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A remark on the topological entropies of covers and partitions

by

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Abstract. We study if the combinatorial entropy of a finite cover can be computed using finite partitions finer than the cover. This relates to an unsolved question in [R] for open covers. We explicitly compute the topological entropy of a fixed clopen cover showing that it is smaller than the infimum of the topological entropy of all finer clopen partitions.

1. Introduction. We consider a topological dynamical system (TDS) (X,T) to be a continuous invertible map $T : X \to X$ on a compact metric space X. In this work we focus our attention on the notion of topological entropy for TDS that was first introduced in [AKMc]. Topological entropy is formally defined for finite open covers but the same definition applies for any finite cover as a combinatorial notion.

Let $\mathcal{C}_X$ be the set of finite covers of X and $\mathcal{U},\mathcal{V} \in \mathcal{C}_X$. Then $\mathcal{U}$ is finer than $\mathcal{V}$ ($\mathcal{U} \succeq \mathcal{V}$) if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subseteq V$. The refinement of $\mathcal{U}$ and $\mathcal{V}$ is defined as $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$.

For $\mathcal{U} \in \mathcal{C}_X$ by standard subadditivity arguments the combinatorial entropy of $\mathcal{U}$ with respect to T exists and is defined as:

$$h(\mathcal{U},T) = \lim_{N\to\infty} \frac{1}{N} \log \mathcal{N}(\mathcal{U}_0^{N-1}) = \inf_{N\in\mathbb{N}} \frac{1}{N} \log \mathcal{N}(\mathcal{U}_0^{N-1}). \tag{1.1}$$

Here $\mathcal{N}(\mathcal{U})$ is the minimal cardinality of a subcover of $\mathcal{U}$ and $\mathcal{U}_0^{N-1} = \bigvee_{n=0}^{N-1} T^{-n}\mathcal{U}$. When $\mathcal{U} \in \mathcal{C}_X$ is a partition (we write $\mathcal{U} \in \mathcal{P}_X$) then $\mathcal{N}(\mathcal{U})$ is just the cardinality $|\mathcal{U}|$, making the computations far easier.

In this work we study if the combinatorial entropy of a finite cover can be computed using finite partitions finer than the cover. This can be stated as:

$$h(\mathcal{U},T) = \inf_{\substack{\alpha\succeq\mathcal{U}\\ \alpha\in\mathcal{P}_X}} h(\alpha,T)\ ? \tag{1.2}$$

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From this moment on we restrict ourselves to a particular class of TDS called shift dynamical systems. Given an alphabet $\mathcal{D}$ of D symbols we consider the space $X_{\mathcal{D}} = \mathcal{D}^{\mathbb{Z}}$ of two-sided sequences $\{x_n\}_{n\in\mathbb{Z}}$ (the *full shift space* over $\mathcal{D}$). This is a compact metric space in the product topology. The shift map $\sigma_{\mathcal{D}} : X_{\mathcal{D}} \to X_{\mathcal{D}}$ defined as $\sigma_{\mathcal{D}}(\{x_n\}_{n\in\mathbb{Z}}) = \{x_{n+1}\}_{n\in\mathbb{Z}}$ is a bicontinuous function. So $(X_{\mathcal{D}}, \sigma_{\mathcal{D}})$ is an invertible 0-dimensional TDS. Here "0-dimensional" means a space with a countable base of open-closed sets. From here on, when no confusion can arise, we shall denote $\sigma_{\mathcal{D}}$ simply by σ.

We do not prove or disprove equation (1.2), but we prove that for the TDS $(X_{\mathcal{D}}, \sigma)$ and the cover $\mathcal{U}_1$ (see (3.4) for definition) we have

$$h(\mathcal{U}_1, \sigma) < \inf_{\substack{\alpha\succeq\mathcal{U} \\ \alpha\in\mathcal{P}^c_{X_{\mathcal{D}}}}} h(\alpha, \sigma), \tag{1.3}$$

where $\mathcal{P}^c_{X_{\mathcal{D}}}$ is the set of finite clopen partitions of $X_{\mathcal{D}}$. In the case of open covers the combinatorial entropy is also called topological entropy and equality (1.2) was stated as an open question in [R]. This question relates, as many others, to an analogous result for measure-theoretical entropy.

The equality (1.2) in question is motivated by a measure-theoretical equality regarding measure-theoretical entropy first defined by Kolmogorov in [K] in the setting of measurable partitions. A local variational principle was established in [BGH]. Following this idea in [R] two notions of measure-theoretical entropy for open covers were developed and the equality between them was left as an open question. A consequence of [HMRY] and another variational principle established in [GW] is that these two notions are one and the same.

More precisely, for any open cover $\mathcal{U}$ and a T-invariant measure μ,

$$h^-_\mu(\mathcal{U}, T) := \inf_{N\in\mathbb{N}} \inf_{\substack{\alpha\succeq\mathcal{U}_0^{N-1} \\ \alpha\in\mathcal{P}_X}} \frac{1}{N} H_\mu(\alpha) = \inf_{\substack{\alpha\succeq\mathcal{U} \\ \alpha\in\mathcal{P}_X}} h_\mu(\alpha, T) =: h^+_\mu(\mathcal{U}, T). \tag{1.4}$$

Here $H_\mu(\alpha) = -\sum_{A\in\alpha} \mu(A)\log\mu(A)$ (with $0\cdot\log 0 = 0$) and $h_\mu(\alpha, T) = \inf_{N\in\mathbb{N}} N^{-1} H_\mu(\alpha_0^{N-1})$.

It is a simple exercise to see that

$$\forall\mathcal{U} \in \mathcal{C}_X, \quad \mathcal{N}(\mathcal{U}) = \min_{\substack{\alpha\succeq\mathcal{U} \\ \alpha\in\mathcal{P}_X}} |\alpha|$$

and thus

$$h(\mathcal{U}, T) = \inf_{N\in\mathbb{N}} \min_{\substack{\alpha\succeq\mathcal{U} \\ \alpha\in\mathcal{P}_X}} \frac{1}{N} \log|\alpha_0^{N-1}|. \tag{1.5}$$

Notice that $h^-_\mu(\mathcal{U}, T)$ is obtained as the combinatorial entropy simply by replacing $\log|\alpha|$ with $H_\mu(\alpha)$ in equation (1.5). A similar replacement in the definition of $h_\mu^+(\mathcal{U},T)$ gives the following combinatorial definition:

$$h^+(\mathcal{U},T) = \inf_{\substack{\alpha\succeq\mathcal{U}\\ \alpha\in\mathcal{P}_X}} \inf_{N\in\mathbb{N}} \frac{1}{N}\log|\alpha_0^{N-1}| = \inf_{\substack{\alpha\succeq\mathcal{U}\\ \alpha\in\mathcal{P}_X}} h(\alpha,T). \tag{1.6}$$

So question (1.2) can be restated as:

$$\forall\mathcal{U}\in\mathcal{C}_X,\qquad h^+(\mathcal{U},T) = h(\mathcal{U},T)\ ? \tag{1.7}$$

Notice that the equalities (1.1) and (1.6) coincide for partitions and $h(\mathcal{U},T)\le h^+(\mathcal{U},T)$.

In the 0-dimensional case, by standard approximation arguments, (1.4) implies that for every clopen cover $\mathcal{U}$,

$$h_\mu^-(\mathcal{U},T) = \inf_{\substack{\alpha\succeq\mathcal{U}\\ \alpha\in\mathcal{P}_X^c}} h_\mu(\alpha,T). \tag{1.8}$$

The main result in this paper proves that the topological equivalent of equation (1.8) is false in general:

THEOREM 1.1. *For any $D\in\mathbb{N}$ and $(X_{\mathcal{D}},\sigma)$ we have*

$$h(\mathcal{U}_1,\sigma) = \log\left(\frac{D}{D-1}\right) < \inf_{\substack{\alpha\succeq\mathcal{U}_1\\ \alpha\in\mathcal{P}_{X_{\mathcal{D}}}^c}} h(\alpha,\sigma) = \log 2,$$

where $\mathcal{U}_1=\{U_1,\dots,U_D\}$ and $U_d=\{\{x_n\}_{n\in\mathbb{Z}}\in X_{\mathcal{D}} : x_0\neq d\}$.

2. Preliminaries. Since entropy is an increasing function on the set of partitions endowed with the partial order $\succeq$, the following topological version of Lemma 3 in [HMRY] will be extremely useful in our calculations. For $\mathcal{U}=\{U_1,\dots,U_M\}\in\mathcal{C}_X$ define

$$\mathcal{U}^* = \{\{A_1,\dots,A_M\}\in\mathcal{P}_X : \forall m\in\{1,\dots,M\},\ A_m\subseteq U_m\}. \tag{2.1}$$

Here A_m can be empty for some values of m and so $|\alpha|\le|\mathcal{U}|$ for $\alpha\in\mathcal{U}^*$.

LEMMA 2.1. *Let (X,T) be a TDS, $M\in\mathbb{N}$ and $\mathcal{U}=\{U_1,\dots,U_M\}\in\mathcal{C}_X$. For every function $G:\mathcal{P}_X\to\mathbb{R}$ such that $\alpha\succeq\beta$ implies that $G(\alpha)\ge G(\beta)$ one has*

$$\inf_{\substack{\alpha\succeq\mathcal{U}\\ \alpha\in\mathcal{P}_X}} G(\alpha) = \inf_{\alpha\in\mathcal{U}^*} G(\alpha).$$

Proof. ($\le$) Just notice that $\alpha\in\mathcal{U}^*$ implies $\alpha\succeq\mathcal{U}$.

($\ge$) Consider $N\in\mathbb{N}$ and $\beta=\{B_1,\dots,B_N\}\succeq\mathcal{U}$. By definition for every $n\in\{1,\dots,N\}$ there exists $j_n\in\{1,\dots,M\}$ such that $B_n\subseteq U_{j_n}$.

There exist $K\le M$ and $\{i_1,\dots,i_K\}$ such that $\{i_1,\dots,i_K\}=\{j_1,\dots,j_N\}$ and $i_k\neq i_l$ if $k\neq l\in\{1,\dots,K\}$.

Define $\alpha = \{A_1, \dots, A_M\}$ as follows:

$$A_{i_k} = \bigcup_{\substack{n\in\{1,\dots,N\}\\ j_n = i_k}} B_n \quad \text{for } k \in \{1, \dots, K\},$$

$$A_m = \emptyset \quad \text{for } m \in \{1, \dots, M\} \setminus \{i_1, \dots, i_K\}.$$

By construction $A_m \subseteq U_m$ for all $m \in \{1, \dots, M\}$ and $\beta \succeq \alpha$ so one has $G(\beta) \geq G(\alpha)$. ■

Remark. Actually this result is stronger since we can assume that if given $\{A_1, \dots, A_M\} \in \mathcal{U}^*$ there exists $k \in \{1, \dots, M\} \setminus \{m\}$ such that $A_m \subseteq U_k$ then we can eliminate A_m and replace A_k by $A_m \cup A_k \subseteq U_k$ obtaining a coarser partition in $\mathcal{U}^*$. However, we will not be needing this stronger result.

A basis for the product topology are cylinder sets. For $M \in \mathbb{N}$ denote by $\mathcal{D}^M$ the set of M-words u on the alphabet $\mathcal{D}$, that is, $u = u_1 \dots u_M$ where $u_m \in \mathcal{D}$ for all $m \in \{1, \dots, M\}$.

The *k-coordinate cylinder set* corresponding to $u \in \mathcal{D}^M$ is defined as

$$[u]_k = \{\{x_n\}_{n\in\mathbb{Z}} \in X_{\mathcal{D}} : x_k \dots x_{M+k-1} = u_1 \dots u_M\}.$$

Given $u, v \in \mathcal{D}^M$ we say that u *overlaps* v ($u \preceq v$) iff $u_2 \dots u_M = v_1 \dots v_{M-1}$.

Let $G = (V, E)$ be a simple undirected connected graph that consists of a finite set V of vertices (or states) together with a finite set E of symmetric subsets of $V \times V$ called edges. The *order* of the graph is $|V|$. The *degree* of a vertex is the number of edges using that vertex, and the *minimal degree* of G is the minimal degree of a vertex.

A set $S \subseteq V$ is *dominant* for G if for each $v \in V \setminus S$ there exists $u \in S$ such that $(u, v) \in E$. The *domination number* of G, denoted by $\gamma(G)$, is the minimal cardinality of a dominant set for G. A set $S \subseteq V$ is *total dominant* for G if for each $v \in V$ there exists $u \in S$ such that $(u, v) \in E$. As before the *total domination number* of G, denoted by $\gamma_t(G)$, is the minimal cardinality of a total dominant set for G.

We state the following upper bound for the domination number, sufficient for our purposes.

Lemma 2.2 (see [AS]). *If G is a simple undirected graph of order n and minimum degree δ, then*

$$\gamma(G) \leq \frac{1}{\delta + 1}(1 + \log(\delta + 1))n. \ ■$$

3. Entropy of cylinder covers. On the full shift space $(X_{\mathcal{D}}, \sigma)$, for $M \in \mathbb{N}$, let C_M be the algebra generated by the set of cylinders of length M.

Notice that for $N \le M$ we have $C_N \subseteq C_M$ since every N-cylinder is a finite disjoint union of M-cylinders.

$\mathcal{U} \in \mathcal{C}_{X_\mathcal{D}}$ is an *M-cylinder cover* if $U \in C_M$ for every $U \in \mathcal{U}$. Denote the set of M-cylinder covers by $\mathcal{C}^M_{X_\mathcal{D}}$, and let $\mathcal{P}^M_{X_\mathcal{D}}$ be the subset of M-cylinder partitions.

For $\mathcal{U} \in \mathcal{C}_{X_\mathcal{D}}$ and $M \in \mathbb{N}$, define

$$h^M(\mathcal{U},\sigma) = \min_{\substack{\alpha \succeq \mathcal{U} \\ \alpha \in \mathcal{P}^M_{X_\mathcal{D}}}} h(\alpha,\sigma). \tag{3.1}$$

Clearly for all $\mathcal{U} \in \mathcal{C}_{X_\mathcal{D}}$ and $M \in \mathbb{N}$, $h^{M+1}(\mathcal{U},\sigma) \le h^M(\mathcal{U},\sigma)$, and

$$h^+_c(\mathcal{U},\sigma) := \min_{M\in\mathbb{N}} h^M(\mathcal{U},\sigma) \ge h^+(\mathcal{U},\sigma). \tag{3.2}$$

Given $M \in \mathbb{N}$ denote by $\mathcal{L}^M_\mathcal{D}$ the set of M-labellings of $\mathcal{D}$, that is, functions $\mathcal{L} : \mathcal{D}^M \to \{1,\dots,D\}$.

For $\alpha = \{A_1,\dots,A_D\} \in \mathcal{P}^M_{X_\mathcal{D}}$ define $\mathcal{L}_\alpha \in \mathcal{L}^M_\mathcal{D}$ by $\mathcal{L}_\alpha(x_1\dots x_M) = d$ iff $[x_1\dots x_M]_0 \in A_d$. For any $\mathcal{L} \in \mathcal{L}^M_\mathcal{D}$ define $\alpha_\mathcal{L} = \{A_1,\dots,A_D\} \in \mathcal{P}^M_{X_\mathcal{D}}$ by

$$A_d = \bigcup_{\substack{x\in\mathcal{D}^M \\ \mathcal{L}(x)=d}} [x]_0 \qquad \text{for } d \in \{1,\dots,D\}.$$

Clearly there is a one-to-one correspondence between $\mathcal{L}^M_\mathcal{D}$ and partitions in $\mathcal{P}^M_{X_\mathcal{D}}$ with at most D atoms. To be explicit, $\alpha_{(\mathcal{L}_\alpha)} = \alpha$ for $\alpha = \{A_1,\dots,A_D\} \in \mathcal{P}^M_{X_\mathcal{D}}$, and $\mathcal{L}_{(\alpha_\mathcal{L})} = \mathcal{L}$ for $\mathcal{L} \in \mathcal{L}^M_\mathcal{D}$.

LEMMA 3.1. *For all $M \in \mathbb{N}$, and $\alpha = \{A_1,\dots,A_D\} \in \mathcal{P}^M_{X_\mathcal{D}}$ and $N \in \mathbb{N}$,*

$$|\alpha_0^{N-1}| = |\{\mathcal{L}_\alpha(u^1)\dots\mathcal{L}_\alpha(u^N) : u^1,\dots,u^N \in \mathcal{D}^M \\ \text{for } u^n \preceq u^{n+1} \text{ and } 0 \le n \le N-1, |\}.$$

Proof. Fix $N \in \mathbb{N}$.

($\le$) For any $\emptyset \ne A \in \alpha_0^{N-1}$, there are unique $d_1\dots d_N \in \{1,\dots,D\}^N$ and $x_0\dots x_{N+M-2} \in \mathcal{D}^{N+M-1}$ such that $[x_0\dots x_{N+M-2}]_0 \in A = \bigcap_{n=1}^N \sigma^{-n}A_{d_n}$. So $[x_{n-1}\dots x_{n+M-2}]_0 \in A_{d_n}$ for all $n \in \{1,\dots,N\}$.

By definition then

$$\mathcal{L}_\alpha(x_0\dots x_{M-1})\dots\mathcal{L}_\alpha(x_{N-1}\dots x_{N+M-2}) = d_1\dots d_N.$$

($\ge$) Consider $u^1,\dots,u^N \in \mathcal{D}^M$ such that $u^n \preceq u^{n+1}$ for $0 \le n \le N-1$. Define $x = x_1\dots x_{N+M-1} \in \mathcal{D}^{N+M-1}$ by

$$x_n = \begin{cases} u^n_0, & n \in \{1,\dots,N-1\}, \\ u^N_{n-N+1}, & n \in \{N, N+M-1\}. \end{cases}$$

By definition of $\mathcal{L}_\alpha$, for all $n \in \{1,\dots,N\}$, $[x_n\dots x_{n+M-1}]_0 \in A_{\mathcal{L}_\alpha(u^n)}$ and so $[x_1\dots x_{N+M-1}]_0 \in \bigcap_{n=0}^{N-1} \sigma^{-n}A_{\mathcal{L}_\alpha(u^n)}$. ∎

Given $\mathcal{L} \in \mathcal{L}_{\mathcal{D}}^M$ define $i_{\mathcal{L}} : \mathcal{D}^M \to \mathcal{P}(\{1, \ldots, D\})$ as

$$i_{\mathcal{L}}(u) = \{\mathcal{L}(v) : v \in \mathcal{D}^M,\ v \preceq u\}. \tag{3.3}$$

In the following lemma we establish a simple combinatorial condition for a partition in $\mathcal{P}_{X_{\mathcal{D}}}^M$ with at most D atoms to have entropy greater than $\log 2$.

LEMMA 3.2. *For* $\mathcal{L} \in \mathcal{L}_{\mathcal{D}}^M$, *if* $\{u \in \mathcal{D}^M : |i_{\mathcal{L}}(u)| = 1\} = \emptyset$ *then for all* $N \in \mathbb{N}$,

$$|\{\mathcal{L}(u^1)\ldots\mathcal{L}(u^N) : u^1, \ldots, u^N \in \mathcal{D}^M \text{ and } u^n \preceq u^{n+1} \text{ for } 0 \le n \le N-1\}| \ge 2^N.$$

Proof. Start with $u^0 \in \mathcal{D}^M$. There exist $u^{0,1}, u^{0,2} \in i_{\mathcal{L}}(u^0)$ such that $\mathcal{L}(u^{0,1}) \ne \mathcal{L}(u^{0,2})$. This proves the case of $N = 1$.

As before for $i = \{1, 2\}$, there are $u^{0,i,1}, u^{0,i,2} \in i_{\mathcal{L}}(u^{0,i})$ such that $\mathcal{L}(u^{0,i,1}) \ne \mathcal{L}(u^{0,i,2})$. This proves the case of $N = 2$ using $\{u^{0,1,1}u^{0,1}, u^{0,1,2}u^{0,1}, u^{0,2,1}u^{0,2}, u^{0,2,2}u^{0,2}\}$. The same argument works for larger N. ∎

The cover $\mathcal{U}_1$ defined in Theorem 1.1 can be rewritten as

$$\mathcal{U}_1 = \{[d]_0^c : d \in \mathcal{D}\}. \tag{3.4}$$

Using the previous lemmas we prove the following theorem.

THEOREM 3.3. *For all* $N \in \mathbb{N}$ *and* $\mathcal{D}$ (*with* $D > 1$), $h_c^+(\mathcal{U}_1, \sigma) = \log 2$.

Proof. Fix $M \in \mathbb{N}$. By Lemma 2.1 to compute $h^M(\mathcal{U}_1, \sigma)$ we consider only partitions in $\mathcal{U}_1^*$. For any $\alpha = \{A_1, \ldots, A_D\} \in \mathcal{U}_1^* \cap \mathcal{P}_{X_{\mathcal{D}}}^M$ and $u \in \mathcal{D}^M$ we have $|i_{\mathcal{L}_\alpha}(u)| \ge 2$ and so $\{u \in \mathcal{D}^M : |i_{\mathcal{L}_\alpha}(u)| = 1\} = \emptyset$. From Lemmas 3.1 and 3.2 we conclude that $|\alpha_0^{N-1}| \ge 2^N$ for all $N \in \mathbb{N}$.

Consider $\alpha = \{A_i, A_j\} \in \mathcal{U}_1^* \cap \mathcal{P}_{X_{\mathcal{D}}}^M$ with $i, j \in \{1, \ldots, D\}$, $i \ne j$. Clearly $|\alpha_0^{N-1}| \le 2^N$ and so for such α we have $|\alpha_0^{N-1}| = 2^N$.

By (3.1) this proves that for every $M \in \mathbb{N}$, $h^M(\mathcal{U}_1, \sigma) = \log 2$. ∎

4. Topological entropy of $\mathcal{U}_1$. In this section we compute the topological entropy of the cover $\mathcal{U}_1$.

For $N \in \mathbb{N}$, $k \in \mathbb{Z}$ and $u \in \mathcal{D}^N$ define

$$[u]_k^* = \bigcap_{n=0}^{N-1} [u_{n+1}]_{n+k}^c.$$

We state the following simple lemma without proof.

LEMMA 4.1. *For all* $k \in \mathbb{Z}$, $M \in \mathbb{N}$ *and* $u, v \in \mathcal{D}^M$,

$$[u]_k \cap [v]_k^* = \begin{cases} [u]_k & \text{if } u_{m+k} \ne v_{m+k} \text{ for all } m \in \{1, \ldots, M\}, \\ \emptyset & \text{if } u_{m+k} = v_{m+k} \text{ for some } m \in \{1, \ldots, M\}. \end{cases}$$

Proof. Left to the reader. ∎

Notice that $\mathcal{U}_N := (\mathcal{U}_1)_0^{N-1} = \{[u]_0^* : u \in \mathcal{D}^N\}$ for all $N \in \mathbb{N}$. In Lemma 4.3 we compute a lower bound for $\mathcal{N}(\mathcal{U}_N)$ by a simple counting argument, and an upper bound by translating the problem to finding the minimal cardinality of a total dominant set for a finite undirected graph.

For $N \in \mathbb{N}$ we define the simple undirected graph $G_N = (V_N, E_N)$ by setting $V_N = \mathcal{D}^N$ and given $u, v \in \mathcal{D}^N$, $(u, v) \in E_N \Leftrightarrow \forall n \in \{1, \ldots, N\}$, $u_n \neq v_n$.

LEMMA 4.2. *For each $N \in \mathbb{N}$, $S \subseteq V_N$ is a total dominant set if and only if $\{[u]_0^* : u \in S\}$ is a subcover of $\mathcal{U}_N$.*

Proof. ($\Rightarrow$) Given $v \in \mathcal{D}^N$, since S is a total dominant set there exists $u \in S$ with $(u, v) \in E_N$. By Lemma 4.1 we have $[v]_0 \cap [u]_0^* = [v]$ and so $\{[u]_0^* : u \in S\}$ is a subcover.

($\Leftarrow$) Take any $v \in \mathcal{D}^N$. Assume that S is not a total dominant set. Then there exists $v \in \mathcal{D}^N$ such that $(u, v) \notin E_N$ for all $u \in S$. Once again by Lemma 4.1 we have $[v]_0 \cap [u]_0^* = \emptyset$ and so $\{[u]_0^* : u \in S\}$ is not a subcover. ∎

LEMMA 4.3. *For all $N \in \mathbb{N}$,*

$$\left(\frac{D}{D-1}\right)^N \leq \mathcal{N}(\mathcal{U}_N) \leq 2(1 + N \log D)\left(\frac{D}{D-1}\right)^N.$$

Proof. ($\geq$) Given $N \in \mathbb{N}$ the number of words of length N is D^N. The total number of different words that belong to any $U \in \mathcal{U}_N$ is $(D-1)^N$. So at least $\lceil DN/(D-1)^N \rceil$ elements of $\mathcal{U}_N$ are needed to obtain a subcover.

($\leq$) From Lemma 4.2 we know that $\mathcal{N}(\mathcal{U}_N) = \gamma_t(G_N)$. It is clear that $|V_N| = D^N$ and every node has degree $(D-1)^N$. From Lemma 2.2 and the fact that $\gamma_t(G) \leq 2\gamma(G)$ for any graph G (just add one neighbour to each element of a dominant set and you obtain a total dominant set), we have

$$\begin{aligned}\gamma_t(G_N) &\leq \frac{2}{(D-1)^N + 1}\,(1 + \log((D-1)^N + 1))D^N \\ &\leq 2(1 + N \log D) \cdot \left(\frac{D}{D-1}\right)^N. \ \blacksquare\end{aligned}$$

Now we are ready to prove the main result.

Proof of Theorem 1.1. For $D \in \mathbb{N}$, $D > 2$, consider $(X_{\mathcal{D}}, \sigma)$ and the cover $\mathcal{U}_1$. From Lemma 4.3 and Theorem 3.3,

$$h(\mathcal{U}_1, \sigma) = \log\left(\frac{D}{D-1}\right) < \log 2 = h_c^+(\mathcal{U}_1, \sigma).$$

5. Conclusions and open questions. Theorem 1.1 shows that it is not possible to compute the topological entropy of clopen covers by just using finer clopen partitions. Since in the 0-dimensional case it is not clear that the combinatorial entropy of any measurable partition can be approximated using clopen partitions, this does not answer question (1.7). However, it is interesting that both in the measure and topological setting h and h_μ^- are computed over a set of partitions with atoms in the σ-fields $\mathcal{U}_0^{N-1}$ with $N \in \mathbb{N}$. That is, given $\mathcal{U} \in \mathcal{C}_X$ and $N \in \mathbb{N}$ let $\mathcal{P}_{\mathcal{U}}^N$ be the set of partitions with atoms in the σ-field generated by $\mathcal{U}_0^{N-1}$. A consequence of the work done in [R] is that

$$h(\mathcal{U},T) = \inf_{N\in\mathbb{N}} \inf_{\substack{\alpha\succeq\mathcal{U}_0^{N-1} \\ \alpha\in\mathcal{P}_{\mathcal{U}}^N}} \frac{1}{N}\log|\alpha|,$$

$$h_\mu^-(\mathcal{U},T) = \inf_{N\in\mathbb{N}} \inf_{\substack{\alpha\succeq\mathcal{U}_0^{N-1} \\ \alpha\in\mathcal{P}_{\mathcal{U}}^N}} \frac{1}{N} H_\mu(\alpha).$$

In the topological case Theorem 1.1 proves that this is false for h^+ since the σ-fields $(\mathcal{U}_1)_0^{N-1}$ with $N \in \mathbb{N}$ generate the topology. It is natural to ask if this is also false in the measure-theoretical setting. This can be stated as the following question:

$$h_\mu^+(\mathcal{U},T) = \inf_{N\in\mathbb{N}} \inf_{\substack{\alpha\succeq\mathcal{U} \\ \alpha\in\mathcal{P}_{\mathcal{U}}^N}} h_\mu(\alpha,T)\ ? \tag{5.1}$$

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