H∞ functional calculus
in real interpolation spaces, II

by

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Abstract. Let $A$ be a linear closed one-to-one operator in a complex Banach space $X$, having dense domain and dense range. If $A$ is of type $\omega$ (i.e. the spectrum of $A$ is contained in a sector of angle $2\omega$, symmetric about the real positive axis, and $\|\lambda(\lambda I - A)^{-1}\|$ is bounded outside every larger sector), then $A$ has a bounded $H^\infty$ functional calculus in the real interpolation spaces between $X$ and the intersection of the domain and the range of the operator itself.

1. Introduction. In this paper we consider the $H^\infty$ functional calculus on a sector for a closed, linear, one-to-one operator $A$ on a complex Banach space $X$, having dense domain and dense range, with resolvent set that contains $\mathbb{R}^-$ and resolvent that decreases in a maximal way on $\mathbb{R}^-$ (i.e. $\|\lambda(\lambda I - A)^{-1}\|$ is bounded). This is a sequel to [1], in which it was proved that such an operator $A$ has a bounded $H^\infty$ functional calculus in the real interpolation spaces between $X$ and $\mathcal{D}(A)$, provided that $0 \in \varrho(A)$. When $0 \notin \varrho(A)$ this theorem is not true, since in particular if $A$ is bounded (i.e. $\mathcal{D}(A) = X$) then every real interpolation space between $X$ and $\mathcal{D}(A)$ coincides with $X$, but there are bounded operators without a bounded $H^\infty$ functional calculus on a sector.

In the present paper we consider the case $0 \notin \varrho(A)$ and we prove that $A$ has a bounded $H^\infty$ functional calculus in the real interpolation spaces between $X$ and $\mathcal{D}(A) \cap \mathcal{R}(A)$.

We refer to [1] for notations and definitions, in particular for the definition of $S_\omega$, $S_\omega^0$, $H^\infty(S_\mu^0)$, $\Psi(S_\mu^0)$, operator of type $\omega$, $H^\infty$ functional calculus, $L_p^p(\mathbb{R}^+)$, real interpolation space.

2. Preliminary results. Let $A$ be a one-to-one operator of type $\omega$ ($\omega \in [0, \pi]$). If $z \in \mathbb{C} \setminus S_\omega$ then also $z^{-1} \in \mathbb{C} \setminus S_\omega$, therefore it is easy to show

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that \( z \in \mathcal{O}(A^{-1}) \) and
\[
(zI - A^{-1})^{-1} = -z^{-1}A(z^{-1}I - A)^{-1}.
\]
Thus if \( \theta \in ]\omega, \pi[ \) and \( M_\theta \) is such that \( \|(zI - A)^{-1}\| \leq M_\theta/|z| \) for \( z \in \mathbb{C} \setminus S_\theta \) then
\[
\|(zI - A^{-1})^{-1}\| = \|z^{-1}A(z^{-1}I - A)^{-1}\|
= \|z^{-1}(z^{-1}I - z^{-1}I + A)(z^{-1}I - A)^{-1}\|
\leq \|z^{-2}(z^{-1}I - A)^{-1}\| + \|z^{-1}\| \leq \frac{M_\theta + 1}{|z|}.
\]
Therefore \( A^{-1} \) is an operator of type \( \omega \).

We denote by \( \mathcal{D}(A; \alpha, p) \) the real interpolation space \((X, \mathcal{D}(A))_{\alpha, p}\) (with \( \alpha \in ]0, 1[ \) and \( p \in [1, \infty] \)); moreover, we denote by \( \mathcal{R}(A; \alpha, p) \) the real interpolation space \((X, \mathcal{R}(A))_{\alpha, p}\) (with \( \|x\|_{\mathcal{D}(A)} = \|x\|_X + \|Ax\|_X \) and \( \|x\|_{\mathcal{R}(A)} = \|x\|_X + \|A^{-1}x\|_X \)).

The norm of \( x \) in \( \mathcal{D}(A; \alpha, p) \) is equivalent to
\[
\|x\|_X + \|t^\alpha A(tI + A)^{-1}x\|_{L_p^\alpha(\mathbb{R}^+)}
\]
(see [2], Definition 1.1 and Theorem 3.1). We note that when \( 0 \in \mathcal{O}(A) \) the term \( \|x\|_X \) can be disregarded, while if \( A \) has unbounded inverse this term is essential.

Since \( \mathcal{D}(A^{-1}) = \mathcal{R}(A) \) we have \( \mathcal{R}(A; \alpha, p) = \mathcal{D}(A^{-1}; \alpha, p) \), therefore an equivalent norm on \( \mathcal{R}(A; \alpha, p) \) is \( \|x\|_X + \|t^\alpha A^{-1}(tI + A^{-1})^{-1}x\|_{L_p^\alpha(\mathbb{R}^+)} \).

But
\[
t^\alpha A^{-1}(tI + A^{-1})^{-1} = t^\alpha A^{-1}t^{-1}A(t^{-1}I + A)^{-1} = t^\alpha(t^{-1}I + A)^{-1},
\]
therefore this norm is equivalent to \( \|x\|_X + \|t^\alpha(t^{-1}I + A)^{-1}x\|_{L_p^\alpha(\mathbb{R}^+)} \).

Let \( E \) and \( F \) be Banach spaces (embedded in the same vector space). The space \( E \cap F \) is a Banach space if endowed with the norm \( \|x\|_{E \cap F} = \|x\|_E + \|x\|_F \).

From now on we will drop the subscript in the notation \( \| \cdot \|_X \).

**Theorem 2.1.** Let \( A \) be a one-to-one operator of type \( \omega \) with dense domain and dense range. Let \( \alpha \in ]0, 1[ \) and \( p \in [1, \infty] \). Then the norm on \( \mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p) \) is equivalent to
\[
\|t^\alpha A(tI + A)^{-1}x\|_{L_p^\alpha(\mathbb{R}^+)} + \|t^\alpha A(tI + A)^{-1}x\|_{L_p^\alpha(\mathbb{R}^+)}.
\]

**Proof.** From the above observations it follows immediately that the norm of \( \mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p) \) is equivalent to
\[
\|x\| + \|t^\alpha A(tI + A)^{-1}x\|_{L_p^\alpha(\mathbb{R}^+)} + \|t^\alpha A(tI + A)^{-1}x\|_{L_p^\alpha(\mathbb{R}^+)},
\]
therefore in order to prove the theorem it is sufficient to show that there
exists $C \in \mathbb{R}^+$ such that for every $x \in \mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p)$ we have

$$
\|x\| \leq C(\|t \mapsto t^\alpha A(tI + A)^{-1}x\|_{L^p(\mathbb{R}^+)} + \|t \mapsto t^{1-\alpha}(tI + A)^{-1}x\|_{L^p(\mathbb{R}^+)})
$$

If $p < \infty$, then since

$$
\int_{\mathbb{R}^+} (e^{-|\alpha|\log t})^p \frac{dt}{t} = \frac{2}{\alpha p}
$$

and for every $t \in \mathbb{R}^+$, $x \in X$,

$$
x = A(tI + A)^{-1}x + t(tI + A)^{-1}x,
$$

we have

$$
\|x\| = \left( \frac{\alpha p}{2} \int_{\mathbb{R}^+} \|e^{-|\alpha|\log t} x\|_p \frac{dt}{t} \right)^{1/p} 
$$

$$
\leq \left( \frac{\alpha p}{2} \int_{\mathbb{R}^+} \|e^{-|\alpha|\log t} A(tI + A)^{-1}x\|_p \frac{dt}{t} \right)^{1/p} 
$$

$$
+ \left( \frac{\alpha p}{2} \int_{\mathbb{R}^+} \|e^{-|\alpha|\log t} t(tI + A)^{-1}x\|_p \frac{dt}{t} \right)^{1/p} 
$$

$$
\leq \left( \frac{\alpha p}{2} \int_{\mathbb{R}^+} \|t^\alpha A(tI + A)^{-1}x\|_p \frac{dt}{t} \right)^{1/p} 
$$

$$
+ \left( \frac{\alpha p}{2} \int_{\mathbb{R}^+} \|t^{1-\alpha}(tI + A)^{-1}x\|_p \frac{dt}{t} \right)^{1/p} 
$$

If $p = \infty$, then

$$
\|x\| \leq \|1^\alpha A(I + A)^{-1}x\| + \|1^{1-\alpha}(I + A)^{-1}x\| 
$$

$$
\leq \sup_{t \in \mathbb{R}^+} \|t^\alpha A(tI + A)^{-1}x\| + \sup_{t \in \mathbb{R}^+} \|t^{1-\alpha}(tI + A)^{-1}x\|.
$$

This concludes the proof.

**Theorem 2.2.** Let $A$ be a one-to-one operator of type $\omega$. Let $B$ be the operator from $\mathcal{D}(A) \cap \mathcal{R}(A)$ to $X$ such that $Bx = (2I + A + A^{-1})x$. Then $B$ is a closed operator of type $\omega_0$ (for a suitable $\omega_0$) and $0 \in \sigma(B)$. Moreover, if $A$ has dense domain and dense range then $\mathcal{D}(B)$ is dense.

**Proof.** Obviously, $B = (I + A)^2 A^{-1}$ but $(I + A)^2$ has bounded inverse and $A^{-1}$ is closed, therefore $B$ is closed and its inverse is $A(I + A)^{-2}$, hence $0 \in \sigma(B)$.

For $t \in \mathbb{R}^+$ put

$$
\tau_t = \frac{t + 2 + \sqrt{t^2 + 4t}}{2}.
$$
We have
\[(t + 2)I + A + A^{-1})^{-1} = \tau_t(\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}.
\]
Indeed, \(t + 2 = \tau_t + \tau_t^{-1}\), thus for \(x \in \mathcal{D}(A) \cap \mathcal{R}(A)\) we have
\[
\tau_t(\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}((t + 2)I + A + A^{-1})x
= \tau_t(\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}((\tau_t + \tau_t^{-1})I + A + A^{-1})x
= (\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}((\tau_t^2 + 1)I + \tau_t A + \tau_t A^{-1})x
= (\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}(\tau_t I + A^{-1})(\tau_t I + A)x = x
\]
and analogously, for every \(x \in X\),
\[(t + 2)I + A + A^{-1})\tau_t(\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}x = x;
\]
therefore \(\mathbb{R}^{-} \subseteq \theta(B)\). Moreover, we have
\[
\|t(I + B)^{-1}\| = \|\tau_t(\tau_t I + A)^{-1}(\tau_t I + A^{-1})^{-1}\|
\leq \|(\tau_t I + A)^{-1}A(\tau_t^{-1} I + A)^{-1}\| \leq \frac{C}{\tau_t} \leq \frac{C}{t + 1},
\]
therefore \(B\) is of type \(\omega_0\) for some \(\omega_0\).

Suppose now that \(\mathcal{D}(A)\) and \(\mathcal{R}(A)\) are dense in \(X\). Then for every \(x \in X\) we have
\[
\|tA(tI + A)^{-1}(t^{-1}I + A)^{-1}x - x\|
\leq \|tA(tI + A)^{-1}(t^{-1}I + A)^{-1}x - t(tI + A)^{-1}x + t(tI + A)^{-1}x - x\|
\leq \|t(tI + A)^{-1}\| \cdot \|A(t^{-1}I + A)^{-1}x - x\| + \|t(tI + A)^{-1}x - x\| \xrightarrow{t \to 0^+} 0.
\]
But
\[tA(tI + A)^{-1}(t^{-1}I + A)^{-1}x \in \mathcal{D}(A) \cap \mathcal{R}(A),\]
so \(x\) is a limit of elements of \(\mathcal{D}(A) \cap \mathcal{R}(A)\), therefore \(\mathcal{D}(B) = \mathcal{D}(A) \cap \mathcal{R}(A)\) is dense in \(X\). This proves the theorem.

Note that
\[
\|x\|_{\mathcal{D}(B)} = \|x\| + \|Bx\| = \|x\| + \|(2I + A + A^{-1})x\|
\leq 3\|x\| + \|Ax\| + \|A^{-1}x\| = \|x\|_X + \|x\|_{\mathcal{D}(A)} + \|x\|_{\mathcal{R}(A)}
\leq \|x\|_{\mathcal{D}(A) \cap \mathcal{R}(A)}.
\]
Therefore the vector spaces \(\mathcal{D}(A) \cap \mathcal{R}(A)\) and \(\mathcal{D}(B)\) are equal and the first space is continuously embedded in the second one; by the open mapping theorem the reverse embedding is continuous and the two norms are equivalent.

It follows that \((X, \mathcal{D}(A) \cap \mathcal{R}(A))_{\alpha, p} = \mathcal{D}(B; \alpha, p)\) with equivalent norms.
THEOREM 2.3. Let \( A \) be a one-to-one operator of type \( \omega \) with dense domain and dense range. Let \( \alpha \in [0, 1[ \) and \( p \in [1, \infty]. \) Then \( \mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p) = (X, \mathcal{D}(A) \cap \mathcal{R}(A))_{\alpha,p} \) with equivalent norms.

Proof. Since \( \mathcal{D}(A) \cap \mathcal{R}(A) \) is continuously embedded in \( \mathcal{D}(A) \) and in \( \mathcal{R}(A), \) by interpolation we deduce that \( (X, \mathcal{D}(A) \cap \mathcal{R}(A))_{\alpha,p} \) is continuously embedded in \( \mathcal{D}(A; \alpha, p) \) and in \( \mathcal{R}(A; \alpha, p) \) and also in their intersection.

As we have already observed, \( \mathcal{D}(A) \cap \mathcal{R}(A) = \mathcal{D}(B) \) (with \( B \) as in Theorem 2.2); therefore, in order to prove the inverse embedding, it is sufficient to prove that \( \mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p) \) is continuously embedded in \( \mathcal{D}(B; \alpha, p). \)

Let \( x \in \mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p); \) we have

\[
\| t \mapsto t^\alpha B(tI + B)^{-1}x\|_{L^p_\omega(\mathbb{R}^+)} \\
\leq \| t \mapsto t^\alpha (2I + A + A^{-1})(tI + B)^{-1}x\|_{L^p_\omega(\mathbb{R}^+)} \\
\leq \| t \mapsto t^\alpha 2(tI + B)^{-1}x\|_{L^p_\omega(\mathbb{R}^+)} \\
+ \| t \mapsto t^\alpha A(tI + A)^{-1}(tI + A)(tI + B)^{-1}x\|_{L^p_\omega(\mathbb{R}^+)} \\
+ \| t \mapsto t^\alpha A^{-1}(tI + A^{-1})(tI + A^{-1})(tI + B)^{-1}x\|_{L^p_\omega(\mathbb{R}^+)} \\
\leq C \| \frac{t^\alpha}{t + 1}\|_{L^p_\omega(\mathbb{R}^+)} \| x \| \\
+ \sup_{t \in \mathbb{R}^+} \| (tI + A)(tI + B)^{-1}\| \cdot \| t \mapsto t^\alpha A(tI + A)^{-1}x\|_{L^p_\omega(\mathbb{R}^+)} \\
+ \sup_{t \in \mathbb{R}^+} \| (tI + A^{-1})(tI + B)^{-1}\| \cdot \| t \mapsto t^\alpha A^{-1}(tI + A^{-1})x\|_{L^p_\omega(\mathbb{R}^+)}.}
\]

For \( t \in \mathbb{R}^+ \) we have

\[
\| (tI + A)(tI + B)^{-1}\| \\
= \| (tI + A)A(I + A)^{-2}B(tI + B)^{-1}\| \\
\leq \| (tI + A)(I + A)^{-1}\| \cdot \| A(I + A)^{-1}\| \cdot \| B(tI + B)^{-1}\| \leq C
\]
since \( A \) and \( B \) are of type \( \omega \) and \( \omega_0 \) respectively. In a similar way one can estimate the term \( \| (tI + A^{-1})(tI + B)^{-1}\|; \) thus the second summand is less than or equal to a constant times \( \| t \mapsto t^\alpha A(tI + A)^{-1}x\|_{L^p_\omega(\mathbb{R}^+)} \) and the third one is less than or equal to a constant times \( \| t \mapsto t^\alpha A^{-1}(tI + A^{-1})x\|_{L^p_\omega(\mathbb{R}^+)} \).

We can conclude that if \( x \in \mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p) \) then

\[
\| t \mapsto t^\alpha B(tI + B)^{-1}x\|_{L^p_\omega(\mathbb{R}^+)} \leq C(\| x \| + \| t \mapsto t^\alpha A(tI + A)^{-1}x\|_{L^p_\omega(\mathbb{R}^+)} + \| t \mapsto t^\alpha A^{-1}(tI + A^{-1})x\|_{L^p_\omega(\mathbb{R}^+)}) \\
\leq C \| x \|_{\mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p)} < \infty.
\]

This proves that the space \( \mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p) \) is continuously embedded in \( (X, \mathcal{D}(A) \cap \mathcal{R}(A))_{\alpha,p}. \)
We denote by $A_{\alpha,p}$ the part of the operator $A$ in $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)$, i.e. the operator such that

$$\mathcal{D}(A_{\alpha,p}) = \{x \in \mathcal{D}(A) \cap \mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p) : Ax \in \mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)\}$$

$$= \{x \in \mathcal{D}(A) \cap \mathcal{R}(A;\alpha,p) : Ax \in \mathcal{D}(A;\alpha,p)\},$$

$$A_{\alpha,p}x = Ax.$$

We note that if $0 \in \varrho(A)$ then $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p) = \mathcal{D}(A;\alpha,p)$ and this definition of $A_{\alpha,p}$ coincides with the one in [1].

**Theorem 2.4.** If $A$ is a one-to-one operator of type $\omega$, then for $\alpha \in [0,1[$ and $p \in [1,\infty[$, $A_{\alpha,p}$ is a one-to-one operator of type $\omega$ in $\mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p)$. Moreover, if $A$ has dense domain and dense range and $p < \infty$ then $A_{\alpha,p}$ has dense domain and dense range.

**Proof.** Obviously, $A_{\alpha,p}$ is a closed operator and it is one-to-one. If $\lambda \in \varrho(A)$ then $(\lambda - A)^{-1}$ restricted to $\mathcal{D}(A;\alpha,p)\cap\mathcal{R}(A;\alpha,p)$ is the inverse operator of $\lambda - A_{\alpha,p}$, thus $\lambda \in \varrho(A_{\alpha,p})$, therefore $\sigma(A_{\alpha,p}) \subseteq S_{\omega}$.

Moreover, if $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ then $(\lambda - A)^{-1}x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ and

$$\|(\lambda I - A)^{-1}x\|_{\mathcal{D}(A)\cap\mathcal{R}(A)} \leq C\|B(\lambda I - A)^{-1}x\|_{\mathcal{X}} = C\|Bx\|_{\mathcal{X}} \leq C_1\|Bx\|_{\mathcal{X}} \leq C_2\|Bx\|_{\mathcal{X}} \leq \mathcal{C}(\lambda I - A)^{-1}\|Bx\|_{\mathcal{X}} \leq \mathcal{C}(\lambda I - A)^{-1}\|x\|_{\mathcal{D}(A)\cap\mathcal{R}(A)}$$

(with $B$ as in Theorem 2.2); this proves that the restriction of $(\lambda I - A)^{-1}$ to $\mathcal{D}(A) \cap \mathcal{R}(A)$ belongs to $\mathcal{L}(\mathcal{D}(A) \cap \mathcal{R}(A))$ and its norm in this space is less than or equal to a constant times its norm in $\mathcal{L}(\mathcal{X})$. By interpolation, taking into account Theorem 2.3, the same is true in $\mathcal{L}(\mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p))$. Since $A$ is of type $\omega$ we can conclude that $A_{\alpha,p}$ is of type $\omega$.

Suppose that $p < \infty$ and that $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are dense in $\mathcal{X}$. In order to prove the density of $\mathcal{D}(A_{\alpha,p})$ and $\mathcal{R}(A_{\alpha,p})$ we shall prove that $\mathcal{D}(A^2)\cap\mathcal{R}(A^2)$ is dense in $\mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p)$ and that it is included in $\mathcal{D}(A_{\alpha,p})$ and in $\mathcal{R}(A_{\alpha,p})$.

By Theorem 2.2, $\mathcal{D}(B)$ is dense in $\mathcal{X}$, therefore (see the proof of Theorem 2.2 of [1]) $\mathcal{D}(B^2)$ is dense in $\mathcal{D}(B;\alpha,p)$. We have

$x \in \mathcal{D}(B^2) \Leftrightarrow x \in \mathcal{D}(B)$ and $Bx \in \mathcal{D}(B)$

$x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ and $(2I + A + A^{-1})x \in \mathcal{D}(A) \cap \mathcal{R}(A)$

$x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ and $Ax + A^{-1}x \in \mathcal{D}(A) \cap \mathcal{R}(A)$

$x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ and $Ax \in \mathcal{D}(A)$ and $A^{-1}x \in \mathcal{R}(A)$

$x \in \mathcal{D}(A^2) \cap \mathcal{R}(A^2)$,

thus $\mathcal{D}(B^2) = \mathcal{D}(A^2)\cap\mathcal{R}(A^2)$; therefore $\mathcal{D}(A^2)\cap\mathcal{R}(A^2)$ is dense in the space $\mathcal{D}(B;\alpha,p)$, that is, in $\mathcal{D}(A;\alpha,p) \cap \mathcal{R}(A;\alpha,p)$.
If there exists a one-to-one operator $A$ with dense domain and dense range. Let $x \in D(A^2) \cap \mathcal{R}(A^2)$ then $x \in D(A)$ and $x \in \mathcal{R}(A) \subseteq \mathcal{R}(A; \alpha, p)$. Hence $x \in D(A) \cap \mathcal{R}(A; \alpha, p)$, and $Ax \in D(A) \subseteq D(A; \alpha, p)$, therefore $D(A^2) \cap \mathcal{R}(A^2) \subseteq D(A, \alpha, p)$. If we consider the operator $A^{-1}$ then the domain and the range are interchanged and $A^{-1} = (A^{-1})_{\alpha, p}$, therefore we also have $D(A^2) \cap \mathcal{R}(A^2) \subseteq \mathcal{R}(A, \alpha, p)$.

In this way we have proved that $D(A, \alpha, p)$ and $\mathcal{R}(A, \alpha, p)$ are dense in $D(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p)$.

3. $H^\infty$ functional calculus

**Theorem 3.1.** Let $A$ be a one-to-one operator of type $\omega$ with dense domain and dense range. Let $\mu \in [\omega, \pi],\alpha \in [0, 1]$ and $p \in [1, \infty]$. If $f \in \Psi(S_\mu^0)$ and $x \in D(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p)$, then $f(A)x \in D(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p)$ and there exists $C_{\alpha, p} \in \mathbb{R}^+$ (independent of $f$ and $x$) such that

$$\|f(A)x\|_{D(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p)} \leq C_{\alpha} \|f\|_\infty \|x\|_{D(A; \alpha, \infty)}.$$

**Proof.** First of all we consider the case $p = \infty$.

By the same argument of the proof of Theorem 3.1 of [1] we find that there exists $C_{\alpha} \in \mathbb{R}^+$ such that for $x \in D(A; \alpha, \infty)$ we have

$$\sup_{t \in \mathbb{R}^+} \|t^{1-\alpha}(tI + A)^{-1}f(A)x\| \leq C_{\alpha} \|f\|_\infty \|x\|_{D(A; \alpha, \infty)}.$$

Analogously, for $x \in \mathcal{R}(A; \alpha, \infty)$ and $t \in \mathbb{R}^+$ we have

$$\|t^{1-\alpha}(tI + A)^{-1}f(A)x\|$$

$$\leq \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{t^{1-\alpha} \|f\|_\infty \|((qe^{i\theta}I - A)^{-1}x\|} |t + qe^{i\theta}| d\theta$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{t^{1-\alpha} \|f\|_\infty \|((qe^{i\theta}I - A)^{-1}x\|} |t + qe^{i\theta}| d\theta$$

$$\leq \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{t^{1-\alpha} \|f\|_\infty \|\sup_{q \in \mathbb{R}^+} \|q^{1-\alpha}(qe^{i\theta}I - A)^{-1}x\|} |t + qe^{i\theta}| d\theta$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{\|f\|_\infty \|\sup_{q \in \mathbb{R}^+} \|q^{1-\alpha}(qe^{i\theta}I - A)^{-1}x\|} |t + qe^{i\theta}| d\theta.$$
The operators $A$, $-e^{i\theta}A$ and $-e^{-i\theta}A$ have the same range and for every $x \in \mathcal{R}(A)$ we have $\|A^{-1}x\| = \|(-e^{i\theta}A)^{-1}x\| = \|(-e^{-i\theta}A)^{-1}x\|$ so that the spaces $\mathcal{R}(A)$, $\mathcal{R}(-e^{i\theta}A)$ and $\mathcal{R}(-e^{-i\theta}A)$ coincide and have equal norms. Therefore $\mathcal{R}(A; \alpha, \infty) = \mathcal{R}(-e^{i\theta}A; \alpha, \infty) = \mathcal{R}(-e^{-i\theta}A; \alpha, \infty)$ (with equal norms). It follows that there exists a constant $C$ such that for $x \in \mathcal{R}(A; \alpha, \infty)$ we have
\[
\sup_{\rho \in \mathbb{R}^+} \|\rho^\alpha e^{i\theta}A(\rho - e^{i\theta}A)^{-1}x\| \leq C\|x\|_{\mathcal{R}(A; \alpha, \infty)},
\]
\[
\sup_{\rho \in \mathbb{R}^+} \|\rho^\alpha e^{-i\theta}A(\rho - e^{-i\theta}A)^{-1}x\| \leq C\|x\|_{\mathcal{R}(A; \alpha, \infty)},
\]
therefore there exists $C_\alpha \in \mathbb{R}^+$ such that for $x \in \mathcal{R}(A; \alpha, \infty)$ we have
\[
\sup_{t \in \mathbb{R}^+} \|t^{-1-\alpha}(tI + A)^{-1}f(A)x\| \leq C_\alpha \|f\|_{\infty} \|x\|_{\mathcal{R}(A; \alpha, \infty)}.
\]

In this way, taking into account Theorem 2.1, we have proved that for $x \in \mathcal{D}(A; \alpha, \infty) \cap \mathcal{R}(A; \alpha, \infty)$ we have $f(A)x \in \mathcal{D}(A; \alpha, \infty) \cap \mathcal{R}(A; \alpha, \infty)$ and there exists $C_\alpha \in \mathbb{R}^+$ (independent of $f$ and $x$) such that
\[
\|f(A)x\|_{\mathcal{D}(A; \alpha, \infty) \cap \mathcal{R}(A; \alpha, \infty)} \leq C_\alpha \|f\|_{\infty} \|x\|_{\mathcal{D}(A; \alpha, \infty) \cap \mathcal{R}(A; \alpha, \infty)}.
\]

If $p < \infty$ choose $\alpha_0 \in [0, \alpha[$ and $\alpha_1 \in ]\alpha, 1]$; then, as a consequence of the reiteration theorem for real interpolation ([3], Theorem 1.10.2) and of Theorem 2.3, we have
\[
\mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p) = (X, \mathcal{D}(A) \cap \mathcal{R}(A))_{\alpha,p} = (X, \mathcal{D}(A) \cap \mathcal{R}(A))_{\alpha_0, \infty} \cap (X, \mathcal{D}(A) \cap \mathcal{R}(A))_{\alpha_1, \infty} = (\mathcal{D}(A; \alpha_0, \infty) \cap \mathcal{R}(A; \alpha_0, \infty), \mathcal{D}(A; \alpha_1, \infty) \cap \mathcal{R}(A; \alpha_1, \infty))_{\alpha_0, \infty} \cap \mathcal{R}(A; \alpha_1, \infty)\]
with equivalence of norms. Since we have proved that $f(A)$ is a bounded operator in $\mathcal{D}(A; \alpha_0, \infty) \cap \mathcal{R}(A; \alpha_0, \infty)$ and in $\mathcal{D}(A; \alpha_1, \infty) \cap \mathcal{R}(A; \alpha_1, \infty)$, with norm not greater than $C_{\alpha_0}\|f\|_{\infty}$ and $C_{\alpha_1}\|f\|_{\infty}$ respectively, we can conclude, by interpolation, that $f(A)$ is a bounded operator in $\mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p)$ with norm less than or equal to a constant (depending only on $\alpha$ and $p$) times $\|f\|_{\infty}$.

**Theorem 3.2.** Let $A$ be a one-to-one operator of type $\omega$ with dense domain and dense range. Let $\mu \in ]\omega, \pi[\, \alpha \in ]0, 1[$ and $p \in [1, \infty[$. Then the operator $A_{\alpha,p}$ has a bounded $H^\infty(S^0_\mu)$ functional calculus.

**Proof.** By Theorem 2.4, $A_{\alpha,p}$ is a one-to-one operator of type $\omega$ with dense domain and dense range. If $f \in \Psi(S^0_\mu)$ then $f(A_{\alpha,p})$ is the restriction of $f(A)$ to $\mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p)$. From Theorem 3.1 we deduce that
\[
\|f(A_{\alpha,p})\|_{\mathcal{L}(\mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p))} \leq C_{\alpha}\|f\|_{\infty}
\]
and the conclusion follows from Theorem 2.1 of [1].
As in [1], the following theorem is an immediate consequence of the existence of a bounded $H^\infty$ functional calculus.

**Theorem 3.3.** Let $A$ be a one-to-one operator of type $\omega$ with dense domain and dense range. Let $\alpha \in ]0, 1[\)$ and $p \in ]1, \infty[\)$. For every $s \in \mathbb{R}$ the operator $A^{\alpha, p}$ is bounded in $\mathcal{D}(A; \alpha, p) \cap \mathcal{R}(A; \alpha, p)$ and for every $\mu > \omega$ there exists $C_\mu$ such that $\|A^{\alpha, p}_s\| \leq C_\mu e^{\mu |s|}$.

**References**


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