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H^∞ functional calculus in real interpolation spaces, II

by

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Abstract. Let A be a linear closed one-to-one operator in a complex Banach space X, having dense domain and dense range. If A is of type ω (i.e. the spectrum of A is contained in a sector of angle 2ω, symmetric about the real positive axis, and $\|\lambda(\lambda I - A)^{-1}\|$ is bounded outside every larger sector), then A has a bounded H^∞ functional calculus in the real interpolation spaces between X and the intersection of the domain and the range of the operator itself.

1. Introduction. In this paper we consider the H^∞ functional calculus on a sector for a closed, linear, one-to-one operator A on a complex Banach space X, having dense domain and dense range, with resolvent set that contains $\mathbb{R}^-$ and resolvent that decreases in a maximal way on $\mathbb{R}^-$ (i.e. $\|\lambda(\lambda I - A)^{-1}\|$ is bounded). This is a sequel to [1], in which it was proved that such an operator A has a bounded H^∞ functional calculus in the real interpolation spaces between X and $\mathcal{D}(A)$, provided that $0 \in \varrho(A)$. When $0 \notin \varrho(A)$ this theorem is not true, since in particular if A is bounded (i.e. $\mathcal{D}(A) = X$) then every real interpolation space between X and $\mathcal{D}(A)$ coincides with X, but there are bounded operators without a bounded H^∞ functional calculus on a sector.

In the present paper we consider the case $0 \notin \varrho(A)$ and we prove that A has a bounded H^∞ functional calculus in the real interpolation spaces between X and $\mathcal{D}(A) \cap \mathcal{R}(A)$.

We refer to [1] for notations and definitions, in particular for the definition of S_ω, S_ω^0, $H^\infty(S_\mu^0)$, $\Psi(S_\mu^0)$, operator of type ω, H^∞ functional calculus, $L_*^p(\mathbb{R}^+)$, real interpolation space.

2. Preliminary results. Let A be a one-to-one operator of type ω ($\omega \in [0, \pi[$). If $z \in \mathbb{C} \setminus S_\omega$ then also $z^{-1} \in \mathbb{C} \setminus S_\omega$, therefore it is easy to show

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that $z \in \varrho(A^{-1})$ and

$$(zI - A^{-1})^{-1} = -z^{-1}A(z^{-1}I - A)^{-1}.$$

Thus if $\theta \in]\omega, \pi[$ and M_θ is such that $\|(zI - A)^{-1}\| \leq M_\theta/|z|$ for $z \in \mathbb{C} \setminus S_\theta$ then

$$\begin{aligned}\|(zI - A^{-1})^{-1}\| &= \|z^{-1}A(z^{-1}I - A)^{-1}\| \\ &= \|z^{-1}(z^{-1}I - z^{-1}I + A)(z^{-1}I - A)^{-1}\| \\ &\leq \|z^{-2}(z^{-1}I - A)^{-1}\| + \|z^{-1}\| \leq \frac{M_\theta + 1}{|z|}.\end{aligned}$$

Therefore A^{-1} is an operator of type ω.

We denote by $\mathcal{D}(A; \alpha, p)$ the real interpolation space $(X, \mathcal{D}(A))_{\alpha,p}$ (with $\alpha \in]0, 1[$ and $p \in [1, \infty]$); moreover, we denote by $\mathcal{R}(A; \alpha, p)$ the real interpolation space $(X, \mathcal{R}(A))_{\alpha,p}$ (with $\|x\|_{\mathcal{D}(A)} = \|x\|_X + \|Ax\|_X$ and $\|x\|_{\mathcal{R}(A)} = \|x\|_X + \|A^{-1}x\|_X$).

The norm of x in $\mathcal{D}(A; \alpha, p)$ is equivalent to

$$\|x\|_X + \|t \mapsto t^\alpha A(tI + A)^{-1}x\|_{L^p_*(\mathbb{R}^+)}$$

(see [2], Definition 1.1 and Theorem 3.1). We note that when $0 \in \varrho(A)$ the term $\|x\|_X$ can be disregarded, while if A has unbounded inverse this term is essential.

Since $\mathcal{D}(A^{-1}) = \mathcal{R}(A)$ we have $\mathcal{R}(A; \alpha, p) = \mathcal{D}(A^{-1}; \alpha, p)$, therefore an equivalent norm on $\mathcal{R}(A; \alpha, p)$ is $\|x\|_X + \|t \mapsto t^\alpha A^{-1}(tI + A^{-1})^{-1}x\|_{L^p_*(\mathbb{R}^+)}$. But

$$t^\alpha A^{-1}(tI + A^{-1})^{-1} = t^\alpha A^{-1}t^{-1}A(t^{-1}I + A)^{-1} = t^{\alpha-1}(t^{-1}I + A)^{-1},$$

therefore this norm is equivalent to $\|x\|_X + \|t \mapsto t^{\alpha-1}(t^{-1}I + A)^{-1}x\|_{L^p_*(\mathbb{R}^+)}$.

Let E and F be Banach spaces (embedded in the same vector space). The space $E \cap F$ is a Banach space if endowed with the norm $\|x\|_{E\cap F} = \|x\|_E + \|x\|_F$.

From now on we will drop the subscript in the notation $\|\cdot\|_X$.

Theorem 2.1. *Let A be a one-to-one operator of type ω with dense domain and dense range. Let $\alpha \in]0, 1[$ and $p \in [1, \infty]$. Then the norm on $\mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p)$ is equivalent to*

$$\|t \mapsto t^\alpha A(tI + A)^{-1}x\|_{L^p_*(\mathbb{R}^+)} + \|t \mapsto t^{1-\alpha}(tI + A)^{-1}x\|_{L^p_*(\mathbb{R}^+)}.$$

Proof. From the above observations it follows immediately that the norm of $\mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p)$ is equivalent to

$$\|x\| + \|t \mapsto t^\alpha A(tI + A)^{-1}x\|_{L^p_*(\mathbb{R}^+)} + \|t \mapsto t^{1-\alpha}(tI + A)^{-1}x\|_{L^p_*(\mathbb{R}^+)},$$

therefore in order to prove the theorem it is sufficient to show that there exists $C \in \mathbb{R}^+$ such that for every $x \in \mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p)$ we have

$$\|x\| \le C(\|t \mapsto t^\alpha A(tI+A)^{-1}x\|_{L^p_*(\mathbb{R}^+)} + \|t \mapsto t^{1-\alpha}(tI+A)^{-1}x\|_{L^p_*(\mathbb{R}^+)}).$$

If $p < \infty$, then since

$$\int_{\mathbb{R}^+} (e^{-\alpha|\log t|})^p \frac{dt}{t} = \frac{2}{\alpha p}$$

and for every $t \in \mathbb{R}^+$, $x \in X$,

$$x = A(tI+A)^{-1}x + t(tI+A)^{-1}x,$$

we have

$$\begin{aligned}
\|x\| &= \Big(\frac{\alpha p}{2} \int_{\mathbb{R}^+} \|e^{-\alpha|\log t|}x\|^p \frac{dt}{t}\Big)^{1/p} \\
&\le \Big(\frac{\alpha p}{2} \int_{\mathbb{R}^+} \|e^{-\alpha|\log t|}A(tI+A)^{-1}x\|^p \frac{dt}{t}\Big)^{1/p} \\
&\quad + \Big(\frac{\alpha p}{2} \int_{\mathbb{R}^+} \|e^{-\alpha|\log t|}t(tI+A)^{-1}x\|^p \frac{dt}{t}\Big)^{1/p} \\
&\le \Big(\frac{\alpha p}{2} \int_{\mathbb{R}^+} \|t^\alpha A(tI+A)^{-1}x\|^p \frac{dt}{t}\Big)^{1/p} \\
&\quad + \Big(\frac{\alpha p}{2} \int_{\mathbb{R}^+} \|t^{1-\alpha}(tI+A)^{-1}x\|^p \frac{dt}{t}\Big)^{1/p}.
\end{aligned}$$

If $p = \infty$, then

$$\begin{aligned}
\|x\| &\le \|1^\alpha A(I+A)^{-1}x\| + \|1^{1-\alpha}(I+A)^{-1}x\| \\
&\le \sup_{t\in\mathbb{R}^+} \|t^\alpha A(tI+A)^{-1}x\| + \sup_{t\in\mathbb{R}^+} \|t^{1-\alpha}(tI+A)^{-1}x\|.
\end{aligned}$$

This concludes the proof.

THEOREM 2.2. *Let A be a one-to-one operator of type ω. Let B be the operator from $\mathcal{D}(A) \cap \mathcal{R}(A)$ to X such that $Bx = (2I + A + A^{-1})x$. Then B is a closed operator of type ω_0 (for a suitable ω_0) and $0 \in \varrho(B)$. Moreover, if A has dense domain and dense range then $\mathcal{D}(B)$ is dense.*

Proof. Obviously, $B = (I+A)^2A^{-1}$ but $(I+A)^2$ has bounded inverse and A^{-1} is closed, therefore B is closed and its inverse is $A(I+A)^{-2}$, hence $0 \in \varrho(B)$.

For $t \in \mathbb{R}^+$ put

$$\tau_t = \frac{t + 2 + \sqrt{t^2+4t}}{2}.$$

We have

$$((t+2)I + A + A^{-1})^{-1} = \tau_t(\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}.$$

Indeed, $t+2 = \tau_t + \tau_t^{-1}$, thus for $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ we have

$$\begin{aligned}
\tau_t(\tau_t I + A)^{-1}&(\tau_t I + A^{-1})^{-1}((t+2)I + A + A^{-1})x \\
&= \tau_t(\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}((\tau_t + \tau_t^{-1})I + A + A^{-1})x \\
&= (\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}((\tau_t^2 + 1)I + \tau_t A + \tau_t A^{-1})x \\
&= (\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}(\tau_t I + A^{-1})(\tau_t I + A)x = x
\end{aligned}$$

and analogously, for every $x \in X$,

$$((t+2)I + A + A^{-1})\tau_t(\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}x = x;$$

therefore $\mathbb{R}^- \subseteq \varrho(B)$. Moreover, we have

$$\begin{aligned}
\|(tI + B)^{-1}\| &= \|\tau_t(\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}\| \\
&= \|(\tau_t I + A)^{-1}A(\tau_t^{-1} I + A)^{-1}\| \le \frac{C}{\tau_t} \le \frac{C}{t+1},
\end{aligned}$$

therefore B is of type ω_0 for some ω_0.

Suppose now that $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are dense in X. Then for every $x \in X$ we have

$$\begin{aligned}
&\|tA(tI + A)^{-1}(t^{-1}I + A)^{-1}x - x\| \\
&\quad \le \|tA(tI + A)^{-1}(t^{-1}I + A)^{-1}x - t(tI + A)^{-1}x + t(tI + A)^{-1}x - x\| \\
&\quad \le \|t(tI + A)^{-1}\| \cdot \|A(t^{-1}I + A)^{-1}x - x\| + \|t(tI + A)^{-1}x - x\| \xrightarrow[t \to 0^+]{} 0.
\end{aligned}$$

But

$$tA(tI + A)^{-1}(t^{-1}I + A)^{-1}x \in \mathcal{D}(A) \cap \mathcal{R}(A),$$

so x is a limit of elements of $\mathcal{D}(A) \cap \mathcal{R}(A)$, therefore $\mathcal{D}(B) = \mathcal{D}(A) \cap \mathcal{R}(A)$ is dense in X. This proves the theorem.

Note that

$$\begin{aligned}
\|x\|_{\mathcal{D}(B)} &= \|x\| + \|Bx\| = \|x\| + \|(2I + A + A^{-1})x\| \\
&\le 3\|x\| + \|Ax\| + \|A^{-1}x\| = \|x\|_X + \|x\|_{\mathcal{D}(A)} + \|x\|_{\mathcal{R}(A)} \\
&\le \|x\|_{\mathcal{D}(A) \cap \mathcal{R}(A)}.
\end{aligned}$$

Therefore the vector spaces $\mathcal{D}(A) \cap \mathcal{R}(A)$ and $\mathcal{D}(B)$ are equal and the first space is continuously embedded in the second one; by the open mapping theorem the reverse embedding is continuous and the two norms are equivalent.

It follows that $(X, \mathcal{D}(A) \cap \mathcal{R}(A))_{\alpha,p} = \mathcal{D}(B; \alpha, p)$ with equivalent norms.

THEOREM 2.3. *Let A be a one-to-one operator of type ω with dense domain and dense range. Let $\alpha \in]0,1[$ and $p \in [1,\infty]$. Then $\mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p) = (X, \mathcal{D}(A) \cap \mathcal{R}(A))_{\alpha,p}$ with equivalent norms.*

Proof. Since $\mathcal{D}(A) \cap \mathcal{R}(A)$ is continuously embedded in $\mathcal{D}(A)$ and in $\mathcal{R}(A)$, by interpolation we deduce that $(X, \mathcal{D}(A) \cap \mathcal{R}(A))_{\alpha,p}$ is continuously embedded in $\mathcal{D}(A;\alpha,p)$ and in $\mathcal{R}(A;\alpha,p)$ and also in their intersection.

As we have already observed, $\mathcal{D}(A) \cap \mathcal{R}(A) = \mathcal{D}(B)$ (with B as in Theorem 2.2); therefore, in order to prove the inverse embedding, it is sufficient to prove that $\mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p)$ is continuously embedded in $\mathcal{D}(B;\alpha,p)$.

Let $x \in \mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p)$; we have

$$\begin{aligned}
&\|t \mapsto t^\alpha B(tI+B)^{-1}x\|_{L^p_*(\mathbb{R}^+)} \\
&\qquad \le \|t \mapsto t^\alpha (2I + A + A^{-1})(tI+B)^{-1}x\|_{L^p_*(\mathbb{R}^+)} \\
&\qquad \le \|t \mapsto t^\alpha 2(tI+B)^{-1}x\|_{L^p_*(\mathbb{R}^+)} \\
&\qquad\quad + \|t \mapsto t^\alpha A(tI+A)^{-1}(tI+A)(tI+B)^{-1}x\|_{L^p_*(\mathbb{R}^+)} \\
&\qquad\quad + \|t \mapsto t^\alpha A^{-1}(tI+A^{-1})^{-1}(tI+A^{-1})(tI+B)^{-1}x\|_{L^p_*(\mathbb{R}^+)} \\
&\qquad \le C\left\|t \mapsto \frac{t^\alpha}{t+1}\right\|_{L^p_*(\mathbb{R}^+)} \|x\| \\
&\qquad\quad + \sup_{t\in\mathbb{R}^+} \|(tI+A)(tI+B)^{-1}\| \cdot \|t \mapsto t^\alpha A(tI+A)^{-1}x\|_{L^p_*(\mathbb{R}^+)} \\
&\qquad\quad + \sup_{t\in\mathbb{R}^+} \|(tI+A^{-1})(tI+B)^{-1}\| \cdot \|t \mapsto t^\alpha A^{-1}(tI+A^{-1})x\|_{L^p_*(\mathbb{R}^+)}.
\end{aligned}$$

For $t \in \mathbb{R}^+$ we have

$$\begin{aligned}
&\|(tI+A)(tI+B)^{-1}\| \\
&\qquad = \|(tI+A)A(I+A)^{-2}B(tI+B)^{-1}\| \\
&\qquad \le \|(tI+A)(I+A)^{-1}\| \cdot \|A(I+A)^{-1}\| \cdot \|B(tI+B)^{-1}\| \le C
\end{aligned}$$

since A and B are of type ω and ω_0 respectively. In a similar way one can estimate the term $\|(tI+A^{-1})(tI+B)^{-1}\|$; thus the second summand is less than or equal to a constant times $\|t \mapsto t^\alpha A(tI+A)^{-1}x\|_{L^p_*(\mathbb{R}^+)}$ and the third one is less than or equal to a constant times $\|t \mapsto t^\alpha A^{-1}(tI+A^{-1})x\|_{L^p_*(\mathbb{R}^+)}$. We can conclude that if $x \in \mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p)$ then

$$\begin{aligned}
\|t \mapsto t^\alpha B(tI+B)^{-1}x\|_{L^p_*(\mathbb{R}^+)} &\le C(\|x\| + \|t \mapsto t^\alpha A(tI+A)^{-1}x\|_{L^p_*(\mathbb{R}^+)} \\
&\qquad + \|t \mapsto t^\alpha A^{-1}(tI+A^{-1})x\|_{L^p_*(\mathbb{R}^+)}) \\
&\le C\|x\|_{\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)} < \infty.
\end{aligned}$$

This proves that the space $\mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p)$ is continuously embedded in $(X, \mathcal{D}(A) \cap \mathcal{R}(A))_{\alpha,p}$.

We denote by $A_{\alpha,p}$ the part of the operator A in $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)$, i.e. the operator such that

$$\begin{aligned}\mathcal{D}(A_{\alpha,p}) &= \{x\in\mathcal{D}(A)\cap\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p) :\\ &\qquad Ax\in\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)\}\\ &= \{x\in\mathcal{D}(A)\cap\mathcal{R}(A;\alpha,p) : Ax\in\mathcal{D}(A;\alpha,p)\},\\ A_{\alpha,p}x &= Ax.\end{aligned}$$

We note that if $0\in\varrho(A)$ then $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)=\mathcal{D}(A;\alpha,p)$ and this definition of $A_{\alpha,p}$ coincides with the one in [1].

THEOREM 2.4. *If A is a one-to-one operator of type ω, then for $\alpha\in\,]0,1[$ and $p\in[1,\infty]$, $A_{\alpha,p}$ is a one-to-one operator of type ω in $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)$. Moreover, if A has dense domain and dense range and $p<\infty$ then $A_{\alpha,p}$ has dense domain and dense range.*

Proof. Obviously, $A_{\alpha,p}$ is a closed operator and it is one-to-one. If $\lambda\in\varrho(A)$ then $(\lambda-A)^{-1}$ restricted to $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)$ is the inverse operator of $\lambda-A_{\alpha,p}$, thus $\lambda\in\varrho(A_{\alpha,p})$, therefore $\sigma(A_{\alpha,p})\subseteq S_\omega$.

Moreover, if $x\in\mathcal{D}(A)\cap\mathcal{R}(A)$ then $(\lambda-A)^{-1}x\in\mathcal{D}(A)\cap\mathcal{R}(A)$ and

$$\begin{aligned}\|(\lambda I-A)^{-1}x\|_{\mathcal{D}(A)\cap\mathcal{R}(A)} &\le C\|B(\lambda I-A)^{-1}x\|_X = C\|(\lambda I-A)^{-1}Bx\|_X\\ &\le C_1\|(\lambda I-A)^{-1}\|_{\mathcal{L}(X)}\|Bx\|_X\\ &\le C_2\|(\lambda I-A)^{-1}\|_{\mathcal{L}(X)}\|x\|_{\mathcal{D}(A)\cap\mathcal{R}(A)}\end{aligned}$$

(with B as in Theorem 2.2); this proves that the restriction of $(\lambda I-A)^{-1}$ to $\mathcal{D}(A)\cap\mathcal{R}(A)$ belongs to $\mathcal{L}(\mathcal{D}(A)\cap\mathcal{R}(A))$ and its norm in this space is less than or equal to a constant times its norm in $\mathcal{L}(X)$. By interpolation, taking into account Theorem 2.3, the same is true in $\mathcal{L}(\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p))$. Since A is of type ω we can conclude that $A_{\alpha,p}$ is of type ω.

Suppose that $p<\infty$ and that $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are dense in X. In order to prove the density of $\mathcal{D}(A_{\alpha,p})$ and $\mathcal{R}(A_{\alpha,p})$ we shall prove that $\mathcal{D}(A^2)\cap\mathcal{R}(A^2)$ is dense in $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)$ and that it is included in $\mathcal{D}(A_{\alpha,p})$ and in $\mathcal{R}(A_{\alpha,p})$.

By Theorem 2.2, $\mathcal{D}(B)$ is dense in X, therefore (see the proof of Theorem 2.2 of [1]) $\mathcal{D}(B^2)$ is dense in $\mathcal{D}(B;\alpha,p)$. We have

$$\begin{aligned}x\in\mathcal{D}(B^2) &\Leftrightarrow x\in\mathcal{D}(B)\text{ and }Bx\in\mathcal{D}(B)\\ &\Leftrightarrow x\in\mathcal{D}(A)\cap\mathcal{R}(A)\text{ and }(2I+A+A^{-1})x\in\mathcal{D}(A)\cap\mathcal{R}(A)\\ &\Leftrightarrow x\in\mathcal{D}(A)\cap\mathcal{R}(A)\text{ and }Ax+A^{-1}x\in\mathcal{D}(A)\cap\mathcal{R}(A)\\ &\Leftrightarrow x\in\mathcal{D}(A)\cap\mathcal{R}(A)\text{ and }Ax\in\mathcal{D}(A)\text{ and }A^{-1}x\in\mathcal{R}(A)\\ &\Leftrightarrow x\in\mathcal{D}(A^2)\cap\mathcal{R}(A^2),\end{aligned}$$

thus $\mathcal{D}(B^2)=\mathcal{D}(A^2)\cap\mathcal{R}(A^2)$; therefore $\mathcal{D}(A^2)\cap\mathcal{R}(A^2)$ is dense in the space $\mathcal{D}(B;\alpha,p)$, that is, in $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)$.

If $x \in \mathcal{D}(A^2) \cap \mathcal{R}(A^2)$ then $x \in \mathcal{D}(A)$ and $x \in \mathcal{R}(A) \subseteq \mathcal{R}(A;\alpha,p)$. Hence $x \in \mathcal{D}(A) \cap \mathcal{R}(A;\alpha,p)$, and $Ax \in \mathcal{D}(A) \subseteq \mathcal{D}(A;\alpha,p)$, therefore $\mathcal{D}(A^2) \cap \mathcal{R}(A^2) \subseteq \mathcal{D}(A_{\alpha,p})$. If we consider the operator A^{-1} then the domain and the range are interchanged and $A^{-1}_{\alpha,p} = (A^{-1})_{\alpha,p}$, therefore we also have $\mathcal{D}(A^2) \cap \mathcal{R}(A^2) \subseteq \mathcal{R}(A_{\alpha,p})$.

In this way we have proved that $\mathcal{D}(A_{\alpha,p})$ and $\mathcal{R}(A_{\alpha,p})$ are dense in $\mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p)$.

3. H^∞ functional calculus

THEOREM 3.1. *Let A be a one-to-one operator of type ω with dense domain and dense range. Let $\mu \in]\omega,\pi[$, $\alpha \in]0,1[$ and $p \in [1,\infty]$. If $f \in \Psi(S^0_\mu)$ and $x \in \mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p)$, then $f(A)x \in \mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p)$ and there exists $C_{\alpha,p} \in \mathbb{R}^+$ (independent of f and x) such that*

$$\|f(A)x\|_{\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)} \le C_\alpha \|f\|_\infty \|x\|_{\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)}.$$

Proof. First of all we consider the case $p = \infty$.

By the same argument of the proof of Theorem 3.1 of [1] we find that there exists $C_\alpha \in \mathbb{R}^+$ such that for $x \in \mathcal{D}(A;\alpha,\infty)$ we have

$$\sup_{t\in\mathbb{R}^+} \|t^\alpha A(tI+A)^{-1} f(A)x\| \le C_\alpha \|f\|_\infty \|x\|_{\mathcal{D}(A;\alpha,\infty)}.$$

Analogously, for $x \in \mathcal{R}(A;\alpha,\infty)$ and $t \in \mathbb{R}^+$ we have

$$\begin{aligned}
&\|t^{1-\alpha}(tI+A)^{-1} f(A)x\| \\
&\quad = \left\| t^{1-\alpha} \frac{1}{2\pi i} \int_{\Gamma_\theta} \frac{f(\lambda)}{t+\lambda} (\lambda I - A)^{-1} x \, d\lambda \right\| \\
&\quad \le \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{t^{1-\alpha}\|f\|_\infty}{|t+\varrho e^{i\theta}|} \|(\varrho e^{i\theta} I - A)^{-1} x\| \, d\varrho \\
&\qquad + \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{t^{1-\alpha}\|f\|_\infty}{|t+\varrho e^{-i\theta}|} \|(\varrho e^{-i\theta} I - A)^{-1} x\| \, d\varrho \\
&\quad \le \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{t^{1-\alpha}}{\varrho^{1-\alpha}|t+\varrho e^{i\theta}|} \, d\varrho \, \|f\|_\infty \sup_{\varrho\in\mathbb{R}^+} \|\varrho^{1-\alpha}(\varrho e^{i\theta} I - A)^{-1} x\| \\
&\qquad + \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{t^{1-\alpha}}{\varrho^{1-\alpha}} \frac{1}{|t+\varrho e^{-i\theta}|} \, d\varrho \, \|f\|_\infty \sup_{\varrho\in\mathbb{R}^+} \|\varrho^{1-\alpha}(\varrho e^{-i\theta} I - A)^{-1} x\| \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{1}{\sigma^{1-\alpha}|1+\sigma e^{i\theta}|} \, d\sigma \, \|f\|_\infty \sup_{\varrho\in\mathbb{R}^+} \|\varrho^{1-\alpha} e^{-i\theta}(\varrho I - e^{-i\theta} A)^{-1} x\| \\
&\qquad + \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{1}{\sigma^{1-\alpha}} \frac{1}{|1+\sigma e^{-i\theta}|} \, d\sigma \, \|f\|_\infty \sup_{\varrho\in\mathbb{R}^+} \|\varrho^{1-\alpha} e^{i\theta}(\varrho I - e^{i\theta} A)^{-1} x\|.
\end{aligned}$$

The operators A, $-e^{i\theta}A$ and $-e^{-i\theta}A$ have the same range and for every $x \in \mathcal{R}(A)$ we have $\|A^{-1}x\| = \|(-e^{i\theta}A)^{-1}x\| = \|(-e^{-i\theta}A)^{-1}x\|$ so that the spaces $\mathcal{R}(A)$, $\mathcal{R}(-e^{i\theta}A)$ and $\mathcal{R}(-e^{-i\theta}A)$ coincide and have equal norms. Therefore $\mathcal{R}(A;\alpha,\infty) = \mathcal{R}(-e^{i\theta}A;\alpha,\infty) = \mathcal{R}(-e^{-i\theta}A;\alpha,\infty)$ (with equal norms). It follows that there exists a constant C such that for $x \in \mathcal{R}(A;\alpha,\infty)$ we have

$$\sup_{\varrho\in\mathbb{R}^+} \|\varrho^\alpha e^{i\theta}A(\varrho - e^{i\theta}A)^{-1}x\| \le C\|x\|_{\mathcal{R}(A;\alpha,\infty)},$$

$$\sup_{\varrho\in\mathbb{R}^+} \|\varrho^\alpha e^{-i\theta}A(\varrho - e^{-i\theta}A)^{-1}x\| \le C\|x\|_{\mathcal{R}(A;\alpha,\infty)},$$

therefore there exists $C_\alpha \in \mathbb{R}^+$ such that for $x \in \mathcal{R}(A;\alpha,\infty)$ we have

$$\sup_{t\in\mathbb{R}^+} \|t^{1-\alpha}(tI + A)^{-1}f(A)x\| \le C_\alpha\|f\|_\infty\|x\|_{\mathcal{R}(A;\alpha,\infty)}.$$

In this way, taking into account Theorem 2.1, we have proved that for $x \in \mathcal{D}(A;\alpha,\infty)\cap\mathcal{R}(A;\alpha,\infty)$ we have $f(A)x \in \mathcal{D}(A;\alpha,\infty)\cap\mathcal{R}(A;\alpha,\infty)$ and there exists $C_\alpha \in \mathbb{R}^+$ (independent of f and x) such that

$$\|f(A)x\|_{\mathcal{D}(A;\alpha,\infty)\cap\mathcal{R}(A;\alpha,\infty)} \le C_\alpha\|f\|_\infty\|x\|_{\mathcal{D}(A;\alpha,\infty)\cap\mathcal{R}(A;\alpha,\infty)}.$$

If $p < \infty$ choose $\alpha_0 \in \,]0,\alpha[$ and $\alpha_1 \in \,]\alpha,1[$; then, as a consequence of the reiteration theorem for real interpolation ([3], Theorem 1.10.2) and of Theorem 2.3, we have

$$\begin{aligned}
&\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p) = (X,\mathcal{D}(A)\cap\mathcal{R}(A))_{\alpha,p}\\
&= ((X,\mathcal{D}(A)\cap\mathcal{R}(A))_{\alpha_0,\infty},(X,\mathcal{D}(A)\cap\mathcal{R}(A))_{\alpha_1,\infty})_{(\alpha-\alpha_0)/(\alpha_1-\alpha_0),p}\\
&= (\mathcal{D}(A;\alpha_0,\infty)\cap\mathcal{R}(A;\alpha_0,\infty),\mathcal{D}(A;\alpha_1,\infty)\cap\mathcal{R}(A;\alpha_1,\infty))_{(\alpha-\alpha_0)/(\alpha_1-\alpha_0),p}
\end{aligned}$$

with equivalence of norms. Since we have proved that $f(A)$ is a bounded operator in $\mathcal{D}(A;\alpha_0,\infty)\cap\mathcal{R}(A;\alpha_0,\infty)$ and in $\mathcal{D}(A;\alpha_1,\infty)\cap\mathcal{R}(A;\alpha_1,\infty)$, with norm not greater than $C_{\alpha_0}\|f\|_\infty$ and $C_{\alpha_1}\|f\|_\infty$ respectively, we can conclude, by interpolation, that $f(A)$ is a bounded operator in $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)$ with norm less than or equal to a constant (depending only on α and p) times $\|f\|_\infty$.

THEOREM 3.2. *Let A be a one-to-one operator of type ω with dense domain and dense range. Let $\mu \in \,]\omega,\pi[$, $\alpha \in \,]0,1[$ and $p \in [1,\infty[$. Then the operator $A_{\alpha,p}$ has a bounded $H^\infty(S^0_\mu)$ functional calculus.*

Proof. By Theorem 2.4, $A_{\alpha,p}$ is a one-to-one operator of type ω with dense domain and dense range. If $f \in \Psi(S^0_\mu)$ then $f(A_{\alpha,p})$ is the restriction of $f(A)$ to $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)$. From Theorem 3.1 we deduce that

$$\|f(A_{\alpha,p})\|_{L(\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p))} \le C_\alpha\|f\|_\infty$$

and the conclusion follows from Theorem 2.1 of [1].

As in [1], the following theorem is an immediate consequence of the existence of a bounded H^∞ functional calculus.

THEOREM 3.3. *Let A be a one-to-one operator of type ω with dense domain and dense range. Let $\alpha \in \,]0,1[$ and $p \in [1,\infty[$. For every $s \in \mathbb{R}$ the operator $A^{is}_{\alpha,p}$ is bounded in $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)$ and for every $\mu > \omega$ there exists C_μ such that $\|A^{is}_{\alpha,p}\| \le C_\mu e^{\mu|s|}$.*

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