

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Received November 10, 2006
Revised version June 14, 2007 (6043)