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On the norm of a projection onto the space of compact operators

by

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Abstract. Let X and Y be Banach spaces and let $\mathcal{A}(X,Y)$ be a closed subspace of $\mathcal{L}(X,Y)$, the Banach space of bounded linear operators from X to Y, containing the subspace $\mathcal{K}(X,Y)$ of compact operators. We prove that if Y has the metric compact approximation property and a certain geometric property $M^*(a,B,c)$, where $a,c \geq 0$ and B is a compact set of scalars (Kalton's property $(M^*) = M^*(1,\{-1\},1)$), and if $\mathcal{A}(X,Y) \neq \mathcal{K}(X,Y)$, then there is no projection from $\mathcal{A}(X,Y)$ onto $\mathcal{K}(X,Y)$ with norm less than $\max|B| + c$. Since, for given λ with $1 < \lambda < 2$, every Y with separable dual can be equivalently renormed to satisfy $M^*(a,B,c)$ with $\max|B| + c = \lambda$, this implies and improves a theorem due to Saphar.

1. Introduction. Let X and Y be Banach spaces over the same, either real or complex, field $\mathbb{K}$. We denote by $\mathcal{L}(X,Y)$ the Banach space of bounded linear operators from X to Y and by $\mathcal{K}(X,Y)$ its subspace of compact operators.

A classical long-standing open question is the following (see, e.g., [11], [13], [14] for results and references): is it true that either $\mathcal{L}(X,Y) = \mathcal{K}(X,Y)$, or there is no bounded linear projection from $\mathcal{L}(X,Y)$ onto $\mathcal{K}(X,Y)$?

The answer is positive in some special cases; for instance, when Y has an unconditional basis, as was proven by Tong and Wilken [26] already in 1971. In 1999, the following result was established by Saphar [25].

THEOREM (Saphar). *Let Y be a real Banach space whose dual space Y^* is separable and has the approximation property. If λ is a scalar with $1 < \lambda < 2$, then Y can be equivalently renormed so that, for any real Banach space X with $\mathcal{L}(X,Y) \neq \mathcal{K}(X,Y)$, there is no projection from $\mathcal{L}(X,Y)$ onto $\mathcal{K}(X,Y)$ with norm less than λ.*

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Throughout this paper, $B \subset \mathbb{K}$ will be a compact set and $a, c \geq 0$. We say that a Banach space Y has *property* $M^*(a, B, c)$ if

$$\limsup_{\nu} \|ay_\nu^* + by^* + cz^*\| \leq \limsup_{\nu} \|y_\nu^*\| \quad \forall b \in B$$

whenever (y_ν^*) is a bounded net converging weak* to y^* in Y^* and $\|z^*\| \leq \|y^*\|$. Property $M^*(a, B, c)$ was introduced and studied in [22] (see also [21]) by the second-named author.

The following theorem is the main result of this paper.

THEOREM 1. *Let Y be a Banach space satisfying property $M^*(a, B, c)$ with* $\max|B| + c > 1$ *and having the metric compact approximation property. Let X be a Banach space and let $\mathcal{A}(X, Y)$ be a closed subspace of $\mathcal{L}(X, Y)$ containing $\mathcal{K}(X, Y)$. If $\mathcal{A}(X, Y) \neq \mathcal{K}(X, Y)$, then there is no projection from $\mathcal{A}(X, Y)$ onto $\mathcal{K}(X, Y)$ with norm less than* $\max|B| + c$.

In [22, Proposition 1.2], it is proved, relying on Zippin's theorem [29], that if $0 < r < 1$, then any Banach space Y with separable dual can be equivalently renormed to have property $M^*(1, B, 0)$ with $B = \{b : |b + 1| \leq r\}$. In this case, $\max|B| + c = 1 + r$.

On the other hand, it is a well-known result, due to Grothendieck [7], that the approximation property of separable Y^* implies the metric approximation property of Y^*, which in turn implies the metric approximation property of Y. Therefore Theorem 1 implies Saphar's theorem, giving also some insight into it.

Let us fix more notation and terminology. We denote the unit sphere of a Banach space X by S_X and the closed unit ball by B_X. A Banach space X is considered, without special notation, as a subspace of its bidual X^{**}. We denote by I_X the identity operator on X.

A net (K_α) of finite-rank operators on X is called an *approximation of the identity* provided $K_\alpha \to I_X$ strongly (i.e. $K_\alpha x \to x$ for any $x \in X$). If the operators K_α are allowed to be compact, then (K_α) is called a *compact approximation of the identity*. If moreover, $K_\alpha^* \to I_{X^*}$ strongly, then (K_α) is called *shrinking* (this notion can be regarded as a generalization of shrinking bases).

If there is an approximation of the identity (respectively, compact approximation of the identity) (K_α) with $\sup\|K_\alpha\| \leq 1$, then X is said to have the *metric approximation property* (respectively, the *metric compact approximation property*). In this case, we shall say that (K_α) is a *metric approximation of the identity* (respectively, *metric compact approximation of the identity*). If, moreover, (K_α) happens to be shrinking, then X^* is said to have the *metric approximation property with conjugate operators* (respectively, the *metric compact approximation property with conjugate operators*).

2. Proof of Theorem 1. We shall develop ideas also used in the proof of Saphar's theorem [25]. However, unlike in [25], we shall not apply the method of generalized Godun sets. Instead, we shall rely on the following result established in [22]. Moreover, results from [3] and [18] enable us to consider the more general situation of the metric compact approximation property instead of the metric approximation property.

THEOREM 2 (see [22, Theorem 3.5]). *Suppose that* $\max|B|+c>1$. *Then the following assertions are equivalent for a Banach space* Y.

1° Y *has the metric* (*respectively*, *the metric compact*) *approximation property and property* $M^*(a,B,c)$.

2° *For any* $S\in B_{\mathcal{K}(Y,Y)}$, *there exists a shrinking metric* (*respectively*, *a shrinking metric compact*) *approximation of the identity* (K_α) *satisfying*

$$\limsup_{\alpha}\|aI_Y+bK_\alpha+cS\|\le 1\quad \forall b\in B.$$

Let X be a Banach space and V a subspace of X^*. Recall that the *characteristic* $r(V)$ of V is defined by

$$r(V)=\inf_{x\in S_X}\sup_{x^*\in B_V}|x^*(x)|$$

(cf. [2, Theorem 7]). Obviously $r(V)\le 1$. On the other hand, if $V=\ker f$ for some $f\in X^{**}$, then we have the following estimate from below, which is probably known.

LEMMA 3. *Let* X *be a Banach space*, $f\in X^{**}$, *and* $\varrho\ge 0$. *Suppose that* $\|x+\lambda f\|_{X^{**}}\ge\varrho$ *for all* $x\in S_X$ *and* $\lambda\in\mathbb{K}$. *Then* $r(\ker f)\ge\varrho$.

Proof. Set $V=\ker f$. Using the canonical identification $V^*=X^{**}/V^\perp$, we have, for all $x\in S_X\subset X^{**}$,

$$\sup_{x^*\in B_V}|x^*(x)|=\|x|_V\|_{V^*}=\|x+V^\perp\|_{X^{**}/V^\perp}.$$

But $V^\perp=\operatorname{span}\{f\}$, since $V=\operatorname{span}\{f\}^\perp$. Therefore

$$\sup_{x^*\in B_V}|x^*(x)|=\inf_{\lambda\in\mathbb{K}}\|x+\lambda f\|\ge\varrho$$

and hence also $r(V)\ge\varrho$. ∎

Proof of Theorem 1. First note that, according to Theorem 2 (take, e.g., $S=0$ in 2°), Y^* has the metric compact approximation property with conjugate operators and, by [22, Corollary 1.6], the Radon–Nikodým property.

Let us consider the trace mapping τ from the projective tensor product $X^{**}\mathbin{\widehat{\otimes}}Y^*$ to $(\mathcal{K}(X,Y))^*$, defined by

$$(\tau v)(S)=\operatorname{trace}(S^{**}v),\quad v\in X^{**}\mathbin{\widehat{\otimes}}Y^*,\ S\in\mathcal{K}(X,Y);$$

that is, if $v=\sum_{n=1}^{\infty} x_n^{**}\otimes y_n^*$, then

$$(\tau v)(S)=\sum_{n=1}^{\infty}(S^{**}x_n^{**})(y_n^*)=\sum_{n=1}^{\infty}x_n^{**}(S^*y_n^*),\quad S\in\mathcal{K}(X,Y).$$

Since Y^* has the metric compact approximation property with conjugate operators, by [18, Theorem 3.8, (a)⇒(b′)], there exists an into isometry $U:\mathcal{L}(X,Y^{**})\to(\mathcal{K}(X,Y))^{**}$ such that $\tau^*(U(T))=T^{**}$ for all $T\in\mathcal{L}(X,Y)$, and, moreover, $U(S)=S$ for all $S\in\mathcal{K}(X,Y)$.

As Y^* has the Radon–Nikodým property, by the description of $(\mathcal{K}(X,Y))^*$ due to Feder and Saphar [3, Theorem 1], τ is a quotient mapping; more precisely, for all $\varphi\in(\mathcal{K}(X,Y))^*$ there exists $v\in X^{**}\widehat{\otimes}Y^*$ such that $\varphi=\tau v$ and $\|\varphi\|=\|v\|_\pi$.

Finally, recall (see, e.g., [1, p. 230] or [24, p. 24]) that $(X^{**}\widehat{\otimes}Y^*)^*$ and $\mathcal{L}(X^{**},Y^{**})$ are canonically isometrically isomorphic under the duality

$$\langle v,A\rangle=\operatorname{trace}(Av)=\sum_{n=1}^{\infty}(Ax_n^{**})(y_n^*),$$
$$v=\sum_{n=1}^{\infty}x_n^{**}\otimes y_n^*\in X^{**}\widehat{\otimes}Y^*,\ A\in\mathcal{L}(X^{**},Y^{**}).$$

Therefore, for all $v\in X^{**}\widehat{\otimes}Y^*$,

$$\langle v,T^{**}\rangle=\langle v,\tau^*(U(T))\rangle=(U(T))(\tau v),\qquad T\in\mathcal{L}(X,Y),$$

and

$$\langle v,S^{**}\rangle=(\tau v)(S),\qquad S\in\mathcal{K}(X,Y).$$

Let now P be a bounded linear projection from $\mathcal{A}(X,Y)$ onto $\mathcal{K}(X,Y)$. Then $\ker P\neq\{0\}$, since $\mathcal{A}(X,Y)\neq\mathcal{K}(X,Y)$. To show that

$$\|P\|\geq\max|B|+c,$$

let $T\in\ker P$ with $\|T\|=1$. Define

$$V=\ker(U(T))\subset(\mathcal{K}(X,Y))^*.$$

Since for all $\varphi\in(\mathcal{K}(X,Y))^*$ there exists $v\in X^{**}\widehat{\otimes}Y^*$ such that $\varphi=\tau v$, we have

$$V=\{\tau v: v\in X^{**}\widehat{\otimes}Y^*,\ \langle v,T^{**}\rangle=0\}.$$

Obviously, $S=P(S+\lambda T)$ for all $S\in S_{\mathcal{K}(X,Y)}$ and $\lambda\in\mathbb{K}$. Therefore

$$1/\|P\|\leq\|S+\lambda T\|=\|U(S+\lambda T)\|=\|S+\lambda U(T)\|\quad\forall S\in S_{\mathcal{K}(X,Y)},\ \forall\lambda\in\mathbb{K}.$$

Hence, by Lemma 3,

$$1/\|P\|\leq r(V).$$

So, to complete the proof, we need to show that

$$r(V) \le \frac{1}{\max|B| + c}.$$

Let $|b| = \max|B| = b \operatorname{sign} b$ for some $b \in B$ and let $0 < \varepsilon < 1$. (Here $\operatorname{sign} b$ is defined to be $|b|/b$ if $b \neq 0$ and 1 if $b = 0$.) First, choose $x \in B_X$ such that $\|Tx\| \ge \varepsilon$. Then choose $y^* \in Y^*$ such that $\|y^*\| = 1/\|Tx\|$ and $y^*(Tx) = 1$. Now consider the rank one operator

$$S = \frac{1}{\operatorname{sign} b}\, y^* \otimes Tx.$$

Obviously, $S \in S_{\mathcal{K}(Y,Y)}$. Notice that

$$\begin{aligned}\|bT + cST\| \ge \|bTx + cSTx\| &= \left\| bTx + \frac{c}{\operatorname{sign} b}\, y^*(Tx)Tx \right\| \\ &= \left| b + \frac{c}{\operatorname{sign} b} \right| \|Tx\| \ge (|b| + c)\varepsilon.\end{aligned}$$

Using Theorem 2 again, we find, for the operator S, a shrinking metric approximation of the identity (K_α) that satisfies

$$\limsup_{\alpha} \|aI_Y + bK_\alpha + cS\| \le 1$$

and therefore also

$$\limsup_{\alpha} \|aT + bK_\alpha T + cST\| \le 1.$$

Since

$$\langle x^{**} \otimes y^*, K_\alpha T\rangle = x^{**}(T^* K_\alpha^* y^*) \underset{\alpha}{\rightarrow} x^{**}(T^* y^*) = \langle x^{**} \otimes y^*, T\rangle$$

for all $x^{**} \in X^{**}$ and $y^* \in Y^*$, and the net $(K_\alpha T)$ is bounded,

$$K_\alpha T \underset{\alpha}{\rightarrow} T$$

in the weak* topology of $\mathcal{L}(X^{**}, Y^{**})$ induced by the duality with $X^{**} \widehat{\otimes} Y^*$.

By the definition of the characteristic, we have, for all α,

$$\begin{aligned} r(V) &\le \sup_{\tau v \in B_V} \left| (\tau v)\left(\frac{bK_\alpha T + cST}{\|bK_\alpha T + cST\|} \right) \right| \\ &= \sup_{\tau v \in B_V} \left| \left\langle v, \frac{bK_\alpha^{**} T^{**} + cS^{**} T^{**}}{\|bK_\alpha T + cST\|} \right\rangle \right| \\ &= \sup_{\tau v \in B_V} \left| \left\langle v, \frac{aT^{**} + bK_\alpha^{**} T^{**} + cS^{**} T^{**}}{\|bK_\alpha T + cST\|} \right\rangle \right| \\ &= \sup_{\tau v \in B_V} \left| \left(U\left(\frac{aT + bK_\alpha T + cST}{\|bK_\alpha T + cST\|} \right) \right)(\tau v) \right| \\ &\le \frac{\|aT + bK_\alpha T + cST\|}{\|bK_\alpha T + cST\|}. \end{aligned}$$

Since $bK_\alpha T + cST \to_\alpha bT + cST$ in the weak* topology of $\mathcal{L}(X^{**}, Y^{**})$, by the weak* lower semicontinuity of conjugate norms, we have

$$\liminf_\alpha \|bK_\alpha T + cST\| \ge \|bT + cST\| \ge (|b| + c)\varepsilon.$$

Therefore

$$r(V) \le \frac{\limsup_\alpha \|aT + bK_\alpha T + cST\|}{\liminf_\alpha \|bK_\alpha T + cST\|} \le \frac{1}{(|b| + c)\varepsilon}.$$

This inequality holds for every positive $\varepsilon < 1$, so $r(V) \le 1/(|b| + c)$ as desired. ∎

REMARK. In the special case when Y^* has the metric approximation property, we need not use [18, Theorem 3.8] and [3, Theorem 1], but some Grothendieck's classics instead. In fact, the proof begins by applying the "metric" part, instead of the "metric compact" part, of Theorem 2. We find that Y^* has the metric approximation property and the Radon–Nikodým property. Therefore, by Grothendieck's classics (see [1, p. 247] or [24, p. 114]), the trace mapping τ is already an isometric isomorphism between $X^{**} \widehat{\otimes} Y^*$ and $(\mathcal{K}(X, Y))^*$. Also, $(\mathcal{K}(X, Y))^{**} = (X^{**} \widehat{\otimes} Y^*)^*$ and $\mathcal{L}(X^{**}, Y^{**})$ are canonically isometrically isomorphic, and $\mathcal{L}(X, Y)$ is canonically embedded in $(\mathcal{K}(X, Y))^{**} = \mathcal{L}(X^{**}, Y^{**})$ under the isometry $T \mapsto T^{**}$.

3. Applications

3.1. *Saphar's theorem.* The following result contains Saphar's theorem (see the Introduction) giving also its extension to the complex case and compact approximation properties.

THEOREM 4. *Let Y be a Banach space whose dual is separable and has the compact approximation property with conjugate operators. If λ is a scalar with $1 < \lambda < 2$, then Y can be equivalently renormed so that, for any Banach space X and for any closed subspace $\mathcal{A}(X, Y)$ of $\mathcal{L}(X, Y)$ containing $\mathcal{K}(X, Y)$ with $\mathcal{A}(X, Y) \ne \mathcal{K}(X, Y)$, there is no projection from $\mathcal{A}(X, Y)$ onto $\mathcal{K}(X, Y)$ with norm less than λ.*

Proof. It is known (this is an extension of Grothendieck's classics) that whenever a dual space has the Radon–Nikodým property (in particular, is separable) and the compact approximation property with conjugate operators, it also has the metric compact approximation property with conjugate operators (see [5, Corollary 1.6 and its proof] or, for an alternative proof, [17, Corollary 5.3]). Therefore, any equivalent renorming of Y has the metric compact approximation property, and the claim follows from [22, Proposition 1.2] and Theorem 1, as was indicated in the Introduction. ∎

Concerning hypotheses of Saphar's theorem and Theorem 4, let us recall that if Y^* has the approximation property, then it has the approximation property with conjugate operators (this is clear from the principle of local reflexivity). By an example due to Grønbæk and Willis [6, Example 4.3], the compact approximation property of Y^* does not imply the compact approximation property with conjugate operators (even if Y^* is separable).

Theorem 4 applies (but Saphar's theorem does not) to the separable reflexive Banach space of Willis [27] which has the metric compact approximation property, but fails the approximation property. There also exists a non-reflexive Banach space Y such that its odd duals $Y^*, Y^{***}, \dots$ are separable and have the compact approximation property with conjugate operators, but fail the approximation property (see [23, Theorem 3.6]).

3.2. *Properties* (M^*) *and* (wM^*). All Banach spaces that have a separable dual can be equivalently renormed to have property $M^*(1, B, 0)$ with $B = \{b : |b+1| \le r\}$ whenever $0 < r < 1$ (see the Introduction). On the other hand, property $M^*(a, B, c)$ does not even imply separability. For example, the spaces $c_0(\Gamma)$ and $\ell_p(\Gamma)$, $1 < p < \infty$, where Γ is an uncountable set, are not separable. But they have property $M^*(1, \{-2\}, 0)$. This is clear from Theorem 2, $2^\circ \Rightarrow 1^\circ$, if one takes finite subsets of Γ, ordered by inclusion, to be the indices α, and the corresponding natural projections to be K_α.

It can easily be seen that $M^*(1, \{-1\}, 1)$ is precisely *property* (M^*) introduced by Kalton [15], $M^*(1, \{-2\}, 0)$ is *property* (wM^*) introduced by Lima [16], and $M^*(1, \{b : |b+1| = 1\}, 0)$ is the *complex version* of (wM^*) (see [22]). It is straightforward to verify that (M^*) implies $M^*(1, \{b : |b+1| \le 1-c\}, c)$ for any $c \in [0, 1]$. In particular, (M^*) implies (wM^*).

As we saw, the spaces $c_0(\Gamma)$ and $\ell_p(\Gamma)$, $1 < p < \infty$, have property (wM^*). In fact, they have property (M^*), but the Lorentz sequence spaces $d(w, p)$ do not (this well-known fact is clear, for example, from [9, Proposition 4.24 and Theorem 4.17, (i)$\Leftrightarrow$(vii)]). On the other hand (as was noticed in [22, p. 2804]), it is straightforward to verify that $d(w, p)$, $1 < p < \infty$, has property $M^*(a, B, c)$ for any fixed $a, c > 0$ such that $a^p + c^p \le 1$ and $B = \{b : |b + a| \le (1 - a^p)^{1/p} - c\}$. The Lorentz sequence spaces $d(w, p)$ and, more generally, Banach spaces with a shrinking 1-unconditional basis enjoy property (wM^*) and, in the case of complex scalars, its complex version (this results from Theorem 2, $2^\circ \Rightarrow 1^\circ$).

An immediate conclusion from Theorem 1 follows.

Corollary 5. *Let X be a Banach space and let Y be a Banach space having the metric compact approximation property and property (wM^*) (its complex version included). Let $\mathcal{A}(X, Y)$ be a closed subspace of $\mathcal{L}(X, Y)$ containing $\mathcal{K}(X, Y)$. If $\mathcal{A}(X, Y) \ne \mathcal{K}(X, Y)$, then there is no projection from $\mathcal{A}(X, Y)$ onto $\mathcal{K}(X, Y)$ with norm less than* 2.

Next, we point out an application to M-ideals of compact operators, a class of compact operators which has been extensively studied by many authors (see, e.g., the monograph [9] for results and references).

Let X be a Banach space and let $\mathcal{A}(X,X)$ be a closed subspace of $\mathcal{L}(X,X)$ containing $\mathcal{K}(X,X)$. Recall that $\mathcal{K}(X,X)$ is an *M-ideal* in $\mathcal{A}(X,X)$ if there exists a projection P on $(\mathcal{A}(X,X))^*$ with $\ker P = (\mathcal{K}(X,X))^\perp = \{f \in (\mathcal{A}(X,X))^* : f|_{\mathcal{K}(X,X)} = 0\}$ such that $\|Pf\| + \|f - Pf\| = \|f\|$ for all $f \in (\mathcal{A}(X,X))^*$.

If $\mathcal{K}(X,X)$ is an M-ideal in $\mathcal{A}(X,X)$ which also contains I_X, then X has the metric compact approximation property (see [8] or [9, p. 299]) and property (M^*) (see [15] and [20], or [9, p. 299]). Therefore the next result is immediate from Corollary 5.

COROLLARY 6. *Let X be a Banach space. Let $\mathcal{A}(X,X)$ be a closed subspace of $\mathcal{L}(X,X)$ containing $\mathcal{K}(X,X)$ and I_X. If $\mathcal{K}(X,X)$ is an M-ideal in $\mathcal{A}(X,X)$, then there is no projection from $\mathcal{A}(X,X)$ onto $\mathcal{K}(X,X)$ with norm less than* 2.

3.3. *Projection constants.* If $\mathcal{K}$ is a closed subspace of a Banach space $\mathcal{A}$, then the *relative projection constant* $\lambda(\mathcal{K},\mathcal{A})$ is defined by

$$\lambda(\mathcal{K},\mathcal{A}) = \inf\{\|P\| : P \text{ is a projection from } \mathcal{A} \text{ onto } \mathcal{K}\}.$$

PROPOSITION 7 (Garling–Gordon [4]). *Let $\mathcal{K}$ be a closed subspace of a Banach space $\mathcal{A}$ and let* $\operatorname{codim}\mathcal{K} = n$ *in $\mathcal{A}$, for some $n \in \mathbb{N}$. Then $\lambda(\mathcal{K},\mathcal{A}) \le \sqrt{n} + 1$.*

See, for example, [28, p. 117] for a proof.

COROLLARY 8. *Let X be a Banach space and let Y be a Banach space having the metric compact approximation property and property (wM^*) (its complex version included). Let $\mathcal{A}(X,Y)$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K}(X,Y)$. If* $\operatorname{codim}\mathcal{K}(X,Y) = n$ *in $\mathcal{A}(X,Y)$, for some $n \in \mathbb{N}$, then $2 \le \lambda(\mathcal{K}(X,Y),\mathcal{A}(X,Y)) \le \sqrt{n} + 1$. In particular, if* $\operatorname{codim}\mathcal{K}(X,Y) = 1$, *then $\lambda(\mathcal{K}(X,Y),\mathcal{A}(X,Y)) = 2$.*

Proof. By Corollary 5, $\|P\| \ge 2$ for any projection P from $\mathcal{A}(X,Y)$ onto $\mathcal{K}(X,Y)$, thus $\lambda(\mathcal{K}(X,Y),\mathcal{A}(X,Y)) \ge 2$. The second inequality is immediate from Proposition 7. ■

In Corollaries 5 and 8, in particular, one may take Y equal to any of the spaces $c_0(\Gamma)$, $\ell_p(\Gamma)$, $1 < p < \infty$, or to any Banach space with a shrinking 1-unconditional basis (like $d(w,p)$, $1 < p < \infty$). The latter include those Banach spaces with a 1-unconditional basis that contain no subspace isomorphic to ℓ_1 (by a well-known result of James [10]; see [19, Theorem 1.c.9]); in particular, all reflexive Banach spaces with a 1-unconditional basis.

Finally, in Corollary 10 below, we see that the projection constant $\lambda(\mathcal{K}(X,Y),\mathcal{A}(X,Y)) = 2$ in Corollary 8 can be attained. To this end, we need the following result, which is surely well known. We present its proof for completeness.

Proposition 9. *Let X be a Banach space and let $\mathcal{A}(X,X)=\mathcal{K}(X,X)\oplus \operatorname{span}\{I_X\}$. If P is the projection from $\mathcal{A}(X,X)$ onto $\mathcal{K}(X,X)$ with $\ker P=\operatorname{span}\{I_X\}$, then $\|P\|\le 2$.*

Proof. We have

$$\|P\|\le \|I_{\mathcal{A}(X,X)}-P\|+\|I_{\mathcal{A}(X,X)}\|=1+\|I_{\mathcal{A}(X,X)}-P\|.$$

Since $\|K+I_X\|\ge 1$ for every $K\in\mathcal{K}(X,X)$ (otherwise K would be invertible), it follows that $\|K+\lambda I_X\|\ge|\lambda|$ for every $K\in\mathcal{K}(X,X)$ and $\lambda\in\mathbb{K}$. Therefore,

$$\begin{aligned}\|I_{\mathcal{A}(X,X)}-P\| &= \sup_{\|K+\lambda I_X\|=1}\|(I_{\mathcal{A}(X,X)}-P)(K+\lambda I_X)\|\\ &= \sup_{\|K+\lambda I_X\|=1}\|\lambda I_X\|\le 1. \ \blacksquare\end{aligned}$$

Corollary 10. *Let X be a Banach space having the metric compact approximation property and property (wM^*) (its complex version included). If P is the projection from $\mathcal{K}(X,X)\oplus\operatorname{span}\{I_X\}$ onto $\mathcal{K}(X,X)$ with $\ker P=\operatorname{span}\{I_X\}$, then $\|P\|=2$.*

Proof. This is immediate from Corollary 8 and Proposition 9. ■

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$$\limsup_{\alpha}\|I_Y-\lambda K_\alpha\|\le 1.$$

References

[1] J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, RI, 1977.

[2] J. Dixmier, *Sur un théorème de Banach*, Duke Math. J. 15 (1948), 1057–1071.

[3] M. Feder and P. D. Saphar, *Spaces of compact operators and their dual spaces*, Israel J. Math. 21 (1975), 38–49.

[4] D. J. H. Garling and Y. Gordon, *Relations between some constants associated with finite dimensional Banach spaces*, ibid. 9 (1971), 346–361.

[5] G. Godefroy and P. D. Saphar, *Duality in spaces of operators and smooth norms on Banach spaces*, Illinois J. Math. 32 (1988), 672–695.

[6] N. Grønbæk and G. A. Willis, *Approximate identities in Banach algebras of compact operators*, Canad. Math. Bull. 36 (1993), 45–53.

[7] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[8] P. Harmand and Å. Lima, *Banach spaces which are M-ideals in their biduals*, Trans. Amer. Math. Soc. 283 (1984), 253–264.

[9] P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.
[10] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. 52 (1950), 518–527.
[11] K. John, *On the uncomplemented subspace $K(X,Y)$*, Czechoslovak Math. J. 42 (117) (1992), 167–173.
[12] —, *Projections from $L(X,Y)$ onto $K(X,Y)$*, Comment. Math. Univ. Carolin. 41 (2000), 765–771.
[13] J. Johnson, *Remarks on Banach spaces of compact operators*, J. Funct. Anal. 32 (1979), 304–311.
[14] N. J. Kalton, *Spaces of compact operators*, Math. Ann. 208 (1974), 267–278.
[15] —, *M-ideals of compact operators*, Illinois J. Math. 37 (1993), 147–169.
[16] Å. Lima, *Property (wM^*) and the unconditional metric compact approximation property*, Studia Math. 113 (1995), 249–263.
[17] Å. Lima, O. Nygaard, and E. Oja, *Isometric factorization of weakly compact operators and the approximation property*, Israel J. Math. 119 (2000), 325–348.
[18] Å. Lima and E. Oja, *Metric approximation properties and trace mappings*, Math. Nachr. 280 (2007), 571–580.
[19] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence Spaces*, Springer, Berlin, 1977.
[20] E. Oja, *A note on M-ideals of compact operators*, Tartu Ül. Toimetised 960 (1993), 75–92.
[21] —, *Géométrie des espaces de Banach ayant des approximations de l'identité contractantes*, C. R. Acad. Sci. Paris Sér. I Math. 328 (1999), 1167–1170.
[22] —, *Geometry of Banach spaces having shrinking approximations of the identity*, Trans. Amer. Math. Soc. 352 (2000), 2801–2823.
[23] —, *Lifting bounded approximation properties from Banach spaces to their dual spaces*, J. Math. Anal. Appl. 323 (2006), 666–679.
[24] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer, London, 2002.
[25] P. D. Saphar, *Projections from $L(E,F)$ onto $K(E,F)$*, Proc. Amer. Math. Soc. 127 (1999), 1127–1131.
[26] A. E. Tong and D. R. Wilken, *The uncomplemented subspace $K(E,F)$*, Studia Math. 37 (1971), 227–236.
[27] G. Willis, *The compact approximation property does not imply the approximation property*, Studia Math. 103 (1992), 99–108.
[28] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Stud. Adv. Math. 25, Cambridge Univ. Press, Cambridge, 1991.
[29] M. Zippin, *Banach spaces with separable duals*, Trans. Amer. Math. Soc. 310 (1988), 371–379.

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