

The Hardy–Lorentz spaces $H^{p,q}(\mathbb{R}^n)$

by

WAEL ABU-SHAMMALA and ALBERTO TORCHINSKY (Bloomington, IN)

Abstract. We deal with the Hardy–Lorentz spaces $H^{p,q}(\mathbb{R}^n)$ where $0 < p \leq 1$, $0 < q \leq \infty$. We discuss the atomic decomposition of the elements in these spaces, their interpolation properties, and the behavior of singular integrals and other operators acting on them.

The real variable theory of the Hardy spaces represents a fruitful setting for the study of maximal functions and singular integral operators. In fact, it is because of the failure of these operators to preserve L^1 that the Hardy space H^1 assumes its prominent place in harmonic analysis. Now, for many of these operators, the role of L^1 can just as well be played by $H^{1,\infty}$, or Weak H^1 . However, although these operators are amenable to H^1 - L^1 and $H^{1,\infty}$ - $L^{1,\infty}$ estimates, interpolation between H^1 and $H^{1,\infty}$ has not been available. Similar considerations apply to H^p and Weak H^p for $0 < p < 1$.

The purpose of this paper is to provide an interpolation result for the Hardy–Lorentz spaces $H^{p,q}$, $0 < p \leq 1$, $0 < q \leq \infty$, including the case of Weak H^p as an end point for real interpolation. Since in this context neither truncations are available nor reiteration applies, the atomic decomposition will be the key ingredient in dealing with interpolation.

The paper is organized as follows. The Lorentz spaces, including criteria that ensure membership in $L^{p,q}$, $0 < p < \infty$, $0 < q \leq \infty$, are discussed in Section 1. In Section 2 we show that distributions in $H^{p,q}$ have an atomic decomposition in terms of H^p atoms with coefficients in an appropriate mixed norm space. An interesting application of this decomposition is to $H^{p,q}$ - $L^{p,\infty}$ estimates for Calderón–Zygmund singular integral operators, $p < q \leq \infty$. Also, by manipulating the different levels of the atomic decomposition, we show that, for $0 < q_1 < q < q_2 \leq \infty$, $H^{p,q}$ is an intermediate space between H^{p,q_1} and H^{p,q_2} . This result applies to Calderón–Zygmund singular integral

2000 *Mathematics Subject Classification*: 42B20, 42B30, 42B35.

Key words and phrases: atomic decomposition, singular integral operators, interpolation.

operators, including those with variable kernels, Marcinkiewicz integrals, and other operators.

1. The Lorentz spaces. The Lorentz space $L^{p,q}(\mathbb{R}^n) = L^{p,q}$, $0 < p < \infty$, $0 < q \leq \infty$, consists of those measurable functions f with finite quasinorm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = \left(\frac{q}{p} \int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t} \right)^{1/q}, \quad 0 < q < \infty, \quad \|f\|_{p,\infty} = \sup_{t>0} [t^{1/p} f^*(t)].$$

The Lorentz quasinorm may also be given in terms of the distribution function $m(f, \lambda) = |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|$, loosely speaking, the inverse of the non-increasing rearrangement f^* of f . Indeed, we have

$$\|f\|_{p,q} = \left(\frac{q}{p} \int_0^\infty \lambda^{q-1} m(f, \lambda)^{q/p} d\lambda \right)^{1/q} \sim \left(\sum_k [2^k m(f, 2^k)^{1/p}]^q \right)^{1/q}$$

when $0 < q < \infty$, and

$$\|f\|_{p,\infty} = \sup_k 2^k m(f, 2^k)^{1/p}$$

Note that, in particular, $L^{p,p} = L^p$, and $L^{p,\infty}$ is weak L^p .

The following two results are useful in verifying that a function is in $L^{p,q}$.

LEMMA 1.1. *Let $0 < p < \infty$ and $0 < q \leq \infty$. Assume that the non-negative sequence $\{\mu_k\}$ satisfies $\{2^k \mu_k\} \in \ell^q$. Further suppose that the non-negative function φ has the following property: there exists $0 < \varepsilon < 1$ such that, given an arbitrary integer k_0 , we have $\varphi \leq \psi_{k_0} + \eta_{k_0}$, where ψ_{k_0} is essentially bounded and satisfies $\|\psi_{k_0}\|_\infty \leq c 2^{k_0}$, and*

$$2^{k_0 \varepsilon p} m(\eta_{k_0}, 2^{k_0}) \leq c \sum_{k=k_0}^\infty [2^{k\varepsilon} \mu_k]^p.$$

Then $\varphi \in L^{p,q}$, and $\|\varphi\|_{p,q} \leq c \|\{2^k \mu_k\}\|_{\ell^q}$.

Proof. It clearly suffices to verify that $\|\{2^k |\{\varphi > \gamma 2^k\}|^{1/p}\}\|_{\ell^q} < \infty$, where γ is an arbitrary positive constant. Now, given k_0 , let ψ_{k_0} and η_{k_0} be as above, and put $\gamma = c + 1$, where c is the constant in the above inequalities; for this choice of γ , $\{\varphi > \gamma 2^{k_0}\} \subseteq \{\eta_{k_0} > 2^{k_0}\}$.

When $q = \infty$, we have

$$2^{k_0 \varepsilon} m(\eta_{k_0}, 2^{k_0})^{1/p} \leq c \left(\sum_{k_0}^\infty [2^{-k(1-\varepsilon)} 2^k \mu_k]^p \right)^{1/p} \leq c 2^{-k_0(1-\varepsilon)} \sup_{k \geq k_0} [2^k \mu_k].$$

Thus, $2^{k_0} m(\eta_{k_0}, 2^{k_0})^{1/p} \leq \sup_{k \geq k_0} [2^k \mu_k]$, and consequently,

$$2^{k_0} m(\varphi, \gamma 2^{k_0})^{1/p} \leq c \|\{2^k \mu_k\}\|_{\ell^\infty} \quad \text{for all } k_0.$$

When $0 < q < \infty$, let $1 - \varepsilon = 2\delta$, and rewrite the right-hand side above as

$$\sum_{k_0}^{\infty} \frac{1}{2^{k\delta p}} [2^{k(1-\delta)} \mu_k]^p.$$

When $p < q$, by Hölder’s inequality with exponent $r = q/p$ and its conjugate r' , this expression is dominated by

$$\left(\sum_{k_0}^{\infty} \frac{1}{2^{k\delta p r'}} \right)^{1/r'} \left(\sum_{k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^{rp} \right)^{1/r} \leq c 2^{-k_0 \delta p} \left(\sum_{k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^q \right)^{p/q},$$

and when $0 < q \leq p$, $r < 1$, we get a similar bound by simply observing that it does not exceed

$$2^{-k_0 \delta p} \left(\sum_{k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^p \right)^{r/r} \leq 2^{-k_0 \delta p} \left(\sum_{k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^q \right)^{p/q}.$$

Hence, continuing with the estimate, we have

$$2^{k_0 \varepsilon p} m(\eta_{k_0}, 2^{k_0}) \leq c 2^{-k_0 \delta p} \left(\sum_{k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^q \right)^{p/q},$$

which yields, since $1 - \varepsilon = 2\delta$,

$$2^{k_0} m(\varphi, \gamma 2^{k_0})^{1/p} \leq c 2^{k_0 \delta} \left(\sum_{k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^q \right)^{1/q}.$$

Thus, raising to the power q and summing, we get

$$\sum_{k_0} [2^{k_0} m(\varphi, \gamma 2^{k_0})^{1/p}]^q \leq c \sum_{k_0} 2^{k_0 \delta q} \sum_{k=k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^q,$$

which, upon changing the order of summation on the right-hand side of the above inequality, is bounded by

$$\sum_k [2^{k(1-\delta)} \mu_k]^q \left[\sum_{k_0=-\infty}^k 2^{k_0 \delta q} \right] \leq c \sum_k [2^k \mu_k]^q. \blacksquare$$

The reader will have no difficulty in verifying that, for Lemma 1.1 to hold, it suffices that ψ_{x_0} satisfies

$$m(\psi_{x_0}, 2^{k_0})^{1/p} \leq c \mu_{k_0} \quad \text{for all } k_0.$$

This holds, for instance, when $\|\psi_{x_0}\|_r^r \leq c 2^{k_0 r} \mu_{k_0}^p$, $0 < r < \infty$. In fact, the assumptions of Lemma 1.1 correspond to the limiting case of this inequality as $r \rightarrow \infty$.

Another useful condition is given by our next result; its proof is left to the reader.

LEMMA 1.2. *Let $0 < p < \infty$, and let the non-negative sequence $\{\mu_k\}$ be such that $\{2^k \mu_k\} \in \ell^q$, $0 < q \leq \infty$. Further, suppose that the non-negative function φ has the following property: there exists $0 < \varepsilon < \min(1, q/p)$ such that, given an arbitrary integer k_0 , we have $\varphi \leq \psi_{k_0} + \eta_{k_0}$, where ψ_{k_0} and η_{k_0} satisfy*

$$2^{k_0 p} m(\psi_{k_0}, 2^{k_0})^\varepsilon \leq c \sum_{-\infty}^{k_0} [2^k \mu_k^\varepsilon]^p, \quad 2^{k_0 \varepsilon} |\{\eta_{k_0} > 2^{k_0}\}| \leq c \sum_{k_0}^{\infty} [2^{k \varepsilon} \mu_k]^p.$$

Then $\varphi \in L^{p,q}$, and $\|\varphi\|_{p,q} \leq c \|\{2^k \mu_k\}\|_{\ell^q}$.

We will also require some basic concepts from the theory of real interpolation. Let A_0, A_1 be a compatible couple of quasinormed Banach spaces, i.e., both A_0 and A_1 are continuously embedded in a larger topological vector space. The Peetre K -functional of $f \in A_0 + A_1$ at $t > 0$ is defined by

$$K(t, f; A_0, A_1) = \inf_{f=f_0+f_1} \|f_0\|_0 + t\|f_1\|_1,$$

where $f = f_0 + f_1$, $f_0 \in A_0$ and $f_1 \in A_1$.

In the particular case of the L^q spaces, the K -functional can be computed by Holmstedt’s formula (see [11]). Specifically, for $0 < q_0 < q_1 \leq \infty$, let α be given by $1/\alpha = 1/q_0 - 1/q_1$. Then

$$K(t, f; L^{q_0}, L^{q_1}) \sim \left(\int_0^{t^\alpha} f^*(s)^{q_0} ds \right)^{1/q_0} + t \left(\int_{t^\alpha}^\infty f^*(s)^{q_1} ds \right)^{1/q_1}.$$

The intermediate space $(A_0, A_1)_{\eta,q}$, $0 < \eta < 1$, $0 < q \leq \infty$, consists of those f ’s in $A_0 + A_1$ with

$$\|f\|_{(A_0, A_1)_{\eta,q}} = \left(\int_0^\infty [t^{-\eta} K(t, f; A_0, A_1)]^q \frac{dt}{t} \right)^{1/q} < \infty, \quad q < \infty,$$

$$\|f\|_{(A_0, A_1)_{\eta,\infty}} = \sup_{t>0} [t^{-\eta} K(t, f; A_0, A_1)] < \infty, \quad q = \infty.$$

Finally, for the L^q and $L^{p,q}$ spaces, we have the following result. Let $0 < q_1 < q < q_2 \leq \infty$, and suppose that $1/q = (1 - \eta)/q_1 + \eta/q_2$. Then $L^q = (L^{q_1}, L^{q_2})_{\eta,q}$, and $L^{p,q} = (L^{p,q_1}, L^{p,q_2})_{\eta,q}$, $0 < p < \infty$ (see [3]).

2. The Hardy–Lorentz spaces $H^{p,q}$. In this paper we adopt the atomic characterization of the Hardy spaces H^p , $0 < p \leq 1$. Recall that an H^p atom with defining cube I is a compactly supported function a with $[n(1/p - 1)]$ vanishing moments, $\text{supp}(a) \subseteq I$, and $|I|^{1/p} |a(x)| \leq 1$. The Hardy space $H^p(\mathbb{R}^n) = H^p$ consists of those distributions f that can be written as $f = \sum \lambda_j a_j$, where the a_j ’s are H^p atoms, $\sum |\lambda_j|^p < \infty$, and the

convergence is in the sense of distributions as well as in H^p . Furthermore,

$$\|f\|_{H^p} \sim \inf \left(\sum |\lambda_j|^p \right)^{1/p},$$

where the infimum is taken over all possible atomic decompositions of f . This last expression has traditionally been called the *atomic H^p -norm* of f .

C. Fefferman, Rivière and Sagher identified the intermediate spaces between the Hardy space H^{p_0} , $0 < p_0 < 1$, and L^∞ as

$$(H^{p_0}, L^\infty)_{\eta,q} = H^{p,q}, \quad 1/p = (1 - \eta)/p_0, \quad 0 < q \leq \infty,$$

where $H^{p,q}$ consists of those distributions f whose radial maximal function $Mf(x) = \sup_{t>0} |(f * \varphi_t)(x)|$ belongs to $L^{p,q}$. Here φ is a compactly supported, smooth function with non-vanishing integral (see [9]). R. Fefferman and Soria studied in detail the space $H^{1,\infty}$, which they called Weak H^1 (see [10]).

Just as in the case of H^p , the spaces $H^{p,q}$ can be characterized in a number of different ways, including in terms of non-tangential maximal functions and Luzin functions. In what follows we will calculate the quasinorm of f in $H^{p,q}$ by means of the expression

$$\|\{2^k m(Mf, 2^k)^{1/p}\}\|_{\ell^q}, \quad 0 < p \leq 1, \quad 0 < q \leq \infty,$$

where Mf is an appropriate maximal function of f .

Passing to the atomic decomposition of $H^{p,q}$, the proof is divided in two parts. First, we construct an essentially optimal atomic decomposition; Parilov has obtained independently this result for $H^{1,q}$ when $1 \leq q$ (see [13]). Also, R. Fefferman and Soria gave the atomic decomposition of Weak H^1 (see [10]), and Alvarez the atomic decomposition of Weak H^p , $0 < p < 1$ (see [1]).

THEOREM 2.1. *Let $f \in H^{p,q}$, $0 < p \leq 1$, $0 < q \leq \infty$. Then f has an atomic decomposition $f = \sum_{j,k} \lambda_{j,k} a_{j,k}$, where the $a_{j,k}$'s are H^p atoms with defining cubes $I_{j,k}$ that have bounded overlap uniformly for each k , the sequence $\{\lambda_{j,k}\}$ satisfies $(\sum_k [\sum_j |\lambda_{j,k}|^p]^{q/p})^{1/q} < \infty$, and the convergence is in the sense of distributions. Furthermore, $(\sum_k [\sum_j |\lambda_{j,k}|^p]^{q/p})^{1/q} \sim \|f\|_{H^{p,q}}$.*

Proof. The idea of constructing an atomic decomposition using Calderón's reproducing formula is well understood, so we will only sketch it here; for further details, see [4] and [17]. Let $Nf(x) = \sup\{|(f * \psi_t)(y)| : |x - y| < t\}$ denote the non-tangential maximal function of f with respect to a suitable smooth function ψ with non-vanishing integral. One considers the open sets $\mathcal{O}_k = \{Nf > 2^k\}$, for all integers k , and builds the atoms with defining cube associated to the cubes of the Whitney decomposition of \mathcal{O}_k , and hence having all the required properties. More precisely, one constructs a se-

quence of bounded functions f_k with L^∞ norm not exceeding $c2^k$ for each k , and such that $f - \sum_{|k| \leq n} f_k \rightarrow 0$ as $n \rightarrow \infty$ in the sense of distributions. These functions have the further property that $f_k(x) = \sum_j \alpha_{j,k}(x)$, where $|\alpha_{j,k}(x)| \leq c2^k$, c is a constant, each $\alpha_{j,k}$ has vanishing moments up to order $[n(1/p - 1)]$ and is supported in $I_{j,k}$ (roughly one of the Whitney cubes), where the $I_{j,k}$'s have bounded overlaps for each k , uniformly in k . It now only remains to scale $\alpha_{j,k}$,

$$\alpha_{j,k}(x) = \lambda_{j,k} a_{j,k}(x),$$

and balance the contribution of each term to the sum. Let $\lambda_{j,k} = 2^k |I_{j,k}|^{1/p}$. Then $a_{j,k}(x)$ is essentially an H^p atom with defining cube $I_{j,k}$, and one has $(\sum_j |\lambda_{j,k}|^p)^{1/p} \sim 2^k |\mathcal{O}_k|^{1/p}$. Thus,

$$\left\| \left(\sum_j |\lambda_{j,k}|^p \right)^{1/p} \right\|_{\ell^q} \sim \left\| \{2^k |\mathcal{O}_k|^{1/p}\} \right\|_{\ell^q} \sim \|f\|_{H^{p,q}}, \quad 0 < q \leq \infty. \blacksquare$$

As an application of this atomic decomposition, the reader should have no difficulty in showing directly the C. Fefferman, Rivière, Sagher characterization of $H^{p,q}$ (see [9]).

Another interesting application of this decomposition is to $H^{p,q}$ - $L^{p,\infty}$ estimates for Calderón–Zygmund singular integral operators T , $p < q \leq \infty$. This approach combines the concept of p -quasiloca operator of Weisz (see [16]) with the idea of variable dilations of R. Fefferman and Soria (see [10]). Intuitively, since Hörmander’s condition implies that T maps H^1 into L^1 , say, for T to be defined in $H^{1,s}$, $1 < s \leq \infty$, some strengthening of this condition is required. This is accomplished by the variable dilations. Moreover, since we will include $p < 1$ in our discussion, as p gets smaller, more regularity of the kernel of T will be required. This justifies the following definition.

Given $0 < p \leq 1$, let $N = [n(1/p - 1)]$, and, associated to the kernel $k(x, y)$ of a Calderón–Zygmund singular integral operator T , consider the modulus of continuity ω_p given by

$$\omega_p(\delta) = \sup_I \frac{1}{|I|} \int_{\mathbb{R}^n \setminus (2/\delta)I} \left[\int_I \left| k(x, y) - \sum_{|\alpha| \leq N} (y - y_I)^\alpha k_\alpha(x, y_I) \right| dy \right]^p dx,$$

where $0 < \delta \leq 1$, and the sup is taken over the collection of arbitrary cubes I of \mathbb{R}^n centered at y_I . Here, for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$k_\alpha(x, y_I) = \frac{1}{\alpha!} D^\alpha k(x, y)|_{y=y_I}.$$

The modulus $\omega_p(\delta)$ controls the behavior of T on atoms. More precisely, if

a is an H^p atom with defining cube I , and $0 < \delta < 1$, observe that

$$T(a)(x) = \int_I \left[k(x, y) - \sum_{|\alpha| \leq N} (y - y_I)^\alpha k_\alpha(x, y_I) \right] a(y) dy,$$

and consequently,

$$\int_{\mathbb{R}^n \setminus (2/\delta)I} |T(a)(x)|^p dx \leq \omega_p(\delta).$$

We are now ready to prove the $H^{p,q}$ - $L^{p,\infty}$ estimate for a Calderón–Zygmund singular integral operator T with kernel $k(x, y)$.

THEOREM 2.2. *Let $0 < p \leq 1$ and $p < q \leq \infty$. Assume that a Calderón–Zygmund singular integral operator T is of weak type (r, r) for some $1 < r < \infty$, and that the modulus of continuity ω_p of the kernel k satisfies a Dini condition of order $q/(q - p)$, namely,*

$$A_{p,q} = \left[\int_0^1 \omega_p(\delta)^{q/(q-p)} \frac{d\delta}{\delta} \right]^{(q-p)/q} < \infty.$$

Then T maps $H^{p,q}$ continuously into $L^{p,\infty}$, and $\|Tf\|_{p,\infty} \leq cA_{p,q}^{1/p} \|f\|_{H^{p,q}}$.

Proof. We need to show that

$$2^{k_0 p} m(Tf, 2^{k_0}) \leq c \|f\|_{H^{p,q}}^p \quad \text{for all } k_0.$$

Let $f = \sum_k \sum_j \lambda_{j,k} a_{j,k}$, be the atomic decomposition of f given in Theorem 2.1, and set $f_1 = \sum_{k \leq k_0} \sum_j \lambda_{j,k} a_{j,k}$ and $f_2 = f - f_1$. Further, let $\mu_k = (\sum_j |\lambda_{j,k}|^p)^{1/p}$, and recall that $\|\{\mu_k\}\|_{\ell^q} \sim \|f\|_{H^{p,q}}$. Now, since $\|f_1\|_r^r \leq c2^{k_0(r-p)} \|f\|_{H^{p,\infty}}^p$, we have

$$2^{p k_0} m(Tf_1, 2^{k_0}) \leq c \|f\|_{H^{p,\infty}}^p.$$

Next, put $I_{j,k}^* = 2^{1/n} (3/2)^{p(k-k_0)/n} I_{j,k}$, and let

$$\Omega = \bigcup_{k > k_0} \bigcup_j I_{j,k}^*.$$

Since $|I_{j,k}^*| = 2(3/2)^{p(k-k_0)} |I_{j,k}| \sim 2^{-k_0 p} (3/4)^{p(k-k_0)} |\lambda_{j,k}|^p$, we get

$$\begin{aligned} |\Omega| &\leq \sum_{k > k_0} \sum_j |I_{j,k}^*| \leq c2^{-k_0 p} \sum_{k > k_0} (3/4)^{p(k-k_0)} \sum_j |\lambda_{j,k}|^p \\ &\leq c2^{-k_0 p} [\sup_{k > k_0} \mu_k]^p \leq c2^{-k_0 p} \|f\|_{H^{p,\infty}}^p. \end{aligned}$$

Also, since $0 < p \leq 1$, it readily follows that

$$|T(f_2)(x)|^p \leq \sum_{k > k_0} \sum_j |\lambda_{j,k}|^p |T(a_{j,k})(x)|^p,$$

and, by Tonelli and the estimate for $T(a)$, we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus \Omega} |T(f_2)(x)|^p dx &\leq \sum_{k>k_0} \sum_j |\lambda_{j,k}|^p \int_{\mathbb{R}^n \setminus I_{j,k}^*} |T(a_{j,k})(x)|^p dx \\ &\leq \sum_{k>k_0} \omega_p((2/3)^{p(k-k_0)/n}) \mu_k^p \\ &\leq \left(\sum_{k>0} \omega_p((2/3)^{pk/n})^{q/(q-p)} \right)^{(q-p)/q} \|\mu_k\|_{\ell^q}^p \\ &\leq c \left[\int_0^1 \omega_p(\delta)^{q/(q-p)} \frac{d\delta}{\delta} \right]^{(q-p)/q} \|f\|_{H^{p,q}}^p. \end{aligned}$$

This bound gives at once

$$2^{pk_0} |\{x \notin \Omega : |T(f_2)(x)| > 2^{k_0}\}| \leq c A_{p,q} \|f\|_{H^{p,q}}^p,$$

which implies that

$$\begin{aligned} 2^{pk_0} m(Tf_2, 2^{k_0-1}) &\leq 2^{pk_0} [|\Omega| + |\{x \notin \Omega : |T(f_2)(x)| > 2^{k_0-1}\}|] \\ &\leq c \|f\|_{H^{p,\infty}}^p + c A_{p,q} \|f\|_{H^{p,q}}^p. \end{aligned}$$

Finally,

$$\begin{aligned} 2^{k_0 p} m(Tf, 2^{k_0}) &\leq 2^{k_0 p} m(Tf_1, 2^{k_0-1}) + 2^{k_0 p} m(Tf_2, 2^{k_0-1}) \\ &\leq c \|f\|_{H^{p,\infty}}^p + c A_{p,q} \|f\|_{H^{p,q}}^p, \end{aligned}$$

and, since $\|f\|_{H^{p,\infty}} \leq c \|f\|_{H^{p,q}}$ for all q , we have finished. ■

We now pass to the converse of Theorem 2.1. It is apparent that a condition that relates the coefficients λ_j with the corresponding atoms a_j involved in an atomic decomposition of the form $\sum_j \lambda_j a_j(x)$ is relevant here. More precisely, if I_j denotes the supporting cube of a_j , let

$$\mathcal{I}_k = \{j : 2^k \leq |\lambda_j|/|I_j|^{1/p} < 2^{k+1}\},$$

and, for $\lambda = \{\lambda_j\}$, put

$$\|\lambda\|_{[p,q]} = \left(\sum_k \left[\sum_{j \in \mathcal{I}_k} |\lambda_j|^p \right]^{q/p} \right)^{1/q}.$$

We then have:

THEOREM 2.3. *Let $0 < p \leq 1$, $0 < q \leq \infty$, and let f be a distribution given by $f = \sum_j \lambda_j a_j(x)$, where the a_j 's are H^p atoms, and the convergence is in the sense of distributions. Further, assume that the family $\{I_j\}$ consisting of the supports of the a_j 's has bounded overlap at each level \mathcal{I}_k uniformly in k , and $\|\lambda\|_{[p,q]} < \infty$. Then $f \in H^{p,q}$, and $\|f\|_{H^{p,q}} \leq c \|\lambda\|_{[p,q]}$.*

Proof. Let $Mf(x) = \sup_{t>0} |(f * \psi_t)(x)|$ denote the radial maximal function of f with respect to a suitable smooth function ψ with support contained

in $\{|x| \leq 1\}$ and non-vanishing integral. We will verify that Mf satisfies the conditions of Lemma 1.1 and is thus in $L^{p,q}$.

Fix an integer k_0 and let

$$g(x) = \sum_{k < k_0} \sum_{j \in \mathcal{I}_k} \lambda_j a_j(x).$$

Since $\|Mg\|_\infty \leq \|g\|_\infty$ it suffices to estimate $|g(x)|$. Let C be the bounded overlap constant for the family of the supports of the a_j 's. Then, for $j \in \mathcal{I}_k$,

$$|\lambda_j| |a_j(x)| = \frac{1}{|I_j|^{1/p}} |\lambda_j| |I_j|^{1/p} |a_j(x)| \leq 2^k \chi_{I_j}(x),$$

and consequently,

$$|g(x)| \leq \sum_{k < k_0} 2^k \sum_j \chi_{I_j}(x) \leq C 2^{k_0}.$$

Next, let

$$h(x) = \sum_{k \geq k_0} \sum_{j \in \mathcal{I}_k} \lambda_j a_j(x).$$

Since a_j has $N = [n(1/p - 1)]$ vanishing moments it is not hard to see that, if I_j is the defining cube of a_j and I_j is centered at x_j , and $\gamma = (n + N + 1)/n > 1/p$, then, with c independent of j , $\varphi_j(x) = M a_j(x)$ satisfies

$$\varphi_j(x) \leq c \frac{|I_j|^{\gamma - 1/p}}{(|I_j| + |x - x_j|^n)^\gamma}.$$

Thus, if $1/\gamma < \varepsilon p < 1$, then

$$Mh(x)^{\varepsilon p} \leq c \sum_{j \in \mathcal{I}_k, k \geq k_0} \frac{(|\lambda_j| |I_j|^{\gamma - 1/p})^{\varepsilon p}}{(|I_j| + |x - x_j|^n)^{\gamma \varepsilon p}},$$

which, upon integration, yields

$$\int_{\mathbb{R}^n} Mh(x)^{\varepsilon p} dx \leq c \sum_{j \in \mathcal{I}_k, k \geq k_0} (|\lambda_j| |I_j|^{\gamma - 1/p})^{\varepsilon p} \int_{\mathbb{R}^n} \frac{1}{(|I_j| + |x - x_j|^n)^{\gamma \varepsilon p}} dx.$$

The integrals on the right-hand side above are of order $|I_j|^{1 - \gamma \varepsilon p}$, and consequently, by Chebyshev's inequality,

$$2^{k_0 \varepsilon p} |\{Mh > 2^{k_0}\}| \leq c \sum_{j \in \mathcal{I}_k, k \geq k_0} |\lambda_j|^{\varepsilon p} |I_j|^{1 - \varepsilon} \leq c \sum_{k \geq k_0} 2^{k \varepsilon p} \sum_{j \in \mathcal{I}_k} |I_j|.$$

Thus, Lemma 1.1 applies with $\varphi = Mf$, $\psi_{k_0} = Mg$, $\eta_{k_0} = Mh$, and $\mu_k = (\sum_{j \in \mathcal{I}_k} |I_j|)^{1/p}$, and we get

$$\|\{2^k m(Mf, 2^k)^{1/p}\}\|_{\ell^q} \leq c \left\| \left\{ 2^k \left(\sum_{j \in \mathcal{I}_k} |I_j| \right)^{1/p} \right\} \right\|_{\ell^q},$$

which, since

$$|I_j| \sim \frac{|\lambda_j|^p}{2^{kp}}, \quad j \in \mathcal{I}_k,$$

is bounded by $c\|\lambda\|_{[p,q]}$, $0 < q \leq \infty$. ■

The next result is of interest because it applies to arbitrary decompositions in $H^{p,q}$. The proof relies on Lemma 1.2, and is left to the reader.

THEOREM 2.4. *Let $0 < p \leq 1$, $0 < q \leq \infty$, and let f be a distribution given by $f = \sum_j \lambda_j a_j(x)$, where the a_j 's are H^p atoms, and the convergence is in the sense of distributions. Further assume that $\|\lambda\|_{[\eta,q]} < \infty$ for some $0 < \eta < \min(p, q)$. Then $f \in H^{p,q}$, and $\|f\|_{H^{p,q}} \leq c\|\lambda\|_{[\eta,q]}$.*

2.1. Interpolation between Hardy–Lorentz spaces. We are now ready to identify the intermediate spaces of a couple of Hardy–Lorentz spaces with the same first index $p \leq 1$.

THEOREM 2.5. *Let $0 < p \leq 1$. Given $0 < q_1 < q < q_2 \leq \infty$, define $0 < \eta < 1$ by the relation $1/q = (1 - \eta)/q_1 + \eta/q_2$. Then, with equivalent quasinorms,*

$$H^{p,q} = (H^{p,q_1}, H^{p,q_2})_{\eta,q}.$$

Proof. Since the non-tangential maximal function Nf of a distribution f in H^{p,q_1} is in L^{p,q_1} , and that of f in H^{p,q_2} is in L^{p,q_2} , we have

$$K(t, Nf; L^{p,q_1}, L^{p,q_2}) \leq cK(t, f; H^{p,q_1}, H^{p,q_2}).$$

Thus,

$$\|Nf\|_{p,q} \sim \|Nf\|_{(L^{p,q_1}, L^{p,q_2})_{\eta,q}} \leq c\|f\|_{(H^{p,q_1}, H^{p,q_2})_{\eta,q}},$$

and $(H^{p,q_1}, H^{p,q_2})_{\eta,q} \hookrightarrow H^{p,q}$.

To show the other embedding, with the notation in the proof of Theorem 2.1, write $f = \sum_k \sum_j \lambda_{j,k} a_{j,k}$, and recall that for every integer k , the level set $\mathcal{I}_k = \{j : |\lambda_{j,k}|/|I_{j,k}|^{1/p} \sim 2^k\}$ contains exclusively the sequence $\{\lambda_{j,k}\}$. Let $\mu_k^p = \sum_{j \in \mathcal{I}_k} |\lambda_{j,k}|^p$. By construction, $\sum_k \mu_k^q \sim \|f\|_{H^{p,q}}^q$. Now, rearrange $\{\mu_k\}$ into $\{\mu_l^*\}$, and, for each $l \geq 1$, let k_l be such that $\mu_{k_l} = \mu_l^*$. For $l_0 \geq 1$, let $\mathcal{K}_{l_0} = \{k_1, \dots, k_{l_0}\}$, and put $f_{1,l_0} = \sum_{k \in \mathcal{K}_{l_0}} \sum_j \lambda_{j,k} a_{j,k}$ and $f_{2,l_0} = f - f_{1,l_0}$. Then, by Theorem 2.2, $f_{1,l_0} \in H^{p,q_1}$, $f_{2,l_0} \in H^{p,q_2}$, and, with the usual interpretation for $q_2 = \infty$,

$$\|f_{1,l_0}\|_{H^{p,q_1}} \leq c \left(\sum_{l=1}^{l_0} \mu_l^{*q_1} \right)^{1/q_1}, \quad \|f_{2,l_0}\|_{H^{p,q_2}} \leq c \left(\sum_{l=l_0+1}^{\infty} \mu_l^{*q_2} \right)^{1/q_2}.$$

So, for $t > 0$ and every positive integer l_0 , we have

$$K(t, f; H^{p,q_1}, H^{p,q_2}) \leq c \left[\left(\sum_{l=1}^{l_0} \mu_l^{*q_1} \right)^{1/q_1} + t \left(\sum_{l=l_0+1}^{\infty} \mu_l^{*q_2} \right)^{1/q_2} \right].$$

Now, by Holmstedt’s formula, there is a choice of l_0 such that the right-hand side above is $\sim K(t, \{\mu_k\}; \ell^{q_1}, \ell^{q_2})$, and consequently,

$$K(t, f; H^{p,q_1}, H^{p,q_2}) \leq cK(t, \{\mu_k\}; \ell^{q_1}, \ell^{q_2}).$$

Thus,

$$\|f\|_{(H^{p,q_1}, H^{p,q_2})_{\eta,q}} \leq c\|\{\mu_k\}\|_{(\ell^{q_1}, \ell^{q_2})_{\eta,q}} \leq c\|\{\mu_k\}\|_{\ell_q} \leq c\|f\|_{H^{p,q}},$$

and $H^{p,q} \hookrightarrow (H^{p,q_1}, H^{p,q_2})_{\eta,q}$. ■

The reader will have no difficulty in verifying that Theorem 2.5 gives the following result: if T is a continuous, sublinear map from H^1 into L^1 , and from $H^{1,\infty}$ into $L^{1,\infty}$, then $\|Tf\|_{1,q} \leq c\|f\|_{H^{1,q}}$ for $1 < q < \infty$.

This observation has numerous applications. For instance, consider the Calderón–Zygmund singular integral operators with variable kernel defined by

$$T_\Omega(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n} f(y) dy.$$

Under appropriate growth and smoothness assumptions on Ω , T_Ω maps H^1 continuously into L^1 (see [5]), and $H^{1,\infty}$ continuously into $L^{1,\infty}$ (see [7]). Thus, if Ω satisfies the assumptions of both of these results, then T_Ω maps $H^{1,q}$ continuously into $L^{1,q}$ for $1 < q < \infty$. A similar result follows by invoking the characterization of $H^{1,q}$ given by C. Fefferman, Rivière and Sagher. However, in this case the H^p - L^p estimate requires additional smoothness of Ω , as shown, for instance, in [5]. Similar considerations apply to the Marcinkiewicz integral (see [8] and [6]).

Finally, when $p < 1$, our results cover, for instance, the δ -CZ operators satisfying $T^*(1) = 0$ discussed by Alvarez and Milman (see [2]). These operators, as well as a more general related class introduced in [14], preserve H^p and $H^{p,\infty}$ for $n/(n+\delta) < p \leq 1$, and consequently, by Theorem 2.5, they also preserve $H^{p,q}$ for p in that same range, and $q > p$.

References

- [1] J. Alvarez, *H^p and weak H^p continuity of Calderón–Zygmund type operators*, in: Lecture Notes in Pure Appl. Math. 157, Dekker, 1994, 17–34.
- [2] J. Alvarez and M. Milman, *H^p continuity properties of Calderón–Zygmund-type operators*, J. Math. Anal. Appl. 118 (1986), 63–79.
- [3] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, 1976.
- [4] A. P. Calderón, *An atomic decomposition of distributions in parabolic H^p spaces*, Adv. Math. 25 (1977), 216–225.
- [5] J. Chen, Y. Ding, and D. Fan, *A class of integral operators with variable kernels on Hardy spaces*, Chinese Ann. Math. 23 (2002), 289–296.
- [6] Y. Ding, C.-C. Lin, and S. Shao, *On the Marcinkiewicz integral with variable kernels*, Indiana Univ. Math. J. 53 (2004), 805–821.

- [7] Y. Ding, S. Lu, and S. Shao, *Integral operators with variable kernels on weak Hardy spaces*, J. Math. Anal. Appl. 317 (2006), 127–135.
- [8] Y. Ding, S. Lu, and Q. Xue, *Marcinkiewicz integral on Hardy spaces*, Integral Equations Operator Theory 42 (2002), 174–182.
- [9] C. Fefferman, N. M. Rivière, and Y. Sagher, *Interpolation between H^p spaces: the real method*, Trans. Amer. Math. Soc. 191 (1974), 75–81.
- [10] R. Fefferman and F. Soria, *The space Weak H^1* , Studia Math. 85 (1987), 1–16.
- [11] T. Holmstedt, *Interpolation of quasi-normed spaces*, Math. Scand. 26 (1970), 177–199.
- [12] P. Krée, *Interpolation d'espaces vectoriels qui ne sont ni normés ni complets. Applications*, Ann. Inst. Fourier (Grenoble) 17 (1967), 137–174.
- [13] D. V. Parilov, *Two theorems on the Hardy–Lorentz classes $H^{1,q}$* , Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 327 (2005), 150–167 (in Russian); English transl.: J. Math. Sci. (N.Y.) 139 (2006), 6447–6456.
- [14] T. Quek and D. Yang, *Calderón–Zygmund type operators on weighted weak Hardy spaces over \mathbb{R}^n* , Acta Math. Sinica (English Ser.) 16 (2000), 141–160.
- [15] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Dover, 2004.
- [16] F. Weisz, *Summability of Multi-Dimensional Fourier Series and Hardy Spaces*, Kluwer, 2002.
- [17] J. M. Wilson, *On the atomic decomposition for Hardy spaces*, Pacific J. Math. 116 (1985), 201–207.

Department of Mathematics
Indiana University
Bloomington, IN 47405, U.S.A.
E-mail: wabusham@indiana.edu
torchins@indiana.edu

Received March 13, 2007
Revised version August 14, 2007

(6122)