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Higher order local dimensions and Baire category

by

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Abstract. Let X be a complete metric space and write $\mathcal{P}(X)$ for the family of all Borel probability measures on X. The local dimension $\dim_{\mathsf{loc}}(\mu;x)$ of a measure $\mu \in \mathcal{P}(X)$ at a point $x \in X$ is defined by

$$\dim_{\mathsf{loc}}(\mu;x) = \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r}$$

whenever the limit exists, and plays a fundamental role in multifractal analysis. It is known that if a measure $\mu \in \mathcal{P}(X)$ satisfies a few general conditions, then the local dimension of μ exists and is equal to a constant for μ-a.a. $x \in X$. In view of this, it is natural to expect that for a fixed $x \in X$, the local dimension of a typical (in the sense of Baire category) measure exists at x. Quite surprisingly, we prove that this is not the case. In fact, we show that the local dimension of a typical measure fails to exist in a very spectacular way. Namely, the behaviour of a typical measure $\mu \in \mathcal{P}(X)$ is so extremely irregular that, for a fixed $x \in X$, the local dimension function,

$$r \mapsto \frac{\log \mu(B(x,r))}{\log r},$$

of μ at x remains divergent as $r \searrow 0$ even after being "averaged" or "smoothened out" by very general and powerful averaging methods, including, for example, higher order Riesz–Hardy logarithmic averages and Cesàro averages.

1. Statement of the main result. Recall that genericity in a topological sense is defined as follows: we say that a typical element x in a complete metric space M has property P if the set of elements in M that do not have property P is meagre (i.e. of the first Baire category).

Fix a complete metric space X and write

$$\mathcal{P}(X) = \{\mu \mid \mu \text{ is a Borel probability measure on } X\},$$

and equip $\mathcal{P}(X)$ with the weak topology. Then $\mathcal{P}(X)$ is a complete metric space. Consequently, we can apply the above definition of topological genericity to $M = \mathcal{P}(X)$, allowing us to talk about typical probability measures

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on X. Indeed, the typical behaviour, in the sense of Baire, of various fractal and multifractal dimensions of measures has recently been studied by a number of authors [Ge, Haa, Ol, MR]. For example, the local dimension of a typical measure has been studied by Haase [Haa]. The local dimension of a measure is defined as follows. For a measure μ on X and a point $x \in X$, the *lower* and *upper local dimensions* of μ at x are defined by

$$\underline{\dim}_{\mathsf{loc}}(\mu; x) = \liminf_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} \tag{1.1}$$

and

$$\overline{\dim}_{\mathsf{loc}}(\mu; x) = \limsup_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r}, \tag{1.2}$$

respectively. If they coincide, we call their common value the *local dimension* of μ at x and denote it by $\dim_{\mathsf{loc}}(\mu; x)$. The main importance of local dimensions is due to their relationship with multifractal analysis and because of this, local dimensions have attracted an enormous interest during the past 20 years; see, for example, the textbooks [Fa2, Pe] and the references therein.

It is known that if a measure μ satisfies a few general conditions, then the local dimension $\dim_{\mathsf{loc}}(\mu; x) = \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r}$ of μ exists and is equal to a constant for μ-a.a. $x \in X$. For example, Cutler [Cu1, Cu2] proved that if $T : X \to X$ is a transformation satisfying a technical condition that is stronger than continuity but weaker than being locally Lipschitz, and if μ is a T-invariant and ergodic measure, then there is a constant d such that

$$\dim_{\mathsf{loc}}(\mu; x) = d \tag{1.3}$$

for μ-a.a. $x \in X$. In view of this result, it is natural to expect that for a fixed $x \in X$, the local dimension of a typical measure exists at x. However, Haase [Haa] proved that this is not the case. Indeed, he showed that for a fixed $x \in X$, the local dimension of a typical measure does not exist at x. More precisely, Haase proved the following result.

THEOREM A ([Haa, Theorem 3]). *Let X be a complete metric space and $x \in X$. Then a typical measure $\mu \in \mathcal{P}(X)$ satisfies*

$$\underline{\dim}_{\mathsf{loc}}(\mu; x) = 0.$$

If, in addition, x is not an isolated point of X, then a typical measure $\mu \in \mathcal{P}(X)$ satisfies

$$\overline{\dim}_{\mathsf{loc}}(\mu; x) = \infty.$$

Define the function $\Delta_{\mu,x} : [1, \infty) \to [0, \infty]$ by

$$\Delta_{\mu,x}(t) = \frac{\log \mu(B(x, e^{-t}))}{\log e^{-t}}. \tag{1.4}$$

Then

$$\underline{\dim}_{\mathsf{loc}}(\mu;x)=\liminf_{t\to\infty}\varDelta_{\mu,x}(t)$$

and

$$\overline{\dim}_{\mathsf{loc}}(\mu;x)=\limsup_{t\to\infty}\varDelta_{\mu,x}(t),$$

and Theorem A therefore shows that for typical μ, the function $\varDelta_{\mu,x}$ diverges in the worst possible way as $t\to\infty$. In this paper we will prove that the behaviour of the local dimension function $t\mapsto\varDelta_{\mu,x}(t)$ is significantly and spectacularly more irregular than suggested by Theorem A. Namely, there are standard techniques, known as averaging methods, that (at least in some cases) can assign limiting values to divergent functions (the precise definitions will be given below), and the purpose of this paper is to show the following surprising result: not only is $\varDelta_{\mu,x}(t)$ divergent as $t\to\infty$, but the function $\varDelta_{\mu,x}(t)$ diverges so badly as $t\to\infty$, that even exceptionally general and powerful averaging methods (including higher order Riesz–Hardy logarithmic averages and Cesàro averages) are not able to "smoothen out" the irregularities in $\varDelta_{\mu,x}(t)$ as $t\to\infty$.

We start by recall the definition of a positive averaging (or integral summability) method.

Definition. A *positive averaging method* is a family $\varPi=(\varPi_T)_{T\in[s,\infty)}$ of positive measurable functions

$$\varPi_T:[s,\infty)\to[0,\infty)$$

with compact supports and indexed by $T\in[s,\infty)$ for some $s\in\mathbb{R}$. For a positive measurable function $f:[s,\infty)\to[0,\infty)$, we define $\mathsf{A}_\varPi f:[s,\infty)\to[0,\infty]$ by

$$(\mathsf{A}_\varPi f)(T)=\int_s^\infty\varPi_T(t)f(t)\,dt.$$

We say that the function f is *$\varPi$-averagable* with *$\varPi$-average* equal to A if

$$\lim_{T\to\infty}(\mathsf{A}_\varPi f)(T)=A.$$

We will now apply various averaging methods to the function $\varDelta_{\mu,x}$ in (1.4). Namely, for $\mu\in\mathcal{P}(X)$ and a positive averaging method $\varPi=(\varPi_T)_{T\in[s,\infty)}$, we define the *lower* and *upper $\varPi$-local dimension* of μ at x by

$$\underline{\dim}_{\varPi,\mathsf{loc}}(\mu;x)=\liminf_{T\to\infty}(\mathsf{A}_\varPi\varDelta_{\mu,x})(T)\tag{1.5}$$

and

$$\overline{\dim}_{\varPi,\mathsf{loc}}(\mu;x)=\limsup_{T\to\infty}(\mathsf{A}_\varPi\varDelta_{\mu,x})(T),\tag{1.6}$$

respectively. Theorem 1.1 below is the main result in the paper. It shows that for a fixed $x\in X$, the local behaviour of a typical (in the sense of Baire category) measure $\mu \in \mathcal{P}(X)$ is so irregular that the local dimension function $t \mapsto \varDelta_{\mu,x}(t)$ remains divergent as $t \to \infty$ even after being "averaged" using very general and powerful averaging methods $\varPi$ including, for example, higher order Riesz–Hardy logarithmic averages and Cesàro averages.

THEOREM 1.1. *Let X be a complete metric space and $x \in X$. Fix $s > 1$ and let $\varPi = (\varPi_T)_{T\in[s,\infty)}$ be a positive averaging method. Assume that $\varPi$ satisfies the following three conditions.*

(i) *Let S_T denote the support of $\varPi_T$ and set $R_T = \sup S_T$. Then*

$$\lim_{T\to\infty} R_T = \infty,$$

and for all $s \le T$ and all $s \le a \le b$ with $(a,b) \cap S_T \neq \emptyset$, we have

$$0 < \int_a^b \frac{\varPi_T(t)}{t}\, dt.$$

(ii) *For all $s \le T$, we have*

$$\int_s^\infty \frac{\varPi_T(t)}{t}\, dt < \infty.$$

(iii) *We have*

$$\lim_{T\to\infty} \int_s^\infty \frac{\varPi_T(t)}{t}\, dt = 0.$$

Then a typical measure $\mu \in \mathcal{P}(X)$ satisfies

$$\underline{\dim}_{\varPi,\mathsf{loc}}(\mu; x) = 0. \tag{1.7}$$

If, in addition, x is not an isolated point of X, then a typical measure $\mu \in \mathcal{P}(X)$ satisfies

$$\overline{\dim}_{\varPi,\mathsf{loc}}(\mu; x) = \infty. \tag{1.8}$$

The proof of Theorem 1.1 is given in Sections 4–6. Section 4 contains various preliminary results. The proof of (1.8) is given in Section 5 and the proof of (1.7) is given in Section 6. However, before proving Theorem 1.1 we present several applications of Theorem 1.1 to different averaging methods $\varPi$:

- In Section 2 we apply Theorem 1.1 to higher order Riesz–Hardy logarithmic averages. This allows us to compute the higher order Riesz–Hardy logarithmic averages of the local dimension function $\varDelta_{\mu,x}$ of a typical measure μ.
- In Section 3 we apply Theorem 1.1 to Cesàro averages. This allows us to compute the Cesàro averages of the local dimension function $\varDelta_{\mu,x}$ of a typical measure μ.

2. Higher order Riesz–Hardy logarithmic local dimensions of measures. Higher order Riesz–Hardy logarithmic averages were introduced into the study of fractal properties of sets and measures by Fisher [Fi1] and Bedford & Fisher [BF] in the early 1990's (see also [ADF]), and have since been investigated by a large number of authors, including Graf [Gr], Mörters [Mö1–Mö3] and Zähle [Zä]; the precise definition of these averages will be given below. In particular, for a self-similar set K with Hausdorff dimension δ, Bedford & Fisher [BF] studied the higher order Riesz–Hardy logarithmic averages of the density function

$$t \mapsto \frac{\mathcal{H}^\delta(B(x,1/t)\cap K)}{(2/t)^\delta} \tag{2.1}$$

for $\mathcal{H}^\delta$-a.a. $x \in K$ where $\mathcal{H}^\delta$ denotes the δ-dimensional Hausdorff measure. For example, it is well known that if K is the Cantor set, then the density

$$\lim_{t\to\infty} \frac{\mathcal{H}^\delta(B(x,1/t)\cap K)}{(2/t)^\delta}$$

does not exist at any $x \in K$ (see, for example, [Fa1, Section 5.1]). In contrast to this, Bedford & Fisher [BF] proved that the 2nd order Riesz–Hardy logarithmic averages of the density function in (2.1) exist for $\mathcal{H}^\delta$-a.a. $x \in K$. Motivated by this, we will now, for a fixed $x \in X$, study the higher order Riesz–Hardy logarithmic averages of the local dimension function

$$t \mapsto \frac{\log \mu(B(x,1/t))}{\log 1/t}$$

for a typical measure $\mu \in \mathcal{P}(X)$.

We first recall the definition of higher order Riesz–Hardy logarithmic averages. Define $\log_+ : \mathbb{R} \to \mathbb{R}$ by $\log_+(t) = \log(t)$ for $t > 0$ and $\log_+(t) = 0$ for $t \le 0$, and for a function $f : \mathbb{R} \to \mathbb{R}$, define $Ef, Lf : \mathbb{R} \to \mathbb{R}$ by

$$(Ef)(t) = f(e^t), \qquad (Lf)(t) = f(\log_+(t)).$$

Next, define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \begin{cases} 0 & \text{for } t \le 0, \\ e^{-t} & \text{for } t > 0. \end{cases}$$

Finally, for a positive measurable function $f : \mathbb{R} \to [0,\infty)$, we define the function $Hf : \mathbb{R} \to \mathbb{R}$ by

$$Hf = h * f,$$

where $*$ denotes the convolution product, i.e.

$$(Hf)(u) = (h * f)(u) = \int h(u-t) f(t)\, dt = e^{-u} \int_{-\infty}^{u} e^t f(t)\, dt.$$

For a positive measurable function $f : \mathbb{R} \to [0, \infty)$ and a positive integer $n \in \mathbb{N}$, the *lower* and *upper nth order Riesz–Hardy logarithmic averages* are now defined by

$$\underline{A}^n_{\mathsf{RH}} f = \liminf_{T\to\infty} (L^n H E^n f)(T), \qquad \overline{A}^n_{\mathsf{RH}} f = \limsup_{T\to\infty} (L^n H E^n f)(T).$$

It is not difficult to derive explicit formulas for the Riesz–Hardy logarithmic averages. Indeed for a positive integer $n \in \mathbb{N}$, we have

$$(E^n f)(u) = f(\exp^n(u)),$$

and so

$$(HE^n f)(u) = e^{-u} \int_{-\infty}^{u} e^t (E^n f)(t)\,dt = e^{-u} \int_{-\infty}^{u} e^t f(\exp^n(t))\,dt,$$

whence

$$(L^n HE^n f)(u) = (HE^n f)(\log_+^n(u)) = e^{-\log_+^n(u)} \int_{-\infty}^{\log_+^n(u)} e^t f(\exp^n(t))\,dt. \tag{2.2}$$

Writing $\log_+^0(u) = u$ for $u \in \mathbb{R}$ and making the substitution $s = \exp^n(t)$ in (2.2), we conclude that

$$(L^n HE^n f)(u) = \frac{1}{\log_+^{n-1}(u)} \int_0^u \log_+^{n-1}(s) \frac{f(s)}{\prod_{k=0}^{n-1} \log_+^k(s)}\,ds. \tag{2.3}$$

For example, this shows that the 1st, 2nd and 3rd lower Riesz–Hardy logarithmic averages of f are given by

$$\underline{A}^1_{\mathsf{RH}} f = \liminf_{T\to\infty} (L^1 HE^1 f)(T) = \liminf_{T\to\infty} \frac{1}{T} \int_0^T f(s)\,ds, \tag{2.4}$$

$$\underline{A}^2_{\mathsf{RH}} f = \liminf_{T\to\infty} (L^2 HE^2 f)(T) = \liminf_{T\to\infty} \frac{1}{\log_+(T)} \int_0^T \frac{f(s)}{s}\,ds, \tag{2.5}$$

$$\underline{A}^3_{\mathsf{RH}} f = \liminf_{T\to\infty} (L^3 HE^3 f)(T) = \liminf_{T\to\infty} \frac{1}{\log_+(\log_+(T))} \int_0^T \frac{f(s)}{s \log_+(s)}\,ds. \tag{2.6}$$

There are similar formulas for the upper Riesz–Hardy logarithmic averages of f.

Using higher order Riesz–Hardy logarithmic averages, we can now define higher order Riesz–Hardy logarithmic local dimensions as follows. For a measure μ on X and $x \in X$, define $D_{\mu,x} : \mathbb{R} \to \mathbb{R}$ by

$$D_{\mu,x}(t) = \begin{cases} 0 & \text{if } t \le e, \\ \dfrac{\log \mu(B(x, 1/t))}{\log 1/t} & \text{if } t > e. \end{cases}$$

We now define the *nth order lower* and *upper Riesz–Hardy logarithmic local dimensions* of μ at x by

$$\underline{\dim}^n_{\mathsf{RH,loc}}(\mu;x) = \underline{A}^n_{\mathsf{RH}} D_{\mu,x} = \liminf_{T\to\infty} (L^n H E^n D_{\mu,x})(T)$$

and

$$\overline{\dim}^n_{\mathsf{RH,loc}}(\mu;x) = \overline{A}^n_{\mathsf{RH}} D_{\mu,x} = \limsup_{T\to\infty} (L^n H E^n D_{\mu,x})(T),$$

respectively. For example, it follows from (2.4)–(2.6) that the 1st, 2nd and 3rd lower Riesz–Hardy logarithmic local dimensions of μ at x are given by

$$\underline{\dim}^1_{\mathsf{RH,loc}}(\mu;x) = \liminf_{T\to\infty} \frac{1}{T}\int_e^T D_{\mu,x}(s)\,ds,$$

$$\underline{\dim}^2_{\mathsf{RH,loc}}(\mu;x) = \liminf_{T\to\infty} \frac{1}{\log_+(T)}\int_e^T \frac{D_{\mu,x}(s)}{s}\,ds,$$

$$\underline{\dim}^3_{\mathsf{RH,loc}}(\mu;x) = \liminf_{T\to\infty} \frac{1}{\log_+(\log_+(T))}\int_e^T \frac{D_{\mu,x}(s)}{s\log_+(s)}\,ds.$$

There are similar formulas for the upper Riesz–Hardy logarithmic local dimensions. The higher order Riesz–Hardy logarithmic local dimensions form a natural double infinite hierarchy in (at least) countably infinitely many levels, namely, we have

$$\begin{aligned}(2.7)\qquad \underline{\dim}_{\mathsf{loc}}(\mu;x) \le \underline{\dim}^1_{\mathsf{RH,loc}}(\mu;x) \le \underline{\dim}^2_{\mathsf{RH,loc}}(\mu;x) \le \cdots \\ \le \overline{\dim}^2_{\mathsf{RH,loc}}(\mu;x) \le \overline{\dim}^1_{\mathsf{RH,loc}}(\mu;x) \le \overline{\dim}_{\mathsf{loc}}(\mu;x);\end{aligned}$$

indeed, if $X = K$ is a so-called cookie-cutter set in $\mathbb{R}$ with Hausdorff dimension δ and $f : \mathbb{R} \to [0,\infty)$ is defined by $f(t) = 0$ for $t \le e$ and

$$f(t) = \frac{\mathcal{H}^\delta(B(x,1/t)\cap K)}{(2/t)^\delta} \qquad \text{for } t > e$$

where $\mathcal{H}^\delta$ denotes the δ-dimensional Hausdorff measure, then the inequalities

$$\liminf_{t\to\infty} f(t) \le \underline{A}^1_{\mathsf{RH}} f \le \underline{A}^2_{\mathsf{RH}} f \le \cdots \le \overline{A}^2_{\mathsf{RH}} f \le \overline{A}^1_{\mathsf{RH}} f \le \limsup_{t\to\infty} f(t)$$

are announced in [BF, pp. 98–99, Property (1)] and [BF] refers the reader to [Fi2] for further discussions of the proof—however, this discussion suggests that the arguments can be adapted to a broader class of functions f including $f = D_{\mu,x}$, providing a proof of (2.7). As an application of Theorem 1.1, we will now show that for a fixed $x \in X$, the local behaviour of a typical measure $\mu \in \mathcal{P}(X)$ is so irregular that even the higher order Riesz–Hardy hierarchy (2.7) is not sufficiently powerful to "smoothen out" the behaviour of $\mu(B(x,e^{-t}))$ as $t \to \infty$.

THEOREM 2.1. *Let X be a complete metric space and $x \in X$. Then a typical measure $\mu \in \mathcal{P}(X)$ satisfies*

$$\underline{\dim}^n_{\mathsf{RH},\mathsf{loc}}(\mu; x) = 0 \quad \textit{for all } n \in \mathbb{N}.$$

If, in addition, x is not an isolated point of X, then a typical measure $\mu \in \mathcal{P}(X)$ satisfies

$$\overline{\dim}^n_{\mathsf{RH},\mathsf{loc}}(\mu; x) = \infty \quad \textit{for all } n \in \mathbb{N}.$$

Proof. For a positive integer n, we define the positive averaging method $\Pi^n = (\Pi^n_T)_{T\in[\exp^{n-1}(1),\infty)}$ by

$$\Pi^n_T(t) = \frac{1}{\log_+^{n-1}(T)} \exp(t)\log_+^{n-1}(\exp(t)) \frac{1}{\prod_{k=0}^{n-1}\log_+^k(\exp(t))} 1_{[\exp^{n-1}(1),\log(T)]}(t).$$

It follows from (2.3) that if $\mu \in \mathcal{P}(X)$ and $x \in X$, then

$$\begin{aligned}
(2.8)\quad \underline{\dim}^n_{\mathsf{RH},\mathsf{loc}}(\mu; x) &= \underline{A}^n_{\mathsf{RH}} D_{\mu,x} = \liminf_{T\to\infty} (L^H H E^n D_{\mu,x})(T)\\
&= \liminf_{T\to\infty} \frac{1}{\log_+^{n-1}(T)} \int_0^T \log_+^{n-1}(s) \frac{D_{\mu,x}(s)}{\prod_{k=0}^{n-1}\log_+^k(s)}\,ds\\
&= \liminf_{T\to\infty} \frac{1}{\log_+^{n-1}(T)} \int_{\exp^n(1)}^T \log_+^{n-1}(s) \frac{D_{\mu,x}(s)}{\prod_{k=0}^{n-1}\log_+^k(s)}\,ds.
\end{aligned}$$

However, since $D_{\mu,x}(s) = \Delta_{\mu,x}(\log s)$ for $s > e$ (see (1.4)), we conclude from (2.8) that

$$\begin{aligned}
(2.9)\quad \underline{\dim}^n_{\mathsf{RH},\mathsf{loc}}(\mu; x) &= \liminf_{T\to\infty} \frac{1}{\log_+^{n-1}(T)} \int_{\exp^n(1)}^T \log_+^{n-1}(s) \frac{\Delta_{\mu,x}(\log s)}{\prod_{k=0}^{n-1}\log_+^k(s)}\,ds\\
&= \liminf_{T\to\infty} \frac{1}{\log_+^{n-1}(T)} \int_{\exp^{n-1}(1)}^T \exp(t)\log_+^{n-1}(\exp(t)) \frac{\Delta_{\mu,x}(t)}{\prod_{k=0}^{n-1}\log_+^k(\exp(t))}\,dt\\
&= \liminf_{T\to\infty} \int_{\exp^{n-1}(1)}^T \Pi^n_T(t)\Delta_{\mu,x}(t)\,dt = \liminf_{T\to\infty} (\mathsf{A}_{\Pi^n}\Delta_{\mu,x})(T) = \underline{\dim}_{\Pi^n,\mathsf{loc}}(\mu; x).
\end{aligned}$$

Similarly, one shows that

$$(2.10)\qquad \overline{\dim}^n_{\mathsf{RH},\mathsf{loc}}(\mu; x) = \overline{\dim}_{\Pi^n,\mathsf{loc}}(\mu; x).$$

In addition, it is easily seen that Π^n satisfies conditions (i)–(iii) of Theorem 1.1, and the desired conclusion therefore follows from (2.9), (2.10) and Theorem 1.1. ■

3. Cesàro averages of local dimensions of measures. Another commonly used averaging method is the method of forming Cesàro averages. We will now define this method and apply it to the local dimension function

$$t \mapsto \Delta_{\mu,x}(t) = \frac{\log \mu(B(x,e^{-t}))}{\log e^{-t}}.$$

We first recall the definition of Cesàro averages. For a positive measurable function $f : [1,\infty) \to [0,\infty)$, we define $If : [1,\infty) \to [0,\infty)$ by

$$(If)(T) = \int_1^T f(t)\,dt.$$

Using the above definition of If, for a positive integer n, we define the *lower* and *upper nth order Cesàro averages* of f by

$$\underline{A}^n_{\mathsf{C}} f = \liminf_{T\to\infty} \frac{n!}{T^n}(I^n f)(T), \qquad \overline{A}^n_{\mathsf{C}} f = \limsup_{T\to\infty} \frac{n!}{T^n}(I^n f)(T).$$

We can now define higher order Cesàro local dimension of a measure $\mu \in \mathcal{P}(X)$ at $x \in X$ by applying the above procedure to the local dimension map $t \mapsto \Delta_{\mu,x}(t)$. Namely, for a positive integer $n \in \mathbb{N}$, the *lower* and *upper nth order Cesàro local dimensions* of a measure $\mu \in \mathcal{P}(X)$ at $x \in X$, are defined by

$$\underline{\dim}^n_{\mathsf{C},\mathsf{loc}}(\mu;x) = \underline{A}^n_{\mathsf{C}} \Delta_{\mu,x} = \liminf_{T\to\infty} \frac{n!}{T^n}(I^n \Delta_{\mu,x})(T)$$

and

$$\overline{\dim}^n_{\mathsf{C},\mathsf{loc}}(\mu;x) = \overline{A}^n_{\mathsf{C}} \Delta_{\mu,x} = \limsup_{T\to\infty} \frac{n!}{T^n}(I^n \Delta_{\mu,x})(T),$$

respectively. The Cesàro local dimensions also form a natural doubly infinite hierarchy in (at least) countably infinitely many levels, namely,

$$\begin{aligned} \underline{\dim}_{\mathsf{loc}}(\mu;x) &\le \underline{\dim}^1_{\mathsf{C},\mathsf{loc}}(\mu;x) \le \underline{\dim}^2_{\mathsf{C},\mathsf{loc}}(\mu;x) \le \cdots \\ &\le \overline{\dim}^2_{\mathsf{C},\mathsf{loc}}(\mu;x) \le \overline{\dim}^1_{\mathsf{C},\mathsf{loc}}(\mu;x) \le \overline{\dim}_{\mathsf{loc}}(\mu;x); \end{aligned} \tag{3.1}$$

indeed, if $f : [1,\infty) \to [0,\infty)$ is a measurable function, then

$$\liminf_{t\to\infty} f(t) \le \underline{A}^1_{\mathsf{C}} f \le \underline{A}^2_{\mathsf{C}} f \le \cdots \le \overline{A}^2_{\mathsf{C}} f \le \overline{A}^1_{\mathsf{C}} f \le \limsup_{t\to\infty} f(t), \tag{3.2}$$

and (3.1) now follows by applying (3.2) to the function $f = \Delta_{\mu,x}$; while (3.2) is almost certainly well-known, we have been unable to find an explicit reference, and for this reason we will provide a short and direct proof in Appendix A. As a further application of Theorem 1.1, we will now show that for a fixed $x \in X$, the local behaviour of a typical measure $\mu \in \mathcal{P}(X)$ is so irregular that even hierarchy (3.1) formed by taking Cesàro averages of all orders is not sufficiently powerful to "smoothen out" the behaviour of $\mu(B(x,e^{-t}))$ as $t\to\infty$.

THEOREM 3.1. *Let X be a complete metric space and $x\in X$. Then a typical measure $\mu\in\mathcal{P}(X)$ satisfies*

$$\underline{\dim}^n_{\mathsf{C},\mathsf{loc}}(\mu;x)=0\quad \text{for all } n\in\mathbb{N}.$$

If, in addition, x is not an isolated point of X, then a typical measure $\mu\in\mathcal{P}(X)$ satisfies

$$\overline{\dim}^n_{\mathsf{C},\mathsf{loc}}(\mu;x)=\infty\quad \text{for all } n\in\mathbb{N}.$$

Proof. For a positive integer n, we define the positive averaging method $\Pi^n=(\Pi^n_T)_{T\in[1,\infty)}$ by

$$\Pi^n_T(t)=\frac{n}{T^n}(T-t)^{n-1}\,1_{[1,T]}(t).$$

It is well-known (see, for example, [Har]) that if $f:[1,\infty)\to[0,\infty)$ is a positive measurable function, then

$$\underline{A}^n_{\mathsf{C}}f=\liminf_{T\to\infty}(\mathsf{A}_{\Pi^n}f)(T),\qquad \overline{A}^n_{\mathsf{C}}f=\limsup_{T\to\infty}(\mathsf{A}_{\Pi^n}f)(T),$$

and so

$$\begin{aligned}\underline{\dim}^n_{\mathsf{C},\mathsf{loc}}(\mu;x)&=\underline{A}^n_{\mathsf{C}}\Delta_{\mu,x}=\liminf_{T\to\infty}(\mathsf{A}_{\Pi^n}\Delta_{\mu,x})(T)=\underline{\dim}_{\Pi^n,\mathsf{loc}}(\mu;x),\\ \overline{\dim}_{\mathsf{C},\mathsf{loc}}(\mu;x)&=\overline{A}^n_{\mathsf{C}}\Delta_{\mu,x}=\limsup_{T\to\infty}(\mathsf{A}_{\Pi^n}\Delta_{\mu,x})(T)=\overline{\dim}_{\Pi^n,\mathsf{loc}}(\mu;x).\end{aligned}\tag{3.3}$$

Since it is easily seen that Π^n satisfies conditions (i)–(iii) of Theorem 1.1, the desired conclusion now follows from (3.3) and Theorem 1.1. ∎

4. Proof of Theorem 1.1. Preliminary results. It is well-known (cf., for example, [Pa, p. 51, Theorem 6.8]) that if X is a complete metric space, then the weak topology on $\mathcal{P}(X)$ is induced by the metric L on $\mathcal{P}(X)$ defined as follows. Let $\mathrm{Lip}(X)$ denote the family of Lipschitz functions $f:X\to\mathbb{R}$ with $|f|\le 1$ and $\mathrm{Lip}(f)\le 1$ where $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f. The metric L is now defined by

$$L(\mu,\nu)=\sup_{f\in\mathrm{Lip}(X)}\Big|\int f\,d\mu-\int f\,d\nu\Big|$$

for $\mu,\nu\in\mathcal{P}(X)$. We will always equip $\mathcal{P}(X)$ with the metric L and all balls in $\mathcal{P}(X)$ will be with respect to the metric L, i.e. if $\mu\in\mathcal{P}(X)$ and $r>0$, we will write

$$B(\mu,r)=\{\nu\in\mathcal{P}(X)\mid L(\mu,\nu)<r\}$$

for the ball with centre at μ and radius r.

5. Proof of (1.8) in Theorem 1.1. We must prove that

$$\overline{\dim}_{\Pi,\mathsf{loc}}(\mu;x)=\infty \tag{5.1}$$

for a typical $\mu\in\mathcal{P}(X)$; recall that $\overline{\dim}_{\Pi,\mathsf{loc}}(\mu;x)=\limsup_{T\to\infty}(\mathsf{A}_\Pi\Delta_{\mu,x})(T)$, where

$$(\mathsf{A}_\Pi\Delta_{\mu,x})(T)=\int_s^\infty \Pi_T(t)\Delta_{\mu,x}(t)\,dt=\int_s^\infty \Pi_T(t)\frac{\log\mu(B(x,e^{-t}))}{\log e^{-t}}\,dt.$$

We first introduce some notation. Let d denote the metric in X, and for $x\in X$ and $r>0$, let $C(x,r)$ denote the closed ball with centre at x and radius r, i.e.

$$C(x,r)=\{y\in X\mid d(x,y)\le r\}.$$

In order to prove (5.1) it is important (for technical reasons) that the set

$$\left\{\mu\in\mathcal{P}(X)\,\middle|\,\int_s^\infty \Pi_T(t)\frac{\log\mu(B(x,e^{-t}))}{\log e^{-t}}\,dt\le c\right\}$$

is closed. Unfortunately, simple examples show that, in general, it is not. However, the set

$$\left\{\mu\in\mathcal{P}(X)\,\middle|\,\int_s^\infty \Pi_T(t)\frac{\log\mu(C(x,e^{-t}))}{\log e^{-t}}\,dt\le c\right\}$$

is always closed (see Lemma 5.1), and for this reason we now introduce the following definitions replacing the open ball $B(x,e^{-t})$ with the closed ball $C(x,e^{-t})$. For $x\in X$, $T\ge s$ and $\mu\in\mathcal{P}(X)$, we define

$$\Gamma_T(\mu;x)=\int_s^\infty \Pi_T(t)\frac{\log\mu(C(x,e^{-t}))}{\log e^{-t}}\,dt,\qquad \Gamma(\mu;x)=\limsup_{T\to\infty}\Gamma_T(\mu;x).$$

Observe that $\Gamma(\mu;x)\le\overline{\dim}_{\Pi,\mathsf{loc}}(\mu;x)$. Hence, to show that $\overline{\dim}_{\Pi,\mathsf{loc}}(\mu;x)=\infty$ for a typical $\mu\in\mathcal{P}(X)$, it suffices to prove that

$$\Gamma(\mu;x)=\infty \tag{5.2}$$

for a typical $\mu\in\mathcal{P}(X)$. We begin with a lemma.

LEMMA 5.1. *Let X be a complete metric space. Let $c\in\mathbb{R}$, $x\in X$ and $T\ge s$. Then $\{\mu\in\mathcal{P}(X)\mid\Gamma_T(\mu;x)\le c\}$ is a closed subset of $\mathcal{P}(X)$.*

Proof. Let $(\mu_n)_n$ be a sequence in $\mathcal{P}(X)$ with $\Gamma_T(\mu_n;x)\le c$ for all n and let $\mu\in\mathcal{P}(X)$. Assume that $\mu_n\to\mu$. We must prove that $\Gamma_T(\mu;x)\le c$. Since $\mu_n\to\mu$ and $C(x,e^{-t})$ is closed we conclude that $\limsup_n\mu_n(C(x,e^{-t}))\le$

$\mu(C(x,e^{-t}))$, and Fatou's lemma yields

$$\begin{aligned}\mathit{\Gamma}_T(\mu;x) &= \int_s^\infty \mathit{\Pi}_T(t)\frac{\log\mu(C(x,e^{-t}))}{\log e^{-t}}\,dt\\ &\le \int_s^\infty \mathit{\Pi}_T(t)\frac{\log\limsup_n\mu_n(C(x,e^{-t}))}{\log e^{-t}}\,dt\\ &= \int_s^\infty \liminf_n \mathit{\Pi}_T(t)\frac{\log\mu_n(C(x,e^{-t}))}{\log e^{-t}}\,dt\\ &\le \liminf_n \int_s^\infty \mathit{\Pi}_T(t)\frac{\log\mu_n(C(x,e^{-t}))}{\log e^{-t}}\,dt\\ &= \liminf_n \mathit{\Gamma}_T(\mu_n;x) \le c. \ \blacksquare\end{aligned}$$

Proof of (1.8) in Theorem 1.1. We must prove that the set

$$\{\mu\in\mathcal{P}(X)\mid \overline{\dim}_{\mathit{\Pi},\mathsf{loc}}(\mu;x)<\infty\}$$

is meagre. First observe that since $\mathit{\Gamma}(\mu;x)\le\overline{\dim}_{\mathit{\Pi},\mathsf{loc}}(\mu;x)$, it suffices to prove that the set $M=\{\mu\in\mathcal{P}(X)\mid\mathit{\Gamma}(\mu;x)<\infty\}$ is meagre. For $u\in\mathbb{R}$, write

$$M_u=\{\mu\in\mathcal{P}(X)\mid\mathit{\Gamma}(\mu;x)<u\},$$

and note that

$$M=\bigcup_{u\in\mathbb{Q}}M_u.$$

Hence, it suffices to show that M_u is meagre for each $u\in\mathbb{Q}$. To do this, it suffices to construct subsets G_l (with $l\in\mathbb{N}$) of $\mathcal{P}(X)$ such that

(1) G_l is open;
(2) G_l is dense;
(3) $\bigcap_l G_l\subseteq\mathcal{P}(X)\setminus M_u$.

Let T_m be any sequence of real numbers with $T_m\ge s$ and $T_m\to\infty$, and put

$$H_m=\{\mu\in\mathcal{P}(X)\mid u<\mathit{\Gamma}_{T_m}(\mu;x)\},\qquad G_l=\bigcup_{m\ge l}H_m.$$

Claim 1. *G_l is open.*

Proof of Claim 1. This follows immediately from Lemma 5.1.

Claim 2. *G_l is dense in $\mathcal{P}(X)$.*

Proof of Claim 2. Let $\mu\in\mathcal{P}(X)$ and $r>0$. We must show that there is $\nu\in\mathcal{P}(X)$ with $\nu\in B(\mu,r)\cap G_l$. Since x is not an isolated point of X, it is not difficult to see that there are $r_0>0$ and $\nu\in\mathcal{P}(X)$ such that $\nu(C(x,r_0))=0$ and $\nu\in B(\mu,r)$. We will now prove that $\nu\in G_l$. First, recall that S_T

denotes the support of Π_T and $R_T = \sup S_T$. It follows from condition (i) in Theorem 1.1 that we can choose positive integers m_0 and n_0 with $m_0 > n_0 \geq l$ such that $-\log r_0 < R_{T_{n_0}} < R_{T_{m_0}}$. Since $\nu(C(x, e^{-t})) \leq \nu(C(x, r_0)) = 0$ for all $t \geq R_{T_{n_0}} > -\log r_0$, we conclude that

$$\begin{aligned}
\Gamma_{T_{m_0}}(\nu; x) &= \int_s^\infty \Pi_{T_{m_0}}(t) \frac{\log \nu(C(x, e^{-t}))}{\log e^{-t}}\, dt \\
&\geq \int_{R_{T_{n_0}}}^{R_{T_{m_0}}} \Pi_{T_{m_0}}(t) \frac{\log \nu(C(x, e^{-t}))}{\log e^{-t}}\, dt \\
&= \int_{R_{T_{n_0}}}^{R_{T_{m_0}}} \Pi_{T_{m_0}}(t) \frac{\log 0}{\log e^{-t}}\, dt = \infty \int_{R_{T_{n_0}}}^{R_{T_{m_0}}} \frac{\Pi_{T_{m_0}}(t)}{t}\, dt.
\end{aligned} \tag{5.3}$$

Finally, since $\int_{R_{T_{n_0}}}^{R_{T_{m_0}}} \frac{\Pi_{T_{m_0}}(t)}{t}\, dt > 0$ (by condition (i) in Theorem 1.1), it follows from (5.3) that $\Gamma_{T_{m_0}}(\nu; x) = \infty > u$, so that $\nu \in H_{m_0} \subseteq G_l$. This completes the proof of Claim 2.

Claim 3. $\bigcap_l G_l \subseteq \mathcal{P}(X) \setminus M_u$.

Proof of Claim 3. Let $\mu \in \bigcap_l G_l$. Then for each positive integer l, there is an integer $m_l \geq l$ such that $u < \Gamma_{T_{m_l}}(\mu; x)$. Hence clearly $\Gamma(\mu; x) = \limsup_{T\to\infty} \Gamma_T(\mu; x) \geq \limsup_l \Gamma_{T_{m_l}}(\mu; x) \geq u$, and so $\mu \in \mathcal{P}(X) \setminus M_u$.

Now (1.8) follows from Claims 1–3. ∎

6. Proof of (1.7) in Theorem 1.1. We must prove that

$$\underline{\dim}_{\Pi,\mathsf{loc}}(\mu; x) = 0 \tag{6.1}$$

for a typical $\mu \in \mathcal{P}(X)$; recall that $\underline{\dim}_{\Pi,\mathsf{loc}}(\mu; x) = \liminf_{T\to\infty} (\mathsf{A}_\Pi \Delta_{\mu,x})(T)$, where

$$(\mathsf{A}_\Pi \Delta_{\mu,x})(T) = \int_s^\infty \Pi_T(t) \Delta_{\mu,x}(t)\, dt = \int_s^\infty \Pi_T(t) \frac{\log \mu(B(x, e^{-t}))}{\log e^{-t}}\, dt.$$

For $x \in X$, $T \geq s$ and $\mu \in \mathcal{P}(X)$, we define

$$\Lambda_T(\mu; x) = \int_s^\infty \Pi_T(t) \frac{\log \mu(B(x, e^{-t}))}{\log e^{-t}}\, dt, \qquad \Lambda(\mu; x) = \liminf_{T\to\infty} \Lambda_T(\mu; x).$$

Observe that $\underline{\dim}_{\Pi,\mathsf{loc}}(\mu; x) = \Lambda(\mu; x)$. Hence, we must prove that for a typical $\mu \in \mathcal{P}(X)$ we have

$$\Lambda(\mu; x) = 0. \tag{6.2}$$

We begin with four small lemmas. The first is a “reverse” Fatou lemma. Its proof is standard, but we have decided to include it for completeness.

LEMMA 6.1. *Let* $(M,\mathcal{E},\mu)$ *be a measure space and let* $(f_m)_m$ *be a sequence of positive* $\mathcal{E}$*-measurable functions* $f_m : M \to [0,\infty]$. *If* $\int \sup_m f_m\,d\mu < \infty$, *then*

$$\limsup_m \int f_m\,d\mu \le \int \limsup_m f_m\,d\mu.$$

Proof. Write $g_m = \sup_{k\ge m} f_k$ and $g = \sup_k f_k$. Then $g_m \le g$ and $\int g\,d\mu = \int \sup_k f_k\,d\mu < \infty$. Since $g_m \to \limsup_m f_m$, it follows from the Dominated Convergence Theorem that

$$\lim_m \int g_m\,d\mu = \int \limsup_m f_m\,d\mu. \tag{6.3}$$

Next, note that $f_k \le g_m$ for $k \ge m$, whence

$$\sup_{k\ge m} \int f_k\,d\mu \le \int g_m\,d\mu. \tag{6.4}$$

Finally, combining (6.3) and (6.4) gives

$$\limsup_m \int f_m\,d\mu = \lim_m \sup_{k\ge m} \int f_k\,d\mu \le \lim_m \int g_m\,d\mu = \int \limsup_m f_m\,d\mu. \blacksquare$$

Before stating the next three lemmas we introduce the following definition. For $x \in X$, define

$$\mathcal{P}_x(X) = \{\mu \in \mathcal{P}(X) \mid x \in \operatorname{supp}\mu\};$$

here and below, $\operatorname{supp}\mu$ denotes the support of μ.

LEMMA 6.2. *Let* X *be a complete metric space. Fix* $x \in X$ *and* $\mu \in \mathcal{P}_x(X)$. *Let* $T \ge s$. *If* $(\mu_n)_n$ *is a sequence of measures in* $\mathcal{P}(X)$ *with* $\mu_n \to \mu$, *then*

$$\limsup_n \int_s^\infty \varPi_T(t) \frac{\log \mu_n(B(x,e^{-t}))}{\log e^{-t}}\,dt \le \int_s^\infty \limsup_n \varPi_T(t) \frac{\log \mu_n(B(x,e^{-t}))}{\log e^{-t}}\,dt.$$

Proof. First recall that R_T denotes the supremum of $\operatorname{supp}\varPi_T$, whence $\varPi_T(t) = 0$ for all $t > R_T$. Next, since $x \in \operatorname{supp}\mu$, we conclude that

$$\mu(B(x,e^{-R_T})) > 0. \tag{6.5}$$

In particular, $\mu(B(x,e^{-R_T})) > \frac{1}{2}\mu(B(x,e^{-R_T}))$. Since $\mu_n \to \mu$, this implies that $\liminf_n \mu_n(B(x,e^{-R_T})) \ge \mu(B(x,e^{-R_T})) > \frac{1}{2}\mu(B(x,e^{-R_T}))$. It follows that there is a positive integer n_0 such that $\mu_n(B(x,e^{-R_T})) > \frac{1}{2}\mu(B(x,e^{-R_T}))$ for all $n \ge n_0$.

For each $n \geq n_0$, define $f_n : [s, \infty) \to [0, \infty]$ by

$$f_n(t) = \Pi_T(t) \frac{\log \mu_n(B(x, e^{-t}))}{\log e^{-t}}.$$

We now claim that

$$\int_s^\infty \sup_{n \geq n_0} f_n(t)\, dt < \infty. \tag{6.6}$$

First note that since $\Pi_T(t) = 0$ for all $t > R_T$, we have

$$\begin{aligned}\int_s^\infty \sup_{n \geq n_0} f_n(t)\, dt &= \int_s^{R_T} \sup_{n \geq n_0} \Pi_T(t)\, \frac{\log \mu_n(B(x, e^{-t}))}{\log e^{-t}}\, dt \\ &\leq \int_s^{R_T} \sup_{n \geq n_0} \Pi_T(t)\, \frac{\log \mu_n(B(x, e^{-R_T}))}{\log e^{-t}}\, dt.\end{aligned}$$

Since $\mu_n(B(x, e^{-R_T})) > \frac{1}{2}\mu(B(x, e^{-R_T}))$ for all $n \geq n_0$, it follows that

$$\begin{aligned}\int_s^{R_T} \sup_{n \geq n_0} f_n(t)\, dt &\leq \int_s^{R_T} \Pi_T(t)\, \frac{\log \frac{1}{2}\mu(B(x, e^{-R_T}))}{\log e^{-t}}\, dt \\ &= -\int_s^{R_T} \frac{\Pi_T(t)}{t} \log \tfrac{1}{2}\mu(B(x, e^{-R_T}))\, dt \\ &= -\int_s^{R_T} \frac{\Pi_T(t)}{t}\, dt \log \tfrac{1}{2}\mu(B(x, e^{-R_T})).\end{aligned} \tag{6.7}$$

Finally, since $\int_s^{R_T} \frac{\Pi_T(t)}{t}\, dt = \int_s^\infty \frac{\Pi_T(t)}{t}\, dt < \infty$ (by condition (ii) in Theorem 1.1) and $\mu(B(x, e^{-R_T})) > 0$ (by (6.5)), we obtain (6.6).

It now follows from (6.6) and Lemma 6.1 that

$$\begin{aligned}\limsup_n \int_s^\infty \Pi_T(t) \frac{\log \mu_n(B(x, e^{-t}))}{\log e^{-t}}\, dt &= \limsup_n \int_s^\infty f_n(t)\, dt \\ &\leq \int_s^\infty \limsup_n f_n(t)\, dt \\ &= \int_s^\infty \limsup_n \Pi_T(t) \frac{\log \mu_n(B(x, e^{-t}))}{\log e^{-t}}\, dt. \quad \blacksquare\end{aligned}$$

Lemma 6.3. *Let X be a complete metric space. Let $c \in \mathbb{R}$, $x \in X$ and $T \geq s$. Then $\{\mu \in \mathcal{P}_x(X) \mid c \leq \Lambda_T(\mu; x)\}$ is a closed subset of $\mathcal{P}_x(X)$.*

Proof. Let $(\mu_n)_n$ be a sequence in $\mathcal{P}_x(X)$ with $c \leq \Lambda_T(\mu_n; x)$ for all n and let $\mu \in \mathcal{P}_x(X)$. Assume that $\mu_n \to \mu$. We must prove that $c \leq \Lambda_T(\mu; x)$. Since

$\mu_n \to \mu$ and $B(x, e^{-t})$ is open, we conclude that $\liminf_n \mu_n(B(x, e^{-t})) \geq \mu(B(x, e^{-t}))$, and Lemma 6.2 yields

$$\begin{aligned}
\Lambda_T(\mu; x) &= \int_s^\infty \varPi_T(t) \frac{\log \mu(B(x, e^{-t}))}{\log e^{-t}}\, dt \\
&\geq \int_s^\infty \varPi_T(t) \frac{\log \liminf_n \mu_n(B(x, e^{-t}))}{\log e^{-t}}\, dt \\
&= \int_s^\infty \limsup_n \varPi_T(t) \frac{\log \mu_n(B(x, e^{-t}))}{\log e^{-t}}\, dt \\
&\geq \limsup_n \int_s^\infty \varPi_T(t) \frac{\log \mu_n(B(x, e^{-t}))}{\log e^{-t}}\, dt \\
&= \limsup_n \Lambda_T(\mu_n; x) \geq c. \ \blacksquare
\end{aligned}$$

LEMMA 6.4. *Let X be a complete metric space. Let $x \in X$ and $r \geq 0$. Then $\{\mu \in \mathcal{P}(X) \mid \operatorname{dist}(x, \operatorname{supp} \mu) \geq r\}$ is a closed subset of $\mathcal{P}(X)$.*

Proof. Let $\mu \in \mathcal{P}(X)$ and let $(\mu_n)_n \subset \mathcal{P}(X)$ with $\operatorname{dist}(x, \operatorname{supp} \mu_n) \geq r$ for all n and with $\mu_n \to \mu$. We must prove that $\operatorname{dist}(x, \operatorname{supp} \mu) \geq r$, i.e. $\mu(B(x, u)) = 0$ for all $u < r$. We therefore fix $u < r$. Since $\operatorname{dist}(x, \operatorname{supp} \mu_n) \geq r$, we have $\mu_n(B(x, u)) = 0$, whence (using the fact that $\mu_n \to \mu$) $\mu(B(x, u)) \leq \liminf_n \mu_n(B(x, u)) = 0$. $\blacksquare$

Proof of (1.7) in Theorem 1.1. We must prove that $M = \{\mu \in \mathcal{P}(X) \mid \Lambda(\mu; x) > 0\}$ is meagre. For $u > 0$, write

$$M_u = \{\mu \in \mathcal{P}(X) \mid \Lambda(\mu; x) > u\},$$

and note that

$$M = \bigcup_{u \in \mathbb{Q},\, u > 0} M_u.$$

Hence, it suffices to show that each M_u is meagre. To do so, we will construct subsets $G_{l,k}$ (with $l, k \in \mathbb{N}$) of $\mathcal{P}(X)$ such that

(1) $G_{l,k}$ is open;
(2) $G_{l,k}$ is dense;
(3) $\bigcap_{l,k} G_{l,k} \subseteq \mathcal{P}(X) \setminus M_u$.

Let T_m be a sequence of real numbers with $T_m \geq s$ and $T_m \to \infty$. It follows from Lemma 6.3 that $\{\mu \in \mathcal{P}_x(X) \mid u \leq \Lambda_T(\mu; x)\}$ is a closed subset of $\mathcal{P}_x(X)$. Consequently,

$$L_m = \{\mu \in \mathcal{P}_x(X) \mid \Lambda_{T_m}(\mu; x) < u\}$$

is an open subset of $\mathcal{P}_x(X)$. Hence, for each $\mu \in L_m$, we can find $r_\mu > 0$

such that

$$B(\mu, r_\mu) \cap \mathcal{P}_x(X) \subseteq L_m. \tag{6.8}$$

Now put

$$H_m = \bigcup_{\mu \in L_m} B(\mu, r_\mu), \qquad G_l = \bigcup_{m \geq l} H_m.$$

Also, for a positive integer k, let

$$Q_k = \{\mu \in \mathcal{P}(X) \,|\, \operatorname{dist}(x, \operatorname{supp}\mu) < 1/k\}.$$

Finally, define

$$G_{l,k} = G_l \cap Q_k.$$

Claim 1. *$G_{l,k}$ is open.*

Proof of Claim 1. It is clear that G_l is open and it follows from Lemma 6.4 that Q_k is open. Hence $G_{l,k} = G_l \cap Q_k$ is open.

Claim 2. *$G_{l,k}$ is dense in $\mathcal{P}(X)$.*

Proof of Claim 2. Let $\mu \in \mathcal{P}(X)$ and $r > 0$. We must show that there is $\nu \in \mathcal{P}(X)$ with $\nu \in B(\mu, r) \cap G_{l,k} = B(\mu, r) \cap G_l \cap Q_k$. It is clear that we can choose $\nu \in \mathcal{P}(X)$ such that $\nu(\{x\}) > 0$ (in particular $\nu \in Q_k$) and $\nu \in B(\mu, r)$.

It remains to prove $\nu \in G_l$. Since $\nu(\{x\}) > 0$ and $\lim_{T\to\infty} \int_s^\infty \frac{\Pi_T(t)}{t}\,dt = 0$ (by condition (iii) in Theorem 1.1), there is a positive integer $m_0 \geq l$ such that

$$\int_s^\infty \frac{\Pi_{T_{m_0}}(t)}{t}\,dt \log\frac{1}{\nu(\{x\})} < u.$$

Hence,

$$\begin{aligned} \Lambda_{T_{m_0}}(\nu; x) &= \int_s^\infty \Pi_{T_{m_0}}(t) \frac{\log \nu(B(x, e^{-t}))}{\log e^{-t}}\,dt \\ &\leq \int_s^\infty \Pi_{T_{m_0}}(t) \frac{\log \nu(\{x\})}{\log e^{-t}}\,dt \\ &= \int_s^\infty \frac{\Pi_{T_{m_0}}(t)}{t}\,dt \log\frac{1}{\nu(\{x\})} < u. \end{aligned} \tag{6.9}$$

Since also $\nu \in \mathcal{P}_x(X)$ (because $\nu(\{x\}) > 0$), we deduce from (6.9) that $\nu \in L_{m_0} \subseteq H_{m_0} \subseteq G_l$. This completes the proof of Claim 2.

Claim 3. *$\bigcap_{l,k} G_{l,k} \subseteq \mathcal{P}(X) \setminus M_u$.*

Proof of Claim 3. Let $\mu \in \bigcap_{l,k} G_{l,k}$. Hence, for all positive integers l and k, we have $\mu \in G_{l,k} \subseteq Q_k$, whence $\operatorname{dist}(x, \operatorname{supp}\mu) \leq 1/k$. We conclude that $\operatorname{dist}(x, \operatorname{supp}\mu) = 0$, and so $x \in \operatorname{supp}\mu$, i.e.

$$\mu \in \mathcal{P}_x(X). \tag{6.10}$$

Also, for all positive integers l and k, we have $\mu \in G_{l,k} \subseteq G_l$. This shows that for each positive integer l, there is an integer $m_l \geq l$ and a measure $\mu_l \in L_{m_l}$ such that

$$\mu \in B(\mu_l, r_{\mu_l}). \tag{6.11}$$

Now (6.8), (6.10) and (6.11) show that $\mu \in B(\mu_l, r_{\mu_l}) \cap \mathcal{P}_x(X) \subseteq L_{m_l}$, whence $\Lambda_{T_{m_l}}(\mu; x) < u$. This implies that $\Lambda(\mu; x) = \liminf_{T\to\infty} \Lambda_T(\mu; x) \leq \liminf_l \Lambda_{T_{m_l}}(\mu; x) \leq u$, and so $\mu \in \mathcal{P}(X) \setminus M_u$. This completes the proof of Claim 3.

Now (1.7) follows from Claims 1–3. ■

7. Appendix A. The purpose of this appendix is to prove (3.2). While (3.2) is almost certainly well-known, we have been unable to find an explicit reference, and for this reason we provide a short and direct proof.

THEOREM A.1. *Let $f : [1, \infty) \to [0, \infty)$ be a measurable function. Then*

$$\liminf_{t\to\infty} f(t) \leq \underline{A}^1_{\mathsf{C}} f \leq \underline{A}^2_{\mathsf{C}} f \leq \cdots \leq \overline{A}^2_{\mathsf{C}} f \leq \overline{A}^1_{\mathsf{C}} f \leq \limsup_{t\to\infty} f(t).$$

Proof. For a positive integer n, define $\varphi_n : [1, \infty) \to [0, \infty)$ by $\varphi_n(t) = t^n$, and note that if $h : [1, \infty) \to [0, \infty)$ is a measurable function, then

$$\begin{aligned} \liminf_{t\to\infty} h(t) &\leq \liminf_{t\to\infty} \frac{\int_1^t \varphi_n(u)\, h(u)\, du}{\int_1^t \varphi_n(u)\, du} \\ &\leq \limsup_{t\to\infty} \frac{\int_1^t \varphi_n(u)\, h(u)\, du}{\int_1^t \varphi_n(u)\, du} \leq \limsup_{t\to\infty} h(t). \end{aligned} \tag{A.1}$$

Next, for a positive integer n and $t \geq 1$, write $(L_n f)(t) = \frac{n!}{t^n}(I^n f)(t)$ (recall that If is defined in Section 3). Then

$$\begin{aligned} (L_{n+1} f)(t) &= \frac{(n+1)!}{t^{n+1}} (I^{n+1} f)(t) \\ &= \frac{(n+1)!}{t^{n+1}} \int_1^t (I^n f)(u)\, du \\ &= \frac{(n+1)!}{t^{n+1}} \int_1^t \frac{u^n}{n!} (L_n f)(u)\, du \\ &= \left(1 - \frac{1}{t^{n+1}}\right) \frac{\int_1^t \varphi_n(u)\, (L_n f)(u)\, du}{\int_1^t \varphi_n(u)\, du}. \end{aligned} \tag{A.2}$$

Finally, since

$$\underline{A}_{\mathsf{C}}^{m} f = \liminf_{T\to\infty} (L_m f)(T) \quad \text{and} \quad \overline{A}_{\mathsf{C}}^{m} f = \limsup_{T\to\infty} (L_m f)(T)$$

for all $m \in \mathbb{N}$, the desired result follows from (A.1) and (A.2). ■

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