STUDIA MATHEMATICA 145 (1) (2001)

Extendibility of polynomials and analytic functions on ℓ_p

by

Daniel Carando (Buenos Aires)

Abstract. We prove that extendible 2-homogeneous polynomials on spaces with cotype 2 are integral. This allows us to find examples of approximable non-extendible polynomials on ℓ_p $(1 \le p < \infty)$ of any degree. We also exhibit non-nuclear extendible polynomials for $4 < p < \infty$. We study the extendibility of analytic functions on Banach spaces and show the existence of functions of infinite radius of convergence whose coefficients are finite type polynomials but which fail to be extendible.

Introduction. The aim of this note is to study the extendibility of polynomials and analytic functions on ℓ_p. This will allow us to show simple examples of non-extendible polynomials and analytic functions. Recall [12] that a k-homogeneous polynomial $P : E \to F$ is said to be *extendible* if for any Banach space G containing E there exists a polynomial $\widetilde{P} : G \to F$ extending P. The Hahn–Banach extension theorem gives the extendibility of all linear functionals but, even in the scalar-valued case ($F = \mathbb{R}$ or $\mathbb{C}$), this cannot be generalized to polynomials of degree 2 or more. For example, ℓ_2 is contained in $C[0,1]$ but the polynomial $P(x) = \sum_k x_k^2$ on ℓ_2 cannot be extended to $C[0,1]$ (this last space has the Dunford–Pettis property and consequently any polynomial on $C[0,1]$ is weakly sequentially continuous [16]).

The question arises about the existence of weakly sequentially continuous polynomials which fail to be extendible (this question was posed by I. Zalduendo in personal communications). Kirwan and Ryan [12] showed that extendible polynomials on Hilbert spaces are nuclear. In consequence, the polynomial

$$P(x) = \sum_{j=1}^{\infty} \frac{x_j^2}{j}$$

is approximable (and therefore weakly sequentially continuous) but not extendible on ℓ_2 (see [17]). We show that this polynomial is not extendible on any ℓ_p with $p \ge 2$ but it is nuclear (and therefore extendible) for $1 \le p < 2$.

2000 *Mathematics Subject Classification*: 46G20, 46G25.

Partially supported by Fundación Antorchas and AECI.

In the first section, we prove that 2-homogeneous extendible polynomials on ℓ_p are integral for $p = 1$ and nuclear for $1 < p \leq 2$. We give examples of non-nuclear extendible polynomials for $p > 4$. In the second section, we give a characterization, for degree 2, of diagonal nuclear polynomials and prove that diagonal extendible polynomials are nuclear ($1 < p < \infty$). These results allow us to find examples of non-extendible approximable polynomials of any degree (greater than 1) on ℓ_p for $1 \leq p < \infty$. In the last section we define extendible analytic functions and show the existence of analytic functions that are not extendible even though all their coefficients are finite type polynomials.

Throughout, E, F and G are Banach spaces. Although definitions and proofs are given for complex Banach spaces, slight modifications lead to analogous results for the real case.

We recall some definitions. The space of *finite type polynomials* $\mathcal{P}_{\mathrm{f}}({}^kE;F)$ is the subspace of $\mathcal{P}({}^kE;F)$ (the space of all continuous polynomials from E to F) spanned by the monomials $\gamma(\cdot)^k y$ for all $\gamma \in E'$, $y \in F$. The *approximable polynomials* are those which can be approximated by finite type polynomials uniformly on the unit ball of E.

A polynomial $P \in \mathcal{P}({}^kE;F)$ is said to be *nuclear* if it can be written as

$$P(x) = \sum_{i=1}^{\infty} \gamma_i(x)^k y_i \quad \forall x \in E$$

where $\gamma_i \in E'$ and $y_i \in F$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \|\gamma_i\|^k \|y_i\| < \infty$. The infimum of these sums over all the representations of P is the *nuclear norm* $\|P\|_{\mathrm{N}}$.

A polynomial $P \in \mathcal{P}({}^kE;F)$ is said to be *integral* if there exists a regular countably additive F-valued Borel measure μ, of bounded variation on $(B_{E'}, w^*)$, such that

$$P(x) = \int_{B_{E'}} \gamma(x)^k \, d\mu(\gamma) \quad \forall x \in E.$$

The *integral norm* $\|P\|_{\mathrm{I}}$ is the infimum of the norms of all the measures μ that represent P as above.

Finally, for an extendible polynomial $P \in \mathcal{P}({}^kE;F)$, its *extendible norm* is defined as

$$\|P\|_{\mathrm{e}} = \inf\{\|Q\| : Q \in \mathcal{P}({}^k\ell_\infty(B_{E'});F) \text{ is an extension of } P\}.$$

We refer to [10] and [14] for notation and results regarding polynomials.

The author wishes to thank the Departamento de Análisis Matemático de la Universidad Complutense de Madrid (where this note was finished) for its hospitality, and Nacho Zalduendo and Nacho Villanueva for helpful conversations.

1. Extendibility of polynomials on ℓ_p. Pisier (see [15]) showed the existence of spaces X for which the projective and injective tensor products $\otimes^2_{s,\pi} X$ and $\otimes^2_{s,\varepsilon} X$ are isometrically isomorphic. This means that every 2-homogeneous polynomial on X is integral and in particular extendible [5]. Note that, since X is not an $\mathcal{L}_\infty$-space, polynomials on X cannot be extended in a linear and continuous way (see [13, 17]).

Pisier spaces will be useful to exhibit spaces where only integral polynomials are extendible (this result was also obtained independently in [6]):

PROPOSITION 1.1. *Let E be a Banach space with cotype 2. Then extendible 2-homogeneous polynomials are integral* (*and the extendible and integral norms coincide*).

Proof. If E has cotype 2, E can be isometrically embedded in a Pisier space X (see [15]). If P is an extendible polynomial on E, it can be extended to a polynomial $\widetilde{P}$ on X with $\|\widetilde{P}\| \le \|P\|_{\mathrm{e}}$. Since $\otimes^2_{s,\pi} X$ and $\otimes^2_{s,\varepsilon} X$ are isometrically isomorphic, $\widetilde{P}$ is integral and $\|\widetilde{P}\|_{\mathrm{I}} = \|\widetilde{P}\|$. But the restriction of an integral polynomial is integral and also $\|P\|_{\mathrm{I}} \le \|\widetilde{P}\|_{\mathrm{I}} \le \|P\|_{\mathrm{e}}$. The reverse inequality always holds. ∎

If $1 \le p \le 2$ and X is an $\mathcal{L}_p$-space, then X has cotype 2 and every extendible 2-homogeneous polynomial is integral. Moreover, if $1 < p \le 2$ and μ is a measure, since $L_p(\mu)$ is reflexive, its dual has the Radon–Nikodym property and every integral polynomial is nuclear [1]. Consequently, extendible 2-homogeneous polynomials on $L_p(\mu)$ are nuclear (for $p = 2$ this was proved by Kirwan and Ryan [12]). In particular, we have the following:

COROLLARY 1.2. *For $1 < p \le 2$, a 2-homogeneous polynomial on ℓ_p is extendible if and only if it is nuclear.*

Corollary 1.2 is not true for $4 < p < \infty$. We show, for any $p \in (4, \infty)$, an example of an extendible polynomial which fails to be nuclear (we have no example of such a polynomial for $2 < p \le 4$). We follow the idea of [7, 10.4].

EXAMPLE 1.3. Consider the bilinear form A_n on ℓ^n_∞ given by the "Fourier matrix"

$$A_n(x, y) = \frac{1}{\sqrt{n}} \sum_{k,l=1}^{n} e^{2\pi i k l/n} x_k y_l$$

which has norm $\|A_n\| \le n$ but nuclear norm $\|A_n\|_{\mathrm{N}} = n^{3/2}$ [7, Ex. 4.3]. Therefore, the nuclear norm of the polynomial $P_n(x) = A_n(x, x)$ is at least $n^{3/2}$ on ℓ^n_∞. If we consider it on ℓ^n_p, since the identity map $\ell^n_\infty \to \ell^n_p$ has norm $n^{1/p}$, we have

$$\|P_n\|_{\mathrm{N},\ell^n_\infty} \le n^{2/p} \|P_n\|_{\mathrm{N},\ell^n_p}$$

and then

$$\|P_n\|_{\mathrm{N},\ell_p^n} \geq \frac{n^{3/2}}{n^{2/p}} = n^{3/2-2/p}.$$

Fix $1 < d < 2^{1/2-2/p} < \sqrt{2}$ and define $Q_m = (2d)^{-m} P_{2^m}$. Identifying c_0 with $c_0((\ell_\infty^{2^m})_m)$, it is easy to see that the polynomial $Q = \bigoplus_m Q_m$ is well defined and continuous on c_0. Let $i_p : \ell_p \to c_0$ be the canonical inclusion. Since every polynomial on c_0 is extendible, Q is extendible and consequently so is $Q \circ i_p$ (cf. [4]). Let us see that $Q \circ i_p$ is not nuclear. If it were, identifying ℓ_p with $\ell_p((\ell_p^{2^m})_m)$ we would have

$$\|Q_m\|_{\mathrm{N},\ell_p^{2^m}} \leq \|Q \circ i_p\|_{\mathrm{N},\ell_p}.$$

But what we do have is

$$\|Q_m\|_{\mathrm{N},\ell_p^{2^m}} \geq \frac{2^{m(3/2-2/p)}}{(2d)^m} = \left(\frac{2^{1/2-2/p}}{d}\right)^m,$$

which goes to infinity with m. Therefore, $Q \circ i_p$ is extendible but not nuclear on ℓ_p for $4 < p < \infty$.

We end this section with some comments about $\mathcal{L}_1$-spaces. We know that integral polynomials are extendible and that a 2-homogeneous extendible polynomial has an absolutely 2-summing differential [12]. Kirwan and Ryan also showed that, for $\mathcal{L}_1$-spaces, this last condition is sufficient for a polynomial to be extendible. Since $\mathcal{L}_1$-spaces have cotype 2, Proposition 1.1 implies that extendible polynomials are integral. Therefore we have:

COROLLARY 1.4. *If P is a 2-homogeneous polynomial on an $\mathcal{L}_1$-space, the following are equivalent*:

(a) *P is integral.*
(b) *P is extendible.*
(c) *The differential dP is absolutely 2-summing.*

2. Examples of non-extendible polynomials. Throughout, p and q will be such that $1/p + 1/q = 1$. Corollary 1.2 affirms that extendible 2-homogeneous polynomials on ℓ_p are nuclear for $1 < p \leq 2$. Although this is not true for all p (as shown above), we will see that extendible "diagonal" polynomials are nuclear. Therefore, a way to detect non-nuclear polynomials on ℓ_p will be helpful. The following lemma is our first step.

LEMMA 2.1. *Let P be a nuclear 2-homogeneous polynomial on ℓ_p.*

(1) *If $1 < p < 2$, then $(P(e_n))_n \in \ell_{q/2}$.*
(2) *If $2 \leq p < \infty$, then $(P(e_n))_n \in \ell_1$.*

Proof. (1) If P is nuclear then it is continuous for the injective norm. Then, if $(\alpha_k)_k$ is a finite sequence, we have

$$\Big|\sum_k P(\alpha_k e_k)\Big| \le C \sup_{b\in B_{\ell_q}} \Big|\sum_k b(\alpha_k e_k)^2\Big|$$
$$= C \sup_{b\in B_{\ell_q}} \Big|\sum_k b_k^2\alpha_k^2\Big| = C \sup_{c\in B_{\ell_{q/2}}} \Big|\sum_k c_k\alpha_k^2\Big|.$$

Since $|\sum_k \alpha_k^2 P(e_k)| = |\sum_k P(\alpha_k e_k)| \le C\sup_{c\in B_{\ell_{q/2}}} |\sum_k c_k\alpha_k^2|$ for every finite sequence $(\alpha_k)_k$, we conclude that $(P(e_n))_n \in \ell_{q/2}$.

(2) P being nuclear, we can write

$$P(x) = \sum_k \Big(\sum_j b_{k,j}x_k\Big)^2 \quad \text{with} \quad \sum_k \Big(\sum_j |b_{k,j}|^q\Big)^{2/q} < \infty.$$

Since $q < 2$, we have

$$\sum_n |P(e_n)| = \sum_n \Big|\sum_k (b_{k,n})^2\Big| \le \sum_k \Big(\sum_n |b_{k,n}|^q\Big)^{2/q} < \infty$$

and then $(P(e_n))_n \in \ell_1$. ■

We will now consider polynomials on ℓ_p of the form

$$P(x) = \sum_{j=1}^{\infty} a_j x_j^2,$$

which we will call *diagonal*. It is clear that if $\sum_{j=1}^{\infty} |a_j| < \infty$, then P is nuclear. Surprisingly enough, a diagonal polynomial can be nuclear even though the coefficients $(a_j)_j$ are not summable. The extreme case is ℓ_1, where any null sequence $(a_j)_j$ gives a nuclear polynomial. The following two propositions clarify the situation:

PROPOSITION 2.2. *Let $P(x)=\sum_{j=1}^{\infty} a_j x_j^2$ be a diagonal polynomial on ℓ_p.*

(1) *For $1 < p < 2$, P is nuclear if and only if $(a_k)_k \in \ell_{q/2}$.*
(2) *For $2 \le p < \infty$, P is nuclear if and only if $(a_k)_k \in \ell_1$.*

Proof. (1) It only remains to prove sufficiency. Suppose that $(a_k)_k \in \ell_{q/2}$. To see that P is nuclear, it is enough to show that the nuclear norm of the polynomial $\sum_{k=n}^{n+l} a_k x_k^2$ can be made arbitrarily small by taking $n\in\mathbb{N}$ large enough, independently of the size of l (since this means that P can be written as the sum of a sequence of polynomials with summable nuclear norms). Consider the operator $T : \ell_p \to \ell_1^{l+1}$ given by $T(x) = (a_n^{1/2}x_n, \ldots, a_{n+l}^{1/2}x_{n+l})$. The operator T has norm $(\sum_{k=n}^{n+l} |a_k|^{q/2})^{1/q}$. If we show that the polynomial $Q_l(y) = \sum_{k=1}^{l+1} y_k^2$ has unitary nuclear norm on ℓ_1^{l+1} (see also [7]), the composition $Q_l \circ T(x) = \sum_{k=n}^{n+l} a_k x_k^2$ has nuclear norm at most $(\sum_{k=n}^{n+l} |a_k|^{q/2})^{2/q}$

on ℓ_p^n. Since $(a_k)_k \in \ell_{q/2}$, we see that P is a nuclear polynomial. But $Q_l(y)$ can be rewritten as

$$Q_l(y) = \sum_{\varepsilon_1=\pm1,\ldots,\varepsilon_{l+1}=\pm1} \Big(\frac{\varepsilon_1 y_1 + \ldots + \varepsilon_{l+1} y_{l+1}}{2^{(l+1)/2}}\Big)^2$$

(note that if we expand the expression above, the product $y_i y_j$ appears multiplied by 1 as many times as it appears multiplied by -1). Therefore, the nuclear norm of Q_l is not greater than

$$\sum_{\varepsilon_1=\pm1,\ldots,\varepsilon_{l+1}=\pm1} \Big\|\frac{1}{2^{(l+1)/2}}(\varepsilon_1, \ldots, \varepsilon_{l+1})\Big\|_\infty^2 = \sum_{\varepsilon_1=\pm1,\ldots,\varepsilon_{l+1}=\pm1} \frac{1}{2^{l+1}} = 1.$$

Since $\|Q_l\|_N \geq \|Q_l\| = 1$, we have $\|Q_l\|_N = 1$.

(2) One implication follows from Lemma 2.1 and the other is clear. ∎

PROPOSITION 2.3. *Let* $P(x) = \sum_{j=1}^\infty a_j x_j^2$ *be a diagonal polynomial on* ℓ_1. *The following are equivalent*:

(a) P *is nuclear.*

(b) P *is approximable.*

(c) $(a_j)_j$ *is a null sequence.*

Proof. It is clear that (a) implies (b). If P is approximable, then dP is a compact operator [2]. For every j, we find that $a_j e'_j$ belongs to $dP(B_{\ell_1})$, which is a compact subset of ℓ_∞. This forces $(a_j)_j$ to be a null sequence. Now suppose that (c) is true. First observe that the polynomial $Q(x) = \sum_{k=1}^n b_k x_k^2$ on ℓ_1^n has nuclear norm $\max_k |b_k|^2$ (independently of the size of n). Then choose $k_i \in \mathbb{N}$ such that $\max_{k_i \leq k < k_{i+1}} |a_k|^2 \leq 1/2^i$. Thus, P can be written as the sum of a sequence of polynomials with summable nuclear norms, which shows that P is nuclear. ∎

We will now characterize the diagonal 2-homogeneous extendible polynomials on ℓ_p, for $1 < p < \infty$. Together with our characterization of nuclear polynomials, this will allow us to show the existence of approximable non-extendible polynomials on every ℓ_p.

PROPOSITION 2.4. *If* $1 < p < \infty$, *then diagonal extendible* 2-*homogeneous polynomials on* ℓ_p *are nuclear.*

Proof. For $1 < p \leq 2$, Corollary 1.2 affirms that every extendible polynomial is nuclear. For $p > 2$, let $P \in \mathcal{P}_{\mathrm{e}}(^2\ell_p)$ be given by $P(x) = \sum_k a_k x_k^2$ and consider $Q \in \mathcal{P}(^2\ell_2)$ given by the same formula. Since $Q = P \circ i$ (where $i : \ell_2 \to \ell_p$ is the natural inclusion), Q is extendible [4] and therefore nuclear. As we have already seen, this means that $\sum_k |a_k| < \infty$ and consequently P is also nuclear on ℓ_p. ∎

The previous result is not true for $p = 1$, since the polynomial $P(x) = \sum_k x_k^2$ on ℓ_1 is integral (and therefore extendible) but not nuclear. This makes it harder to find approximable non-extendible polynomials on ℓ_1 than on any other ℓ_p, where we can find diagonal polynomials satisfying our requirements.

COROLLARY 2.5. *There are approximable non-extendible* 2*-homogeneous polynomials on every* ℓ_p $(1 \le p < \infty)$.

Proof. For $1 < p \le 2$, the polynomial $P(x) = \sum_k a_k x_k^2$ is approximable whenever a_k is a null sequence, but is not extendible if we take $(a_k)_k \notin \ell_{q/2}$ (note that if $p = 2$, then $q/2 = 1$).

For $p > 2$, the polynomial $P(x) = \sum_k a_k x_k^2$ is well defined and approximable for $(a_k)_k \in \ell_r$ if $r = 1 + 2/(p-2)$. Taking $(a_k)_k$ in ℓ_r but not in ℓ_1, we get a non-nuclear diagonal polynomial which by Proposition 2.4 cannot be extendible.

For $p = 1$, extendible polynomials are integral by Proposition 1.1, so we have to see that there are approximable polynomials which are not integral. If every approximable polynomial is integral, by the closed graph theorem, the inclusion $\mathcal{P}_{\mathrm{A}}(^2\ell_1) \hookrightarrow \mathcal{P}_{\mathrm{I}}(^2\ell_1)$ is continuous. Since we always have $\|P\| \le \|P\|_{\mathrm{I}}$, both norms turn out to be equivalent on $\mathcal{P}_{\mathrm{A}}(^2\ell_1)$. On the other hand, the space $\mathcal{P}_{\mathrm{N}}(^2\ell_1)$ is dense in $(\mathcal{P}_{\mathrm{A}}(^2\ell_1), \|\cdot\|)$ and, by Theorem VIII.3.10 of [8], closed in $(\mathcal{P}_{\mathrm{I}}(^2\ell_1), \|\cdot\|_{\mathrm{I}})$. By the equivalence of the norms, $\mathcal{P}_{\mathrm{N}}(^2\ell_1)$ and $\mathcal{P}_{\mathrm{A}}(^2\ell_1)$ coincide. Taking duals (see [11] and [9]), we obtain $\mathcal{P}(^2\ell_\infty) = \mathcal{P}_{\mathrm{I}}(^2\ell_\infty)$, and in particular, every 2-homogeneous polynomial on c_0 should be integral. Since this is false [5], we conclude that there are approximable polynomials on ℓ_1 which are not integral and consequently fail to be extendible. ■

EXAMPLE 2.6. The polynomial

$$P(x) = \sum_k \frac{x_k^2}{k}$$

is approximable but not extendible on every ℓ_p, $p \ge 2$ (but is nuclear, and therefore extendible, for $1 \le p < 2$). The polynomial

$$Q(x) = \sum_k \frac{x_k^2}{\ln(k+1)}$$

is approximable but not extendible for $1 < p \le 2$ (observe that Q is nuclear if we consider it on ℓ_1 and is not defined for $p > 2$).

We want to generalize Corollary 2.5 to polynomials of any degree. This will be easy with the help of the following:

PROPOSITION 2.7. *Let P be a k-homogeneous polynomial on a Banach space E and $\gamma \in E'$, $\gamma \neq 0$. Then P is extendible if and only if the $(k+1)$-homogeneous polynomial $\gamma(x)P(x)$ is extendible.*

Proof. If P is extendible then it is clear that γP is extendible. Conversely, let Q be an extension of γP to a Banach space F containing E, and $\phi \in F'$ an extension of γ. If $e \in E$ is such that $\gamma(e) = 1$, there exists a k-homogeneous polynomial Q_0 on F such that

$$Q(f) - Q(f - \phi(f)e) = \phi(f)Q_0(f) \quad \text{for } f \in F$$

(see [3]). For $x \in E$, we have

$$(\gamma P)(x) - (\gamma P)(x - \gamma(x)e) = \gamma(x)Q_0(x),$$
$$\gamma(x)P(x) = \gamma(x)Q_0(x)$$

and since γ is non-zero on a dense subset of E we see that Q_0 extends P to F. ■

COROLLARY 2.8. *There are approximable non-extendible polynomials of any degree $k \geq 2$ on ℓ_p for $1 \leq p < \infty$.*

Proof. If P is an approximable non-extendible 2-homogeneous polynomial, then for any $\gamma \in \ell_q$, $\gamma \neq 0$, the polynomial $\gamma^{k-2}P$ is approximable but non-extendible. ■

Note that the previous results not only prove the existence of approximable non-extendible polynomials of any degree, but also show a way to find examples of them. Following the idea of Proposition 2.7 we find that the product of linear functionals with the polynomial exhibited in Example 1.3 will give extendible non-nuclear polynomials of any degree.

Another consequence of Proposition 2.7 is:

COROLLARY 2.9. *If every k-homogeneous polynomial on E is extendible, so is every j-homogeneous polynomial for $1 \leq j \leq k$.*

3. Extendibility of analytic functions. Let U be an open subset of E. We will say that an analytic function $f : U \to F$ is *extendible* at $a \in U$ if for any $G \supset E$ there exists an open subset V of G containing a and an analytic function $\widetilde{f} : V \to F$ which coincides with f on $V \cap E$. If such an f has a Taylor expansion

$$f(x) = \sum_{k=0}^{\infty} P_k(x-a)$$

where $P_k \in \mathcal{P}(^kE, F)$, from the uniqueness of these expansions we deduce that every P_k must be extendible. We will see that the extendibility of the coefficients P_k is not enough to ensure the extendibility of f. First we define the *extendibility radius* of f (at a) as

$$r_{\mathrm{e}} = \frac{1}{\limsup_{k\to\infty} \|P_k\|_{\mathrm{e}}^{1/k}}$$

if every P_k is extendible. Since $\|P_k\| \le \|P_k\|_{\mathrm{e}}$ for all k, the extendibility radius is not greater than the radius of uniform convergence.

PROPOSITION 3.1. *Let $f(x) = \sum_{k=0}^{\infty} P_k(x-a)$ be an analytic function from $U \subset E$ to F. The following conditions are equivalent*:

(a) *f is extendible at a.*

(b) *Every P_k is extendible and the extendibility radius r_{e} is positive.*

Moreover, *if* (a) *and* (b) *occur*, *given G containing E we can extend f to an analytic function on $a \in G$ with radius of uniform convergence at least r_{e}.*

Proof. If f is extendible every P_k is extendible, as we observed above. f being extendible, we extend it to an open subset of $\ell_\infty(B_{E'})$ containing a and call the coefficients of the extension $\overline{P}_k$ (which are extensions of P_k). Therefore, $\|\overline{P}_k\| \ge \|P_k\|_{\mathrm{e}}$ and

$$r_{\mathrm{e}} = \frac{1}{\limsup_{k\to\infty} \|P_k\|_{\mathrm{e}}^{1/k}} \ge \frac{1}{\limsup_{k\to\infty} \|\overline{P}_k\|^{1/k}},$$

which is positive since $\sum_k \overline{P}_k(x-a)$ is analytic at a (and has positive radius of convergence).

Conversely, suppose (b) is true and let G be a Banach space containing E. For any k we take $\varepsilon_k > 0$ such that

$$\limsup_{k\to\infty} \|P_k\|_{\mathrm{e}}^{1/k} = \limsup_{k\to\infty} (\|P_k\|_{\mathrm{e}} + \varepsilon_k)^{1/k}.$$

We also take for each k an extension $\widetilde{P}_k$ of P_k to G such that $\|\widetilde{P}_k\| \le \|P_k\|_{\mathrm{e}} + \varepsilon_k$. If we define $\widetilde{f}(z) = \sum_{k=0}^{\infty} \widetilde{P}_k(z-a)$ (for $z \in F$) we have

$$\frac{1}{\limsup_{k\to\infty} \|\widetilde{P}_k\|^{1/k}} \ge \frac{1}{\limsup_{k\to\infty} (\|P_k\|_{\mathrm{e}} + \varepsilon_k)^{1/k}} = \frac{1}{\limsup_{k\to\infty} \|P_k\|_{\mathrm{e}}^{1/k}} = r_{\mathrm{e}}$$

This means that $\widetilde{f}$ is analytic and has radius of uniform convergence (at a) greater than or equal to r_{e}. In consequence, (a) is true, as is the statement about the convergence radius of the extensions. ■

We have shown that there are approximable non-extendible polynomials of any degree $k \ge 2$ in every ℓ_p ($1 \le p < \infty$). For such a polynomial P, there exists a sequence of finite type polynomials which approximate it in norm. However, this sequence cannot approximate P in extendible norm, since this would mean that P is extendible (because finite type polynomials are extendible and the extendible norm is complete). So we conclude that the usual norm and the extendible norm are not equivalent on the subspace $\mathcal{P}_{\mathrm{f}}({}^k\ell_p)$ of finite type polynomials, for any $k \geq 2$ and $1 \leq p < \infty$. Therefore, for each $k \geq 2$ we can find a finite type polynomial P_k of degree k such that

$$\|P_k\| \leq \frac{1}{k^k} \quad \text{and} \quad \|P_k\|_{\mathrm{e}} \geq k^k.$$

If we define $f(x) = \sum_k P_k(x)$ on ℓ_p (with P_0 and P_1 arbitrary), then f is an analytic function with infinite radius of uniform convergence. All its coefficients are finite type polynomials (and therefore extendible) but f is not extendible since its extendible radius is 0.

Note that the previous idea can be used for any space on which there is an approximable non-extendible polynomial. If we only know that there exists a non-extendible polynomial, we make use of the following fact ([12], see also [4]): the extendible norm is equivalent to the usual norm on $\mathcal{P}_{\mathrm{e}}({}^kE)$ if and only if every k-homogeneous polynomial is extendible. With this result, Proposition 2.7 and a similar construction we can find a non-extendible analytic function with infinite radius of convergence such that every coefficient is extendible. We summarize all this in the following theorem:

THEOREM 3.2. (1) *On any space with an approximable non-extendible polynomial there exists an analytic function* (*of infinite radius of convergence*) *with finite type coefficients that is not extendible.*

(2) *On any space with a non-extendible polynomial there exists an analytic function* (*of infinite radius of convergence*) *with extendible coefficients that is not extendible.*

References

[1] R. Alencar, *On reflexivity and basis for $\mathcal{P}({}^mE)$*, Proc. Roy. Irish Acad. Sect. A 85 (1985), 131–138.

[2] R. Aron, C. Hervés and M. Valdivia, *Weakly continuous mappings on Banach spaces*, J. Funct. Anal. 52 (1983), 189–204.

[3] R. Aron and M. Schottenloher, *Compact holomorphic mappings on Banach spaces and the approximation property*, ibid. 21 (1976), 7–30.

[4] D. Carando, *Extendible polynomials on Banach spaces*, J. Math. Anal. Appl. 233 (1999), 359–372.

[5] D. Carando and I. Zalduendo, *A Hahn–Banach theorem for integral polynomials*, Proc. Amer. Math. Soc. 127 (1999), 241–250.

[6] J. Castillo, R. García and J. Jaramillo, *Extension of bilinear forms on Banach spaces*, preprint.

[7] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Math. Stud. 176, North-Holland, 1993.

[8] J. Diestel and J. Uhl, *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., 1977.
[9] S. Dineen, *Holomorphy types on Banach spaces*, Studia Math. 39 (1971), 240–288.
[10] —, *Complex Analysis in Locally Convex Spaces*, North-Holland Math. Stud. 57, North-Holland, Amsterdam, 1981.
[11] C. Gupta, *Malgrange theorem for nuclearly entire functions of bounded type on a Banach space*, Ph.D. thesis, Univ. of Rochester, 1966.
[12] P. Kirwan and R. Ryan, *Extendibility of homogeneous polynomials on Banach spaces*, Proc. Amer. Math. Soc. 126 (1998), 1023–1029.
[13] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. 48 (1964).
[14] J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland Math. Stud. 120, North-Holland, Amsterdam, 1986.
[15] G. Pisier, *Counterexamples to a conjecture of Grothendieck*, Acta Math. 151 (1983), 181–208.
[16] R. Ryan, *Dunford–Pettis properties*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (1979), 373–379.
[17] I. Zalduendo, *Extending polynomials—a survey*, Publicaciones del Departamento de Análisis Matemático de la Universidad Complutense 41 (1998).

Departamento de Matemática
Universidad de San Andrés
Vito Dumas 284
(1644) Victoria, Argentina
E-mail: daniel@udesa.edu.ar

Received June 13, 2000
Revised version December 18, 2000 (4548)