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John–Nirenberg lemmas for a doubling measure

by

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Abstract. We study, in the context of doubling metric measure spaces, a class of BMO type functions defined by John and Nirenberg. In particular, we present a new version of the Calderón–Zygmund decomposition in metric spaces and use it to prove the corresponding John–Nirenberg inequality.

1. Introduction. Besides the well known class BMO of functions of bounded mean oscillation, F. John and L. Nirenberg defined another, larger class of functions in their paper [11]. We call this space the *John–Nirenberg space* with exponent p and write JN_p. Whereas the classical John–Nirenberg lemma shows that any function of bounded mean oscillation has exponentially decaying distribution function, any function in JN_p belongs to weak L^p.

Unlike BMO, the John–Nirenberg space has not been systematically studied. In this paper we generalize the definition to doubling metric measure spaces by replacing the cubes in the original definition by metric space balls, and, in particular, prove the John–Nirenberg lemma for JN_p in this setting. We also study properties of this space; for example, we show that every p-integrable function is in the John–Nirenberg space with the same exponent, and we provide an example of a function in the weak L^p that is not a John–Nirenberg function.

In the Euclidean case there are a few proofs of the John–Nirenberg inequality for JN_p. The original proof in [11], based on an induction argument, can be found with more details in [8] and [7]. There is an alternative proof on the real line: see [18]. We present here a new proof in the Euclidean case, which is more straightforward than the original argument. The proof is based on iterating a suitable good-λ inequality. It is interesting that this proof generalizes directly to the setting of doubling metric measure spaces via dyadic sets defined by M. Christ; see [1] or [3] for the definition.

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To prove the John–Nirenberg inequality for JN_p in the metric case we have adapted ideas from A. P. Calderón's proof of the classical John–Nirenberg lemma for BMO in the Euclidean setting in [16], and from the aforementioned proof in [18]. To this end, we present a new version of the Calderón–Zygmund decomposition in metric spaces. The advantage of this version is that we are able to iterate it efficiently, which is not trivial in the metric setting. We also get both lower and upper bounds for mean values over the decomposition balls. Existence of a doubling measure is the only assumption we need to impose on the space.

Calderón's method is remarkably flexible as illustrated by a simplified proof of the so-called parabolic John–Nirenberg inequality by E. Fabes and N. Garofalo; see [5]. To further demonstrate this flexibility of Calderón's technique and the use of our decomposition lemma we also give a new proof of the classical John–Nirenberg lemma for BMO in doubling metric measure spaces. The lemma has previously been generalized to doubling metric measure spaces, for example, in [12], [2], [14], [15].

Addendum. After the paper had been accepted, the authors learned that the results in [6] and [13] can also be applied for the class JN_p. For similar results in Orlicz spaces, see [9].

2. Doubling metric measure spaces.

Let (X, d, μ) be a metric space endowed with a metric d and a Borel regular measure μ. We assume that an open ball always comes with a center and a radius, i.e.

$$B = B(x, r) = \{y \in X : d(y, x) < r\}.$$

We denote by λB the λ-*dilate* of B, that is, the ball with the same center as B but λ times its radius. We assume that μ is *doubling*, i.e. all open balls have positive and finite measure whenever $r > 0$ and there exists a constant $c_\mu \geq 1$, called the *doubling constant* of μ, so that

$$\mu(2B) \leq c_\mu \mu(B) \quad \text{for all } B \text{ in } X.$$

The doubling condition implies a covering theorem, sometimes referred to as the Vitali covering theorem. Indeed, given any collection of balls with uniformly bounded radius, there exists a pairwise disjoint, countable subcollection of balls whose 5-dilates cover the union of the original collection. This theorem implies Lebesgue's differentiation theorem, which guarantees that any locally integrable function can be approximated at almost every point by integral averages of the function over a contracting sequence of balls.

The *Hardy–Littlewood maximal function* Mf of a locally integrable function f is defined for every $x \in X$ by

$$Mf(x) = \sup_{B \ni x} \fint_B |f| \, d\mu,$$

where

$$f_B = \fint_B f \, d\mu = \frac{1}{\mu(B)} \int_B f \, d\mu,$$

and the supremum is taken over all balls containing x. The Hardy–Littlewood maximal function satisfies

$$\|Mf\|_p \le c(p, \mu) \|f\|_p \tag{2.1}$$

for every $f \in L^p(X)$ with $1 < p \le \infty$. For the proof of (2.1), the Vitali covering theorem and further information on metric spaces, see, for example, [10] or [4].

3. The second John–Nirenberg inequality for a doubling measure.

We begin by recalling the definition of the John–Nirenberg space in the Euclidean case; see [11]. Let Q_0 be a cube in $\mathbb{R}^n$ and $1 \le p < \infty$. An integrable function f defined on Q_0 belongs to $\mathrm{JN}_p(Q_0)$, the *John–Nirenberg space* with exponent p, if there exists $K_f < \infty$ such that

$$\sum_i |Q_i| \Big[\fint_{Q_i} |f - f_{Q_i}| \, dx \Big]^p \le K_f^p \tag{3.1}$$

independent of the family $\{Q_i\}_{i=1}^{\infty}$, where Q_i are subcubes of Q_0 such that $\bigcup Q_i = Q_0$ and the interiors of Q_i are disjoint.

Observe that the definition in terms of cubes can be directly generalized in metric spaces. Indeed, the dyadic structure of the Euclidean cubes can be transferred to a doubling metric measure space using Christ's construction [3]. Then the natural definition is in terms of these dyadic sets. However, the definition of JN_p in a doubling metric measure space is most natural in terms of balls. Balls cannot be organized in a simple dyadic way in nested generations as cubes in $\mathbb{R}^n$ and we have thus chosen to define the space JN_p so that the definition is compatible with the Vitali covering theorem.

Definition 3.2. Let (X, d, μ) be a metric measure space, $1 < p < \infty$ and $B_0 \subset X$ be a ball. Let f be a locally integrable function defined on $11B_0$. We say that f belongs to the *John–Nirenberg space* with exponent p, and we write $f \in \mathrm{JN}_p(B_0)$, if there exists $K_f < \infty$ such that

$$\sum_i \mu(B_i) \Big(\fint_{B_i} |f - f_{B_i}| \, d\mu \Big)^p \le K_f^p$$

whenever $\{B_i\}$ is a countable collection of balls centered at B_0 and contained in $11B_0$ with the property that the balls $\frac{1}{5}B_i$ are pairwise disjoint. We will call the smallest possible constant K_f the JN_p *norm* of f.

Remark 3.3. Observe that JN_p is a generalization of BMO. Indeed, it follows directly from the definitions that a function is of bounded mean oscillation if and only if its JN_p norm is bounded as p tends to infinity.

The next result shows that there are plenty of functions in John–Nirenberg spaces.

Proposition 3.4. *Let $1 < p < \infty$ and $f \in L^p(11B_0)$. Then $f \in \mathrm{JN}_p(B_0)$.*

Proof. Let B_i be a family of balls that is admissible in the definition of $\mathrm{JN}_p(B_0)$. Write $B_i' = \frac{1}{5}B_i$ for the disjoint balls. We know that for every ball B_i,

$$\fint_{B_i'} Mf\,d\mu \ge \inf_{x\in B_i'} Mf(x) \ge \fint_{B_i} |f|\,d\mu.$$

Hence,

$$\sum_i \mu(B_i)\Big(\fint_{B_i} |f-f_{B_i}|\,d\mu\Big)^p \le 2^p c_\mu^3 \sum_i \mu(B_i')\Big(\fint_{B_i'} Mf\,d\mu\Big)^p.$$

Now by Hölder's inequality

$$\mu(B_i')\Big(\fint_{B_i'} Mf\,d\mu\Big)^p \le \int_{B_i'} (Mf)^p\,d\mu,$$

and by the disjointness of the balls B_i' and the boundedness of the maximal operator we have

$$\sum_i \int_{B_i'} (Mf)^p\,d\mu \le \int_{11B_0} (Mf)^p\,d\mu \le c\int_{11B_0} |f|^p\,d\mu,$$

which is finite by assumption. This completes the proof. ■

Notice that in $\mathbb{R}^n$ Proposition 3.4 follows from the definition simply by using the Hölder inequality.

The John–Nirenberg inequality for $\mathrm{JN}_p(Q_0)$ shows that it is contained in weak $L^p(Q_0)$. The following one-dimensional example shows that the inclusion is strict.

Example 3.5. Consider the function $f(x) = x^{-1/p}$ on $Q_0 = (0,2)$ with $p > 1$. It is clear that this function belongs to weak $L^p(Q_0)$. Let us partition the interval Q_0 as $Q_j = (2^{-j}, 2^{1-j})$, where $j = 0, 1, \ldots$, to see that (3.1) fails. A simple change of variable $x = 2^{-j}y$ shows that $f_{Q_j} = 2^{j/p} f_{Q_0}$. Similarly, we set $I = |f - f_{Q_0}|_{Q_0}$ and conclude that $|f - f_{Q_j}|_{Q_j} = 2^{j/p} I$. Hence, the sum in (3.1) diverges.

The following theorem is our main result.

THEOREM 3.6. *If $f \in \mathrm{JN}_p(B_0)$, then*

$$\mu(\{x \in B_0 : |f(x) - f_{B_0}| > \lambda\}) \le C(K_f/\lambda)^p, \tag{3.7}$$

where C only depends on p and the doubling constant.

To prove the theorem we need two lemmas. The first one is a Calderón–Zygmund decomposition lemma and the second one is a good-λ-type inequality. The key idea behind the proof of Theorem 3.6 stems from the method used in [16].

LEMMA 3.8. *Let f be a non-negative locally integrable function on X. Fix a ball $B_0 = B(x_0, R)$ and assume that*

$$\lambda_0 \ge \frac{1}{\mu(B_0)} \int_{11B_0} f\,d\mu.$$

Then there exists a countable, possibly finite, family $\{B_i\}_i$ of disjoint balls centered in B_0 and satisfying $5B_i \subset 11B_0$ such that

(i) $f(x) \le \lambda_0$ *for μ-a.e. $x \in B_0 \setminus \bigcup_i 5B_i$,*
(ii) $\lambda_0 < \fint_{B_i} f \le c_\mu^3 \lambda_0$,
(iii) $c_\mu^{-3}\lambda_0 < \fint_{5B_i} f \le \lambda_0$.

The balls satisfying the above conditions are called Calderón–Zygmund balls *at level λ_0. Moreover, if $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_N$, then the Calderón–Zygmund balls corresponding to different levels λ_n may be chosen in such a way that each $B_i(\lambda_{n+1})$ is contained in some $5B_j(\lambda_n)$.*

Proof. Define a maximal function

$$M_{B_0} f(x) = \sup_{\substack{B \ni x \\ B \subset B_0}} \fint_B f\,d\mu,$$

where the supremum is taken over all balls containing x and included in B_0. Write

$$E_\lambda = \{x \in B_0 : M_{B_0} f(x) > \lambda\}.$$

Let us first consider λ_N to show how the balls are chosen. By the definition of $M_{B_0} f$, for every $x \in E_{\lambda_N}$ there exists a ball B_x with $x \in B_x \subset B_0$ and

$$\lambda_0 \le \cdots \le \lambda_N < \fint_{B_x} f\,d\mu. \tag{3.9}$$

We now take a look at the balls $5^k B_x$, where $k \in \mathbb{Z}_+$. Note that if a ball B satisfies $B_0 \subset B \subset 11B_0$, then by the choice of λ_N, we have

$$\fint_B f\,d\mu \le \frac{1}{\mu(B_0)} \int_{11B_0} f\,d\mu \le \lambda_0 \le \lambda_N.$$

If B_x has radius r, take k such that $5^{k-1}r \le 2R < 5^k r$. Then $B_0 \subset 5^k B_x \subset 11B_0$ and the average of f over $5^k B_x$ is at most λ_N. Consequently, there exists a smallest $n = n_x \ge 1$ such that

$$\fint_{5^n B_x} f\, d\mu \le \lambda_N. \tag{3.10}$$

Then

$$\lambda_N < \fint_{5^j B_x} f\, d\mu \tag{3.11}$$

for all $j = 0, 1, \ldots, n-1$.

Consider the balls $5^{n_x - 1}B_x$. They form a covering of E_{λ_N} and by the Vitali covering theorem we may pick a countable subfamily of pairwise disjoint balls $B_i = 5^{n_{x_i}-1}B_{x_i}$ with

$$E_{\lambda_N} \subset \bigcup_{i=1}^{\infty} 5B_i.$$

The balls B_i have the required properties. Indeed, by (3.10) and (3.11), we have

$$\lambda_N < \fint_{5^{n-1}B_x} f\, d\mu \le c_\mu^3 \fint_{5^n B_x} f\, d\mu \le c_\mu^3 \lambda_N, \tag{3.12}$$

thus proving (ii). Since $5B_i = 5^n B_{x_i}$, the first inequality in (iii) has already been proved in (3.12), while the second inequality is just (3.10).

It remains to prove (i). We have

$$B_0 \setminus \bigcup_{i=1}^{\infty} 5B_i \subset B_0 \setminus E_{\lambda_N}.$$

This implies that $M_{B_0} f(x) \le \lambda_N$ for μ-a.e. $x \in B_0 \setminus \bigcup_i 5B_i$, from which we get (i) by Lebesgue's differentiation theorem.

We have now constructed the desired decomposition at level λ_N and turn to λ_{N-1}. Since $E_{\lambda_N} \subset E_{\lambda_{N-1}}$, for every $x \in E_{\lambda_N}$ we may start from exactly the same ball B_x satisfying (3.9) as before. For every $x \in E_{\lambda_{N-1}} \setminus E_{\lambda_N}$ we take a ball B_x with $x \in B_x \subset B_0$ and

$$\lambda_0 \le \cdots \le \lambda_{N-1} < \fint_{B_x} f\, d\mu. \tag{3.13}$$

Now for each ball B_x choose the smallest $m = m_x \ge 1$ satisfying

$$\fint_{5^m B_x} f\, d\mu \le \lambda_{N-1}. \tag{3.14}$$

Notice that if B_x is a ball corresponding to an $x \in E_{\lambda_N}$, then $n \le m$ (here n is from (3.10)). Then apply Vitali's theorem to the balls $5^{m-1}B_x$ to obtain a family of balls satisfying conditions (i)–(iii) with λ_0 replaced by λ_{N-1}.

Now let $B_i(\lambda_N)$ by any of the Calderón–Zygmund balls corresponding to λ_N. Then $B_i(\lambda_N) = 5^{n-1}B_{x_i}$ for some $x_i \in E_{\lambda_N}$ and $B_i(\lambda_N) \subset 5^{m-1}B_{x_i}$ (because $n \le m$). The ball $5^{m-1}B_{x_i}$ is not necessarily a Calderón–Zygmund ball corresponding to the level λ_{N-1}, but it is one of the balls in the collection from which the Calderón–Zygmund balls were extracted. Vitali's theorem shows that $5^{m-1}B_{x_i}$ is contained in a 5-dilate of some of them, say, $B_j(\lambda_{N-1})$. Then $B_i(\lambda_N) \subset 5B_j(\lambda_{N-1})$.

We continue this procedure. Next, we consider $E_{\lambda_{N-2}}$. For $x \in E_{\lambda_N}$ we take the same ball B_x which we used in the first step. For $x \in E_{\lambda_{N-1}} \setminus E_{\lambda_N}$ we use the same ball B_x which we used in the second step. For every $x \in E_{\lambda_{N-2}} \setminus E_{\lambda_{N-1}}$ we take a ball B_x with $x \in B_x \subset B_0$ and

$$\lambda_0 \le \cdots \le \lambda_{N-2} < \fint_{B_x} f\,d\mu \tag{3.15}$$

and proceed as previously. ■

Lemma 3.16. *Assume* $f \in \mathrm{JN}_p(B_0)$ *and*

$$\lambda \ge \frac{1}{\mu(B_0)} \int_{11B_0} |f - f_{B_0}|\,d\mu.$$

Consider Calderón–Zygmund balls $\{B_i(\lambda)\}_i$ *and* $\{B_j(2\lambda)\}_j$ *for the function* $|f - f_{B_0}|$ *at levels* λ *and* 2λ, *respectively. Suppose that each* $B_i(2\lambda)$ *is contained in some* $5B_j(\lambda)$. *Then*

$$\sum_j \mu(B_j(2\lambda)) \le \frac{c_\mu^{3/q}K_f}{\lambda}\Big(\sum_i \mu(B_i(\lambda))\Big)^{1/q}, \tag{3.17}$$

where q *is the conjugate exponent of* p, *that is,* $1/p + 1/q = 1$.

Proof. We may assume $K_f = 1$ and $f_{B_0} = 0$. We partition the family $\{B_j(2\lambda)\}_j$ as follows. First collect those balls which are contained in $5B_1(\lambda)$. From the remaining balls we collect those which are contained in $5B_2(\lambda)$ and continue similarly. In other words,

$$\{B_j(2\lambda)\}_j = \bigcup_i \{B_j(2\lambda)\}_{j \in J_i},$$

where

$$\begin{aligned}
J_1 &= \{j : B_j(2\lambda) \subset 5B_1(\lambda)\},\\
J_2 &= \{j : B_j(2\lambda) \subset 5B_2(\lambda),\ j \notin J_1\},\\
J_3 &= \{j : B_j(2\lambda) \subset 5B_3(\lambda),\ j \notin J_1 \cup J_2\},\\
&\ \ \vdots
\end{aligned}$$

We have

$$2\lambda\sum_j \mu(B_j(2\lambda)) \le \sum_j \int_{B_j(2\lambda)} |f|\,d\mu = \sum_i \sum_{j\in J_i} \int_{B_j(2\lambda)} |f|\,d\mu, \tag{3.18}$$

where

$$\begin{aligned}\sum_{j\in J_i} \int_{B_j(2\lambda)} |f|\,d\mu &\le \sum_{j\in J_i} \int_{B_j(2\lambda)} \big||f| + \lambda - |f_{5B_i(\lambda)}|\big|\,d\mu \\ &\le \sum_{j\in J_i} \int_{B_j(2\lambda)} |f - f_{5B_i(\lambda)}|\,d\mu + \sum_{j\in J_i} \int_{B_j(2\lambda)} \lambda\,d\mu \\ &\le \int_{5B_i(\lambda)} |f - f_{5B_i(\lambda)}|\,d\mu + \lambda \sum_{j\in J_i} \mu(B_j(2\lambda)).\end{aligned}$$

Now we sum over i to obtain

$$2\lambda\sum_j \mu(B_j(2\lambda)) \le \sum_i \int_{5B_i(\lambda)} |f - f_{5B_i(\lambda)}|\,d\mu + \lambda\sum_j \mu(B_j(2\lambda)).$$

By Hölder's inequality and the normalization $K_f = 1$ we get

$$\begin{aligned}&\sum_i \int_{5B_i(\lambda)} |f - f_{5B_i(\lambda)}|\,d\mu \\ &\quad= \sum_i \mu(5B_i(\lambda))^{1/q}\mu(5B_i(\lambda))^{-1/q} \int_{5B_i(\lambda)} |f - f_{5B_i(\lambda)}|\,d\mu \\ &\quad\le \Big(\sum_i \mu(5B_i(\lambda))\Big)^{1/q} \Big(\sum_i \mu(5B_i(\lambda))^{-p/q}\Big(\int_{5B_i(\lambda)} |f - f_{5B_i(\lambda)}|\,d\mu\Big)^p\Big)^{1/p} \\ &\quad\le c_\mu^{3/q}\Big(\sum_i \mu(B_i(\lambda))\Big)^{1/q},\end{aligned}$$

whence

$$2\lambda\sum_j \mu(B_j(2\lambda)) \le c_\mu^{3/q}\Big(\sum_i \mu(B_i(\lambda))\Big)^{1/q} + \lambda\sum_j \mu(B_j(2\lambda)).$$

This finishes the proof. ■

Proof of Theorem 3.6. We wish to iterate the estimate (3.17). We still assume $K_f = 1$ and $f_{B_0} = 0$, whence

$$\mu(B_0)\Big(\fint_{B_0} |f|\,d\mu\Big)^p \le 1 \quad\text{and}\quad \mu(11B_0)\Big(\fint_{11B_0} |f - f_{11B_0}|\,d\mu\Big)^p \le 1.$$

Therefore,

$$\begin{aligned}\frac{1}{\mu(B_0)}\int\limits_{11B_0}|f|\,d\mu &\le c_\mu^4 \fint\limits_{11B_0}|f-f_{11B_0}|\,d\mu + c_\mu^4\fint\limits_{B_0}|f_{11B_0}|\,d\mu\\ &\le \frac{c_\mu^4}{\mu(11B_0)^{1/p}} + c_\mu^4\fint\limits_{B_0}|f-f_{11B_0}|\,d\mu + c_\mu^4\fint\limits_{B_0}|f|\,d\mu\\ &\le \frac{c_\mu^4}{\mu(B_0)^{1/p}} + c_\mu^8\fint\limits_{11B_0}|f-f_{11B_0}|\,d\mu + \frac{c_\mu^4}{\mu(B_0)^{1/p}}\\ &\le \frac{C_1}{\mu(B_0)^{1/p}},\end{aligned}$$

where $C_1 = 3c_\mu^8$. We choose

$$\lambda_0 = \frac{C_1}{\mu(B_0)^{1/p}}.$$

Now let $\lambda > \lambda_0$ and take $N \in \mathbb{Z}_+$ such that

$$2^N\lambda_0 < \lambda \le 2^{N+1}\lambda_0. \tag{3.19}$$

Then apply the decomposition lemma at levels $\lambda_0 < 2\lambda_0 < 2^2\lambda < \cdots < 2^N\lambda$ to obtain $N+1$ families of Calderón–Zygmund balls. Observe that for $n = 0, 1, \ldots, N-1$ each $B_i(2^{n+1}\lambda)$ is contained in some $5B_j(2^n\lambda)$.

First notice that

$$\begin{aligned}\mu(\{x\in B_0 : |f(x)|>\lambda\}) &\le \mu(\{x\in B_0 : |f(x)|>2^N\lambda_0\})\\ &\le \sum_j \mu(5B_j(2^N\lambda_0)) \le c_\mu^3\sum_j \mu(B_j(2^N\lambda_0)).\end{aligned}$$

Then use (3.17) and the fact that

$$1+q^{-1}+\cdots+q^{-(N-1)} = p - pq^{-N}$$

to estimate

$$\begin{aligned}\sum_j \mu(B_j(2^N\lambda_0)) &\le \frac{c_\mu^{3/q}}{2^{N-1}\lambda_0}\left(\frac{c_\mu^{3/q}}{2^{N-2}\lambda_0}\right)^{1/q}\left(\frac{c_\mu^{3/q}}{2^{N-3}\lambda_0}\right)^{1/q^2}\cdots\\ &\quad\cdot\left(\frac{c_\mu^{3/q}}{2^0\lambda_0}\right)^{1/q^{N-1}}\cdot\left(\frac{1}{\lambda_0}\int\limits_{11B_0}|f|\,d\mu\right)^{q^{-N}}\\ &= \frac{c_\mu^{3q^{-1}+\cdots+3q^{-N}}}{g(N)}\cdot\left(\frac{1}{\lambda_0}\right)^{p-pq^{-N}}\cdot\left(\frac{1}{\lambda_0}\int\limits_{11B_0}|f|\,d\mu\right)^{q^{-N}}.\end{aligned}$$

Here $g(1)=1$ and for $N\ge 2$,

$$\frac{1}{g(N)}=\frac{2^{q^{-1}+2q^{-2}+\cdots+(N-1)q^{-(N-1)}}}{2^{(N-1)(p-pq^{-N})}}.$$

We have the estimate

$$\frac{c_\mu^{3q^{-1}+\cdots+3q^{-N}}}{g(N)}\le\frac{C}{2^{(N-1)p}},$$

where the constant C only depends on p and the doubling constant.

Moreover, the choice of λ_0 gives

$$\left(\frac{1}{\lambda_0}\right)^{-pq^{-N}}\cdot\left(\frac{1}{\lambda_0}\int_{11B_0}|f|\,d\mu\right)^{q^{-N}}\le\left(\frac{C_1^p}{\mu(B_0)}\right)^{q^{-N}}\cdot\mu(B_0)^{q^{-N}}=C_1^{pq^{-N}}\le C_1^{pq}.$$

Now combine the previous estimates and use (3.19) to get

$$\mu(\{x\in B_0:|f(x)|>\lambda\})\le\frac{C}{2^{(N-1)p}}\left(\frac{1}{\lambda_0}\right)^p=\frac{C}{(2^{N-1}\lambda_0)^p}\le\frac{C}{\lambda^p}.$$

Here C is a constant depending only on p and on the doubling constant. For $0<\lambda<\lambda_0$ we use the trivial estimate

$$\mu(\{x\in B_0:|f(x)|>\lambda\})\le\mu(B_0)=\frac{C_1^p}{\lambda_0^p}\le\frac{C_1^p}{\lambda^p}.\ \blacksquare$$

4. Euclidean case.

In this section we give a new proof for the second John–Nirenberg inequality in $\mathbb{R}^n$. See Lemma 3 in [11].

THEOREM 4.1 (John–Nirenberg inequality II). *If f is a function satisfying* (3.1), *then $f-f_{Q_0}$ is in weak $L^p(Q_0)$, i.e., there exists $C>0$ depending only on n and p such that*

$$|\{x\in Q_0:|f(x)-f_{Q_0}|>\lambda\}|\le C(K_f/\lambda)^p \tag{4.2}$$

for all $\lambda>0$.

Let Q be a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, and denote by $|S|$ the Lebesgue measure of a set S. The *dyadic maximal function* of f is defined as

$$M^df(x)=\sup_{Q\ni x}\fint_Q|f(y)|\,dy, \tag{4.3}$$

where the supremum is taken over all dyadic cubes Q containing x. Moreover, for $\lambda>0$ we define $E_Q(\lambda)=\{x\in Q:M^df(x)>\lambda\}$.

We recall a decomposition lemma; see [17, Chapter IV, Section 3.1].

LEMMA 4.4. *Let Q_0 be a cube and let $f\in L^1(Q_0)$. Suppose that*

$$\fint_{Q_0}|f(x)|\,dx\le\lambda.$$

Then $E_{Q_0}(\lambda) = \bigcup_{k=1}^{\infty} Q_k$, *where* $\{Q_k\}$ *is a collection of cubes whose interiors are disjoint, such that*

(i) $|f(x)| \le \lambda$ *for a.e.* $x \in Q_0 \setminus \bigcup_{k=1}^{\infty} Q_k$,
(ii) $\lambda < \fint_Q |f(x)|\,dx \le 2^n\lambda$ *for all* Q *in the collection* $\{Q_k\}$,
(iii) $|E_{Q_0}(\lambda)| \le \lambda^{-1} \int_{E_{Q_0}(\lambda)} |f(x)|\,dx$.

The following good-λ inequality is the core of our proof.

LEMMA 4.5. *For a function* $f \in \mathrm{JN}_p(Q_0)$ *and a number* $0 < b < 2^{-n}$ *we have*

$$|\{x \in Q_0 : M^d(f - f_{Q_0})(x) > \lambda\}| \le \frac{aK_f}{\lambda} |\{x \in Q_0 : M^d(f - f_{Q_0})(x) > b\lambda\}|^{1/q} \tag{4.6}$$

for all $\lambda \ge b^{-1} \fint_{Q_0} |f(x) - f_{Q_0}|\,dx$, *where* $a = 1/(1 - 2^n b)$.

Proof. Without loss of generality, we assume that $f_{Q_0} = 0$; then (4.6) becomes

$$|E_{Q_0}(\lambda)| \le \frac{aK_f}{\lambda} \cdot |E_{Q_0}(b\lambda)|^{1/q}. \tag{4.7}$$

First, we apply Lemma 4.4 to $|f(x)|$ on Q_0 with λ replaced by $b\lambda$ to get a collection $\{Q_k\}_{k\ge1}$ of countable disjoint dyadic cubes such that $E_{Q_0}(b\lambda) = \bigcup_{k=1}^{\infty} Q_k$. It follows that $E_{Q_0}(\lambda) = \bigcup_{k=1}^{\infty} E_{Q_k}(\lambda)$ since $E_{Q_0}(\lambda) \subset E_{Q_0}(b\lambda)$.

Moreover, let $x \in Q_k$ be such that $M^d f(x) > \lambda$. Then there exists a dyadic cube Q containing x with

$$\fint_Q f\,dx > \lambda. \tag{4.8}$$

Since Q_k is the maximal dyadic cube such that the first inequality in (ii) holds for $b\lambda$, $Q \subset Q_k$ and it follows from (4.8) that $M^d(f\chi_{Q_k})(x) > \lambda$. Moreover, $M^d[(f - f_{Q_k})\chi_{Q_k}](x) > (1-2^n b)\lambda$ by the second inequality in (ii).

Then fix a k; if $\fint_{Q_k} |(f - f_{Q_k})|\,dx \le (1 - 2^n b)\lambda$, we apply Lemma 4.4 to $|(f - f_{Q_k})\chi_{Q_k}|$ on Q_k with λ replaced by $(1 - 2^n b)\lambda$. By (iii) we have

$$\begin{aligned} |E_{Q_k}(\lambda)| &\le |\{x \in Q_k : M^d[(f - f_{Q_k})\chi_{Q_k}](x) > (1 - 2^n b)\lambda\}| \\ &\le \frac{1}{(1 - 2^n b)\lambda} \int_{Q_k} |f - f_{Q_k}|\,dx \\ &= \frac{|Q_k|^{1/q}}{(1 - 2^n b)\lambda} \Big(|Q_k|^{1/p-1} \int_{Q_k} |f - f_{Q_k}|\,dx\Big). \end{aligned} \tag{4.9}$$

Otherwise

$$|Q_k| < \frac{1}{(1-2^n b)\lambda} \int_{Q_k} |f - f_{Q_k}|\,dx,$$

and (4.9) holds as well.

By adding these inequalities for all k, we get, by the Hölder inequality,

$$\begin{aligned}|E_{Q_0}(\lambda)| &\le \sum_k \frac{|Q_k|^{1/q}}{(1-2^n b)\lambda}\Big(|Q_k|^{1/p-1} \int_{Q_k} |f - f_{Q_k}|\,dx\Big) \\ &\le \frac{1}{(1-2^n b)\lambda}\Big(\sum |Q_k|\Big)^{1/q}\Big\{\sum |Q_k|^{1-p}\Big[\int_{Q_k} |f - f_{Q_k}|\,dx\Big]^p\Big\}^{1/p} \\ &\le \frac{1}{(1-2^n b)\lambda}|E_{Q_0}(b\lambda)|^{1/q} K_f\end{aligned}$$

since Q_k are disjoint. ■

We are now ready to prove the John–Nirenberg lemma.

Proof of Theorem 4.1. Without loss of generality we may assume $f_{Q_0}=0$. Let $b = 2^{-(n+1)}$ and define

$$\eta = \frac{K_f}{b|Q_0|^{1/p}}.$$

Let

$$\lambda \ge \frac{1}{b} \fint_{Q_0} |f(x)|\,dx$$

and let j be the smallest integer satisfying $b^{-j}\eta < \lambda$. We iterate the estimate (4.7) j times to get

$$\begin{aligned}|E_\lambda(Q_0)| &\le |E_{Q_0}(b^{-j}\eta)| \\ &\le \Big(\frac{aK_f}{b^{-j}\eta}\Big)\Big(\frac{aK_f}{b^{-j+1}\eta}\Big)^{1/q}\cdots\Big(\frac{aK_f}{b^{-1}\eta}\Big)^{1/q^{j-1}}|E_{Q_0}(\eta)|^{1/q^j} \\ &\le \Big(\frac{aK_f}{b\lambda}\Big)\Big(\frac{aK_f}{b^2\lambda}\Big)^{1/q}\cdots\Big(\frac{aK_f}{b^j\lambda}\Big)^{1/q^{j-1}}\Big[\frac{1}{\eta}\int_{Q_0} |f|\,dx\Big]^{1/q^j},\end{aligned}$$

where the third inequality comes from the weak type inequality (iii) in Lemma 4.4 and from the definition of j.

Observe that

$$1 + \frac{2}{q} + \cdots + \frac{j}{q^{j-1}} \le p^2.$$

By the definition of JN_p and η we have

$$\frac{1}{\eta}\int_{Q_0} |f|\,dx \le b|Q_0|.$$

Hence,

$$\begin{aligned}|E_{Q_0}(\lambda)| &\le \left(\frac{aK_f}{\lambda}\right)^{p(1-q^{-j})} b^{-p^2}(b|Q_0|)^{1/q^j}\\ &= 2^{p(1-q^{-j})}2^{(n+1)(p^2-1/q^j)}\left(\frac{K_f}{\lambda}\right)^{p}\left|\frac{\lambda|Q_0|^{1/p}}{K_f}\right|^{p/q^j}.\end{aligned}$$

By the definition of η and j we have

$$\frac{\lambda|Q_0|^{1/p}}{K_f} \le b^{-j+2} = 2^{(n+1)(j-2)}.$$

Since

$$(j-2)q^{-j} \le q^{-3}p^2,$$

we can now conclude

$$|E_{Q_0}(\lambda)| \le 2^{p+(n+1)(p^2+(p/q)^3)}(K_f/\lambda)^p.$$

This proves the theorem for large values of λ.

For $\lambda \le K_f/(b|Q_0|^{1/p})$, we have

$$|E_{Q_0}(\lambda)| \le |Q_0| \le 2^{(n+1)p}(K_f/\lambda)^p$$

as desired. ■

Observe that this proof can be generalized to the metric setting via Christ's dyadic sets and by a Calderón–Zygmund decomposition lemma by Aimar et al.; see Theorems 2.6 and 3.1 in [1].

5. John–Nirenberg inequality for a doubling measure. In this section we give a new proof of the John–Nirenberg lemma in a doubling metric measure space. The result is by no means sharper or more general than the results in the literature. Nevertheless, we hope that the current proof will further increase the understanding of the phenomenon.

We recall that a locally integrable function $f\colon X \to \mathbb{R}$ is in BMO(X) if there exists a constant c such that

$$\fint_B |f - f_B|\, d\mu \le c \tag{5.1}$$

for all balls B in X. The space is equipped with the seminorm

$$\|f\|_\sharp = \sup_{B\subset X} \fint_B |f - f_B|\, d\mu.$$

If we define an equivalence relation

$$f \sim g \quad \text{if and only if} \quad f - g = \text{constant},$$

then $\mathrm{BMO}(X)/_\sim$ is a normed space. As is common, we continue denoting this space by $\mathrm{BMO}(X)$ and speak of functions instead of equivalence classes.

Theorem 5.2. *Let $f \in \mathrm{BMO}(X)$. Then*

$$\mu(\{x \in B : |f - f_B| > \lambda\}) \leq c_1 \mu(B) e^{-c_2\lambda/\|f\|_\sharp}$$

for all balls $B \subset X$ and $\lambda > 0$ with with c_1, c_2 not depending on f and λ.

Proof. Take $f \in \mathrm{BMO}(X)$. We may assume that $\|f\|_\sharp = 1$. We first notice that

$$\begin{aligned}\frac{1}{\mu(B)} \int_{11B} |f - f_B|\, d\mu &\leq c_\mu^4 \fint_{11B} |f - f_{11B}|\, d\mu + c_\mu^4 |f_B - f_{11B}| \\ &\leq c_\mu^4 + c_\mu^4 \fint_B |f - f_{11B}|\, d\mu \leq 2c_\mu^8.\end{aligned}$$

Thus, the expression on the left-hand side above is bounded uniformly in B. Now fix a ball B_0 and assume $f_{B_0} = 0$. If $\{B_j\}_j$ is the Calderón–Zygmund decomposition at level $\lambda \geq 2c_\mu^8$, given by Lemma 3.8, then

(i) $|f(x)| \leq \lambda$ for μ-a.e. $x \in B_0 \setminus \bigcup_j 5B_j$,
(ii) $\lambda < \fint_{B_j} |f| \leq c_\mu^3 \lambda$,
(iii) $c_\mu^{-3}\lambda < \fint_{5B_j} |f| \leq \lambda$.

We deduce by (i) that

$$\mu(\{x \in B_0 : |f(x)| > \lambda\}) \leq \sum_j \mu(5B_j) \leq c_\mu^3 \sum_j \mu(B_j). \tag{5.3}$$

Analogously to Calderón's proof in [16], we wish to study the size of $\sum_j \mu(B_j)$. Apply the decomposition lemma at levels $\lambda > \gamma \geq 2c_\mu^8$. Denote the corresponding Calderón–Zygmund balls by $\{B_j(\lambda)\}_j$ and $\{B_k(\gamma)\}_k$, which we choose in a similar way to the proof of Lemma 3.16. We write $\{B_j(\lambda)\}_j$ as a disjoint union

$$\{B_j(\lambda)\}_j = \bigcup_k \{B_j(\lambda)\}_{j \in J_k},$$

where J_k's are defined as in the proof of Lemma 3.16, but with 2λ replaced by λ, and λ by γ. By (ii), we may now write

$$\lambda \sum_j \mu(B_j(\lambda)) \leq \sum_j \int_{B_j(\lambda)} |f|\, d\mu = \sum_k \sum_{j \in J_k} \int_{B_j(\lambda)} |f|\, d\mu. \tag{5.4}$$

Moreover, we have

$$\begin{aligned}\sum_{j\in J_k}\int_{B_j}|f|\,dx &\le \sum_{j\in J_k}\int_{B_j}\big||f|+\gamma-|f_{5B_k(\gamma)}|\big|\,dx\\ &\le \sum_{j\in J_k}\int_{B_j}|f-f_{5B_k(\gamma)}|\,dx+\sum_{j\in J_k}\int_{B_j}\gamma\,dx\\ &\le \int_{5B_k(\gamma)}|f-f_{5B_k(\gamma)}|\,dx+\gamma\sum_{j\in J_k}\mu(B_j)\\ &\le \mu(5B_k(\gamma))+\gamma\sum_{j\in J_k}\mu(B_j)\le c_\mu^3\mu(B_k(\gamma))+\gamma\sum_{j\in J_k}\mu(B_j).\end{aligned}$$

Now sum over k and use (5.4) to obtain

$$\lambda\sum_j\mu(B_j(\lambda))\le c_\mu^3\sum_k\mu(B_k(\gamma))+\gamma\sum_j\mu(B_j(\lambda)).$$

Thus, we see that

$$(\lambda-\gamma)\sum_j\mu(B_j(\lambda))\le c_\mu^3\sum_k\mu(B_k(\gamma)) \tag{5.5}$$

whenever $\lambda\ge\gamma\ge 2c_\mu^8$. Now set $a=2c_\mu^8>2c_\mu^3$ and replace λ and γ respectively by $\lambda+a$ and λ. We have shown that if $\lambda\ge a$ and the Calderón–Zygmund balls corresponding to λ and $\lambda+a$ are chosen in such a way that each ball $B_j(\lambda+a)$ is contained in some $5B_k(\lambda)$, then

$$\sum_j\mu(B_j(\lambda+a))\le\frac{1}{2}\sum_k\mu(B_k(\lambda)).$$

Now let $\lambda\ge a$ and take $N\in\mathbb{Z}_+$ such that $Na\le\lambda<(N+1)a$. Then apply the decomposition lemma at each level $a<2a<\cdots<Na$. From the above estimate and (5.3) we get

$$\begin{aligned}\mu(\{x\in B_0:|f(x)|>\lambda\})&\le\mu(\{x\in B_0:|f(x)|>Na\})\\ &\le c_\mu^3\sum_j\mu(B_j(Na))\le c_\mu^3 2^{-N+1}\sum_j\mu(B_j(a))\\ &\le c_\mu^3 2^{-N+1}\mu(11B_0)\le c_\mu^7 e^{(2-\lambda/a)\log 2}\mu(B_0)\\ &=4c_\mu^7 e^{-(\lambda\log 2)/a}\mu(B_0).\end{aligned}$$

For $0<\lambda<a$ we have

$$\begin{aligned}\mu(\{x\in B_0:|f(x)|>\lambda\})&\le\mu(B_0)\le 4c_\mu^7 e^{-\log 2}\mu(B_0)\\ &\le 4c_\mu^7 e^{-(\lambda\log 2)/a}\mu(B_0).\end{aligned}$$

Hence the John–Nirenberg inequality holds with $c_1=4c_\mu^7$ and $c_2=(\log 2)/a$. ∎

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