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Hardy spaces for the Laplacian with lower order perturbations

by

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Abstract. We consider Hardy spaces of functions harmonic on smooth domains in Euclidean spaces of dimension greater than two with respect to the Laplacian perturbed by lower order terms. We deal with the gradient and Schrödinger perturbations under appropriate Kato conditions. In this context we show the usual correspondence between the harmonic Hardy spaces and the L^p spaces (or the space of finite measures if $p = 1$) on the boundary. To this end we prove the uniform comparability of the respective harmonic measures for a class of approximating domains and the relative Fatou theorem for harmonic functions of the perturbed operator.

1. Introduction. Let L be the operator of the form $L = \frac{1}{2}\Delta + b(\cdot) \cdot \nabla$ on a $C^{1,1}$ domain D in $\mathbb{R}^d$, $d \geq 3$, where b is a vector field such that $|b|$ belongs to the Kato class $K_{d+1}^{\text{loc}}(D)$ and $|b|^2$ belongs to $K_d^{\text{loc}}(D)$. We study the Hardy spaces $h_L^p(D)$, $1 \leq p \leq \infty$, of harmonic functions of L on D and we extend the results obtained in [CrZ] for the positive solutions of $Lu \equiv 0$. In the case of the *classical* harmonic functions of the Laplacian, i.e. when $b \equiv 0$, this topic has been intensively studied (see [DK], [JK1], [JK2], [L], [S1], [S2]). The basic properties of the *classical* Hardy spaces on the ball and the half-space in $\mathbb{R}^d$ are described in [ABR]. The case of smooth bounded domains is discussed in [S2], Lipschitz domains are considered in [DK] and [JK2] and the so-called *nontangentially accessible* domains in [JK1]. A typical theorem in the theory of Hardy spaces says that a function u harmonic on D belongs to the Hardy space $h^p(D)$ for a given $p \in (1, \infty]$ if and only if u is the Poisson integral of some function $f \in L^p(\partial D, \sigma)$ where σ is the surface measure. For $p = 1$, f should be a finite complex Borel measure on ∂D. To address the operators more general than $\frac{1}{2}\Delta$, we note that this topic is considered by Widman in [W1] in the case of the half-space and elliptic operators with continuous or Hölder continuous coefficients. The Schrödinger operator $\frac{1}{2}\Delta - c$ with nonnegative potential c is considered in

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[K1] for the ball and in [K2] for the half-space. Recently, similar properties have been obtained by Michalik and Ryznar for Lipschitz domains in the case of the so-called *singular* α-harmonic functions (see [MR1]). We should note there is also a different line of research on Hardy spaces featuring the maximal functions of the corresponding semigroups; see for example [CKS], [DzZ1] or [DzZ2]. The focus of the present note is however in the spirit of [L] and [S2].

The starting point for our results is the paper [CrZ], where the authors have proved that for Lipschitz domains the harmonic measure and Green function of L are comparable with the harmonic measure and Green function of $\frac{1}{2}\Delta$. In the case of $C^{1,1}$ domains similar results were obtained in [HS] for elliptic operators with coefficients which are Hölder continuous up to the boundary of D. In [IR] Ifra and Riahi have shown analogous properties for the operator $\operatorname{div}(A(\cdot)\nabla) + b(\cdot)\cdot\nabla$, where A is a uniformly elliptic matrix with Lipschitz continuous coefficients, but their methods make a restrictive assumption of smallness of the Kato norm of b, at the same time relaxing the condition $|b|^2 \in K_d^{\mathrm{loc}}(D)$. Finally, some of the results were generalized by Kim and Song in [KiS2] to the operators of the form $\mathcal{L} + \mu\cdot\nabla + \nu$, where $\mathcal{L}$ is uniformly elliptic and the measures μ, ν belong to the corresponding Kato classes. We wish to note, however, that the papers mentioned above in this paragraph do not address Hardy spaces.

The main objective of this note is an extension of the h^p-theory to L-harmonic functions on a $C^{1,1}$ domain D, with L satisfying the assumptions of [CrZ]. Our definition of the L-harmonic Hardy spaces $h^p_L(D)$ corresponds to the one introduced by Stein in [S2] (see Section 4), and we prove the following theorem.

THEOREM 1.1. *Let u be L-harmonic on a bounded $C^{1,1}$ domain $D \subset \mathbb{R}^d$, where $d \geq 3$. Then*

(i) *$u \in h^1_L(D)$ if and only if $u = P_L[\mu]$ with a unique $\mu \in \mathcal{M}(\partial D)$. Furthermore, there exists a constant c_1 depending on D and b such that, for every $\mu \in \mathcal{M}(\partial D)$,*

$$c_1^{-1}\|\mu\| \leq \|P_L[\mu]\|_{h^1} \leq c_1\|\mu\|.$$

(ii) *$u \in h^p_L(D)$ for a given $p \in (1, \infty]$ if and only if $u = P_L[f]$ with a unique $f \in L^p(\partial D, \sigma)$. Furthermore, there exists a constant c_2 depending on D and b such that, for every $f \in L^p(\partial D, \sigma)$,*

$$c_2^{-1}\|f\|_p \leq \|P_L[f]\|_{h^p} \leq c_2\|f\|_p.$$

Moreover, our results extend to the operators considered in [HS], [IR], [KiS2] and [R] (see Remark 5.5). In order to study the Hardy spaces of L-harmonic functions we adapt the ideas introduced in [MR1]. As we proceed, we need some strong assertions about the comparability of the harmonic measures of $\frac{1}{2}\Delta$ and L on $C^{1,1}$ domains. In particular we prove that the corresponding Poisson kernels are uniformly comparable for the sets $\{x \in D : \operatorname{dist}(x, \partial D) > r\}$, when r is sufficiently small. Another important tool is the relative Fatou theorem for ratios of positive harmonic functions. We extend it to the operator L applying the methods of [Wu].

At the end of the paper we give an extension of our results to the Schrödinger operator $\frac{1}{2}\Delta + q$, which may be of independent interest. In this part, we work under the assumptions $q \in K_d^{\mathrm{loc}}(D)$ and the gaugeability of (D, q) (see [ChuZ] and [CrFZ]).

The h^p-theory for diffusion operators may also be considered for Lipschitz domains; here the use of probabilistic techniques introduced in [MR1] seems suitable. A part of the relevant properties of Hardy spaces should then follow from martingale theory. On the other hand a purely analytic approach to this problem seems to be difficult and may be an objective of further research based on the Green function estimates of [B] and available estimates of gradients of nonnegative harmonic functions (see for example [BKN]). In this context, some results for the *classical* harmonic functions can be found in [DK], [JK1] and [JK2], but they are based on different characterizations of h^p-spaces than the one considered in Theorem 1.1.

The paper is organized as follows. In Section 2 we give basic definitions and facts concerning the classical potential theory and recall some of the results of [CrZ]. In Section 3 we prove the uniform comparability of the Poisson kernels of L and $\frac{1}{2}\Delta$ on a class of subdomains of D. The correspondence between L-harmonic Hardy spaces and L^p spaces for $1 < p \leq \infty$ and with the space of finite measures for $p = 1$ is given in Section 4. In Section 5 we prove analogous results for the Schrödinger operator $\frac{1}{2}\Delta + q$.

2. Preliminaries. By $\mathbb{R}^d$ we denote the d-dimensional Euclidean space with norm $|\cdot|$. Here and throughout the paper we consider dimensions $d \geq 3$. Let ∂B denote the boundary of B and let $\delta_B(x) = \operatorname{dist}(x, \partial B)$. All constants in this paper are (strictly) positive and we use the convention that constants denoted by small letters may differ in each lemma, while constants denoted by capital letters do not change. The notation $c = c(a, b)$ means that the constant c depends only on a and b. All functions are assumed to be complex valued unless stated otherwise. In this paper, by a domain we mean a connected open subset of $\mathbb{R}^d$.

Let D be a bounded domain. We say that D is a $C^{1,1}$ *domain* if there exist constants r, λ depending only on D such that for every $z \in \partial D$ there is a function $F : \mathbb{R}^{d-1} \to \mathbb{R}$ and an orthonormal coordinate system $y = (y_1, \ldots, y_d)$ such that

$$D \cap B(z, r) = \{y : y_d > F(y_1, \ldots, y_{d-1})\} \cap B(z, r),$$

where F is differentiable and ∇F is a Lipschitz function with the Lipschitz constant not greater than λ.

Equivalently, we may define a $C^{1,1}$ domain D as a bounded domain satisfying the following geometric condition, called the *ball condition* (see [AKSZ]): there exists a constant $r = r(D)$ such that for each $y \in \partial D$ there are balls $B(c_y, r) \subset D$ and $B(\tilde{c}_y, r) \subset D^c$, tangent at y.

From now on D will be a fixed nonempty $C^{1,1}$ domain in $\mathbb{R}^d$. For $r \geq 0$ we define the approximating inner sets D_r as

$$D_r = \{x \in D : \delta_D(x) > r\}.$$

Note that $D_0 = D$. Let σ_r denote the $(d-1)$-dimensional Hausdorff surface measure on ∂D_r, and set $\sigma = \sigma_0$ (occasionally, σ will also denote the Hausdorff surface measure on spheres). The following two lemmas provide some basic properties of D_r (see [MR1] for the proofs).

LEMMA 2.1. *There exists a constant $r_0 = r_0(D)$ such that for every $r \in [0, r_0]$, D_r is a $C^{1,1}$ domain and satisfies the ball condition with radius r_0. Furthermore,*

$$\partial D_r = \{x \in D : \delta_D(x) = r\}.$$

In view of Lemma 2.1, for r sufficiently small and $0 \leq s < r$ we may define the projections $\pi_{s,r} \colon \partial D_s \to \partial D_r$, where $\pi_{s,r}(x)$ means the closest point to x on ∂D_r, and let $\pi_r = \pi_{0,r}$. Then for all $x \in \partial D_s$ we have $|x - \pi_{s,r}(x)| = r - s$, and the ball $B(x, r-s) \subset D_r^c$ is tangent to ∂D_r at $\pi_{s,r}(x)$, while the ball $B(\pi_{s,r}(x), r-s) \subset D_s$ is tangent to ∂D_s at x. In particular we have

$$\pi_r(x) = \pi_{s,r}(\pi_s(x)), \quad x \in \partial D. \tag{1}$$

LEMMA 2.2. *There exist constants c, r_0 depending only on D such that for all $r \in (0, r_0]$, $s \in [0, r)$ and every nonnegative function f on $\overline{D}$ we have*

$$c^{-1} \int_{\partial D_r} f(x)\, d\sigma_r(x) \leq \int_{\partial D_s} f(\pi_{s,r}(y))\, d\sigma_s(y) \leq c \int_{\partial D_r} f(x)\, d\sigma_r(x).$$

Let $G_r(x,y)$ be the Green function of $\frac{1}{2}\Delta$ for D_r. If r is sufficiently small, then D_r is a $C^{1,1}$ domain, and the Poisson kernel of $\frac{1}{2}\Delta$ for D_r is given by

$$P_r(x,z) = -\frac{\partial G_r(x,z)}{\partial \nu_z^r}, \quad x \in D_r,\ z \in \partial D_r, \tag{2}$$

where ν_z^r is the outward unit normal vector at z.

LEMMA 2.3. *There exist constants r_0, c_1, c_2, c_3 depending only on D such that for all $r \in [0, r_0]$, $x, y \in D_r$ and $z \in \partial D_r$,*

$$c_1^{-1} \Phi_r(x,y) \leq G_r(x,y) \leq c_1 \Phi_r(x,y) \tag{3}$$

where

$$\Phi_r(x,y) = \min\left\{ \frac{1}{|x-y|^{d-2}}, \frac{\delta_{D_r}(x)\delta_{D_r}(y)}{|x-y|^d} \right\},$$

and we have

$$c_2^{-1}\frac{\delta_{D_r}(x)}{|x-z|^d} \le P_r(x,z) \le c_2\frac{\delta_{D_r}(x)}{|x-z|^d}, \tag{4}$$

$$|\nabla P_r(x,z)| \le c_3\frac{1}{|x-z|^d}. \tag{5}$$

Proof. With r fixed and the constants c_1, c_2, c_3 depending on D_r, the estimates above follow from [W2], [Z1] and [Z2]. The uniformity of (3) is an immediate consequence of [B, (22)] and Lemma 2.1. The upper bounds of (4) and (5) follow from (2), (3) and gradient estimates for harmonic functions. In the case of (4), it is explained in [Z1, proof of Lemma 1, p. 25] that the constant in the lower bound depends only on the diameter of the domain, the radius r_0 from the ball condition for D_r and c such that

$$\inf\{P_r(x,z) : x \in D_r,\ \delta_{D_r}(x) \ge r_0/2,\ z \in \partial D_r\} \ge c$$

(see [Z1, p. 22]). However, in view of Lemma 2.1 and the Harnack inequality, c can be taken independent of $r \in [0, r_0]$, $z \in \partial D_r$ and $x \in \overline{D}_{2r_0}$. This gives the desired conclusion. ∎

From now on, R_0 will denote the constant r_0 satisfying the conditions of Lemma 2.1–2.3. We will consider $r \in [0, R_0]$. We define the *Martin kernel* for D_r by

$$K_r(x,z) = \lim_{D_r \ni y \to z} \frac{G_r(x,y)}{G_r(x_0,y)}, \qquad x \in D_r,\ z \in \partial D_r,$$

where x_0 is a fixed point in D_{R_0}. By (2) we have

$$K_r(x,z) = \frac{P_r(x,z)}{P_r(x_0,z)}. \tag{6}$$

The following inequalities are known as the *3G Theorem*.

Lemma 2.4. *There exist constants c_1, c_2, c_3 depending only on D such that for all $r \in [0, R_0]$, $x, y, w \in D_r$ and $z \in \partial D_r$ we have*

$$\frac{G_r(x,y)G_r(y,w)}{G_r(x,w)} \le c_1\left(\frac{1}{|x-y|^{d-2}} + \frac{1}{|y-w|^{d-2}}\right),$$

$$\frac{G_r(x,y)K_r(y,z)}{K_r(x,z)} \le c_2\left(\frac{1}{|x-y|^{d-2}} + \frac{1}{|y-z|^{d-2}}\right),$$

$$\frac{G_r(x,y)|\nabla K_r(y,z)|}{K_r(x,z)} \le c_3\left(\frac{1}{|x-y|^{d-1}} + \frac{1}{|y-z|^{d-1}}\right).$$

Proof. It is well known that the constants in the inequalities above depend only on the estimates (3)–(5) and the dimension d (see for example [CrFZ] or [IR]). Thus the conclusion follows from Lemma 2.3. ∎

For $r \geq 0$ we let $(W_t, \mathbb{P}_r^x)$ be the Brownian motion killed on exiting D_r and let $p_r(t,x,y)$ be its transition density. For $z \in \partial D_r$ we define the $K_r(\cdot,z)$-conditional Brownian motion on D_r to be the Markov process with the transition density

$$p_r^z(t,x,y) = \frac{p_r(t,x,y)K_r(y,z)}{K_r(x,z)}.$$

For $z \in D_r$ we consider the $G_r(\cdot,z)$-conditional Brownian motion on $D_r \setminus \{z\}$ defined by

$$p_r^z(t,x,y) = \frac{p_r(t,x,y)G_r(y,z)}{G_r(x,z)}.$$

Also $(W_t, \mathbb{P}_{z,r}^x)$, for $z \in \partial D_r$, will denote the $K_r(\cdot,z)$-conditional Brownian motion on D_r (and the $G_r(\cdot,z)$-conditional Brownian motion if $z \in D_r$). For standard properties of conditional processes see [ChuZ] and [Do2]. Equivalently, we may define these processes as the respective solutions of the following stochastic differential equations:

$$dY_t = dW_t + \frac{\nabla K_r(Y_t,z)}{K_r(Y_t,z)}dt, \quad z \in \partial D_r,$$
$$d\tilde{Y}_t = dW_t + \frac{\nabla G_r(\tilde{Y}_t,z)}{G_r(\tilde{Y}_t,z)}dt, \quad z \in D_r.$$

Let b be a vector field on D with $|b| \in K_{d+1}^{\mathrm{loc}}(D)$ and $|b|^2 \in K_d^{\mathrm{loc}}(D)$, i.e., $|b|$ is uniformly integrable with respect to the measures $\mu(x,dy) = |x-y|^{1-d}dy$, $x \in \overline{D}$, and $|b|^2$ is uniformly integrable with respect to the measures $\nu(x,dy) = |x-y|^{d-2}dy$, $x \in \overline{D}$. As explained in [CrZ], there exists a unique solution X_t of the stochastic differential equation

$$dX_t = dW_t + b(X_t)dt, \qquad X_0 = x,$$

where W_t is the Brownian motion killed on exiting D. The process X_t is a diffusion with the generator $L = \frac{1}{2}\Delta + b(\cdot)\cdot\nabla$. We say that a function u on D is *L-harmonic* on D if it is continuous and for every open set $B \subset D$ with $\overline{B} \subset D$ we have

$$u(x) = \mathbb{E}^x u(X_{\tau_B}), \quad x \in B, \quad \text{where } \tau_B = \inf\{t \geq 0 : X_t \notin B\}. \tag{7}$$

Equivalently, a continuous function u on D is L-harmonic on D if $Lu \equiv 0$ in the weak sense, i.e., for any function $\phi \in C_c^\infty(D)$ we have

$$\frac{1}{2}\int_D u(x)\Delta\phi(x)\,dx = \int_D u(x)b(x)\cdot\nabla\phi(x)\,dx.$$

For $t \geq 0$ we define $M(t) = \int_0^t b(W_s)\,dW_s$, $\langle M\rangle_t = \int_0^t |b(W_s)|^2\,ds$, and $N(t) = \exp\{M(t) - \frac{1}{2}\langle M\rangle_t\}$. Let $H_r(x,dz)$ be the harmonic measure of $\frac{1}{2}\Delta$ for D_r and let $G_r^L(x,y)$, $H_r^L(x,dz)$ be the Green function and the harmonic measure of L for D_r, respectively. As opposed to (7), set $\tau_{D_r} = \inf\{t \ge 0 : W_t \notin D_r\}$. In [CrZ] it is shown that

$$G_r^L(x,y) = \mathbb{E}_{y,r}^x N(\tau_{D_r}) G_r(x,y), \quad x, y \in D_r, \tag{8}$$

and on ∂D_r we have

$$H_r^L(x,dz) = \mathbb{E}_{z,r}^x N(\tau_{D_r}) H_r(x,dz), \quad x \in D_r.$$

Furthermore, there is a constant c depending on b and D_r such that

$$c^{-1} \le \mathbb{E}_{z,r}^x N(\tau_{D_r}) \le c, \quad x \in D_r,\ z \in \overline{D}_r.$$

We conclude that the Poisson kernel of L for D_r satisfies

$$P_r^L(x,z) = \mathbb{E}_{z,r}^x N(\tau_{D_r}) P_r(x,z). \tag{9}$$

In particular,

$$\mathbb{E}_r^x N(\tau_{D_r}) = \int_{\partial D_r} \mathbb{E}_{z,r}^x N(\tau_{D_r}) P_r(x,z)\, d\sigma_r(z) = 1, \tag{10}$$

and

$$c^{-1} P_r(x,z) \le P_r^L(x,z) \le c P_r(x,z). \tag{11}$$

In the next section we will prove that the estimate (11) holds with a constant which does not depend on small r.

3. Estimates for conditional Brownian motion. The objective of this section is to prove the following theorem.

THEOREM 3.1. *There exist constants c, r_0 depending only on b and D such that for all $r \in [0, r_0]$, $x \in D_r$ and $z \in \partial D_r$ we have*

$$c^{-1} \le \mathbb{E}_{z,r}^x N(\tau_{D_r}) \le c.$$

The idea of the proof is the same as in [CrZ], and it originated in the proof of the Conditional Gauge Theorem (see [ChuZ]). However, in the present context we aim at uniform estimates for the variable sets D_r, which makes the calculations a little more complicated. The essential difference in the proof is Lemma 3.6.

Recall that the constants denoted by capital letters do not change from place to place. Using (4) and (6) fix a constant C_1 depending only on D such that

$$C_1^{-1} \frac{\delta_{D_r}(x)}{|x-z|^d} \le K_r(x,z) \le C_1 \frac{\delta_{D_r}(x)}{|x-z|^d} \tag{12}$$

for every $r \in [0, R_0]$, $x \in D_r$ and $z \in \partial D_r$.

LEMMA 3.2. *There exists a constant $c = c(D)$ such that if $0 < \rho' < \rho$, $r \in [0, R_0]$, $z \in \partial D_r$ and $x, y \in D_r$ are such that $|x-y| < \rho'$ and $B(y, 2\rho) \subset D_r$,* *then for every Borel set* $A \subset \partial B(y,\rho)$ *we have*

$$\mathbb{P}^x_{z,r}(W_{\tau_{B(y,\rho)}} \in A) \geq c(\rho - \rho')\sigma(A).$$

Proof. By the definition of $\mathbb{P}^x_{z,r}$ we have

$$\begin{aligned}\mathbb{P}^x_{z,r}(W_{\tau_{B(y,\rho)}} \in A) &= \frac{1}{K_r(x,z)} \int\limits_A K_r(w,z) P_{B(y,\rho)}(x,w)\, d\sigma(w) \\ &= \frac{\Gamma(d/2)}{2\pi^{d/2}\rho K_r(x,z)} \int\limits_A K_r(w,z) \frac{\rho^2 - |x-y|^2}{|x-w|^d}\, d\sigma(w) \\ &\geq \frac{c}{K_r(x,z)} \cdot \frac{\rho - \rho'}{\rho^d} \int\limits_A K_r(w,z)\, d\sigma(w),\end{aligned}$$

where $c = \Gamma(d/2)/(2^{d+1}\pi^{d/2})$. By (12) we obtain

$$\mathbb{P}^x_{z,r}(W_{\tau_{B(y,\rho)}} \in A) \geq \frac{c}{C_1^2} \cdot \frac{|x-z|^d}{\delta_{D_r}(x)} \cdot \frac{\rho - \rho'}{\rho^d} \int\limits_A \frac{\delta_{D_r}(w)}{|z-w|^d}\, d\sigma(w).$$

Since $z \in \partial D_r$ and $B(y, 2\rho) \subset D_r$, we have

$$\begin{aligned}\mathbb{P}^x_{z,r}(W_{\tau_{B(y,\rho)}} \in A) &\geq \frac{c}{C_1^2} \cdot \frac{\rho^{d-1}(\rho - \rho')}{\rho^d} \int\limits_A \frac{\rho}{(\mathrm{diam}(D))^d}\, d\sigma(w) \\ &= \tilde{c}(\rho - \rho')\sigma(A). \ \blacksquare\end{aligned}$$

The next lemma provides some geometric properties of the domains D_r, needed in the proof of Theorem 3.1.

LEMMA 3.3. *For every* $\delta > 0$ *there exist constants* r_1, r_2, r_3 *depending only on* D *and* δ *with the following properties:* $r_1 < R_0$,

$$2r_3 < r_2 < \left(\frac{1}{C_1^2 2^{d+1}}\right)^{\frac{1}{d-1}} r_1,$$

and there exists a compact connected set $F_0 \subset D$ *and, for every* $r \in [0, r_3]$, *a domain* $U_r \subset D_r$ *such that*

(i) $D_r \setminus \overline{D}_{r_1} \subset U_r$, *and thus* $\partial D_r \subset \partial U_r$,
(ii) $\partial D_{r_2} \subset F_0 \subset U_r$,
(iii) $\mathrm{dist}(F_0, \partial D) \geq 2r_3$, *and thus* $\mathrm{dist}(F_0, \partial D_r) \geq 2r_3 - r$,
(iv) $B(x, r_3) \subset U_r$ *for every* $x \in F_0$,
(v) $m(U_r) < \delta$, *where* m *is* d-*dimensional Lebesgue measure.*

Proof. Fix $\delta > 0$. Since D is a $C^{1,1}$ domain, $\partial D = \bigcup_{i=1}^n F_i$, where F_i are closed, disjoint and connected. Choose a constant $r_1 = r_1(D, \delta)$, $0 < r_1 < R_0$, for which the sets $U_i = \{x \in D : \mathrm{dist}(x, F_i) < r_1\}$ are disjoint (and obviously connected), and $m(\bigcup_{i=1}^{n} U_i) < \delta/2$. Take $r_2 > 0$ such that

$$r_2 < r_1\left(\frac{1}{C_1^2 2^{d+1}}\right)^{\frac{1}{d-1}}.$$

By Lemma 2.1,

$$\partial D_{r_2} = \{x \in D : \delta_D(x) = r_2\} = \{y - r_2\nu_y : y \in \partial D\},$$

where ν_y is the outward unit normal vector at y. Obviously, $\partial D_{r_2} \subset \bigcup_{i=1}^{n} U_i$. We have $\partial D_{r_2} = \bigcup_{i=1}^{n} \widetilde{F}_i$, where $\widetilde{F}_i = \{y - r_2\nu_y : y \in F_i\}$ are closed, connected, and $\widetilde{F}_i \subset U_i$ for every i. Fix $x_i \in \widetilde{F}_i$, $i = 1, \ldots, n$. Since D is connected, for every $i = 1, \ldots, n-1$ there exists a curve $\Gamma_i \subset D$ connecting x_i to x_{i+1}, and thus $\widetilde{F}_i$ to $\widetilde{F}_{i+1}$. Set $F_0 = \partial D_{r_2} \cup \bigcup_{i=1}^{n-1} \Gamma_i$. Then F_0 is a compact, connected set contained in D, and there exists a constant r_3 such that $\operatorname{dist}(F_0, \partial D) > 2r_3$. For every $i = 1, \ldots, n-1$ we define a "tube" around Γ_i by $\beta_i = \{x : \operatorname{dist}(x, \Gamma_i) < r_3\}$. Then for every i, β_i is also contained in D and connects U_i to U_{i+1}. Obviously, $r_3 < r_2/2$. Taking $r_3 = r_3(D, \delta)$ sufficiently small we may assume that $m(\bigcup_{i=1}^{n-1} \beta_i) < \delta/2$. For $r \in [0, r_3]$ let

$$U_r = \Big(\bigcup_{i=1}^{n} U_i \cup \bigcup_{i=1}^{n-1} \beta_i\Big) \cap D_r.$$

Then it is easily seen that U_r and F_0 together with the constants r_1, r_2, r_3 satisfy the conditions of the lemma. ∎

LEMMA 3.4. *For $r \in [0, R_0]$ let U_r be a domain contained in D_r with $\partial D_r \subset \partial U_r$. Let $0 < \varepsilon < 1/4$. There exists a $\delta = \delta(\varepsilon, b, D) > 0$ such that if $m(U_r) < \delta$ for every r, then*

$$\sup_{(x,z)\in U_r\times\partial D_r} \mathbb{E}_{z,r}^x \left| M(\tau_{U_r}) - \frac{1}{2}\langle M\rangle_{\tau_{U_r}} \right| < \varepsilon$$

and

$$\sup_{(x,z)\in U_r\times\partial D_r} \mathbb{E}_{z,r}^x N(\tau_{U_r}) < \frac{1}{1-4\varepsilon}.$$

Proof. We follow the proof of [CrZ, Lemma 3.2] to obtain the first estimate. The desired conclusion is then a consequence of Lemma 2.4 and the assumptions on b. The second estimate follows from the first one by the John–Nirenberg inequality. ∎

For $\varepsilon_0 = 1/8$, we fix $\delta_0 = \delta_0(\varepsilon_0, b, D) > 0$ satisfying the conditions of Lemma 3.4. From now on, R_1, R_2, R_3 will denote the constants and F_0, U_r the sets from Lemma 3.3 with respect to δ_0. Then for all $r \in [0, R_3]$ we have $\partial D_r \subset \partial U_r$ and $m(U_r) < \delta_0$, so by Lemma 3.4 it follows that

$$\sup_{(x,z)\in U_r\times\partial D_r} \mathbb{E}_{z,r}^x \left| M(\tau_{U_r}) - \frac{1}{2}\langle M\rangle_{\tau_{U_r}} \right| < \frac{1}{8} \tag{13}$$

and

$$\sup_{(x,z)\in U_r\times\partial D_r} \mathbb{E}^x_{z,r} N(\tau_{U_r}) < 2. \tag{14}$$

Since D_r satisfies the ball condition with radius R_0 and since $R_2 < R_0$, for every $r \in [0, R_2)$ and $z \in \partial D_r$ there exists a unique point $x_z \in \partial D_{R_2}$ for which

$$\delta_{D_r}(x_z) = |z - x_z| = R_2 - r.$$

LEMMA 3.5. *There exists a constant $c < 1$ depending only on D and b such that for all $r \in [0, R_3]$, $z \in \partial D_r$ and $x \in B(x_z, (R_2 - r)/2)$ we have $\mathbb{P}^x_{z,r}(\tau_{U_r} < \tau_{D_r}) \le c$.*

Proof. Let $r \in [0, R_3]$, $z \in \partial D_r$ and denote $B_{z,r} = B(z, R_1 - r) \cap D_r$. By Lemma 3.3, $B_{z,r} \subset U_r$. By (12), for every $x \in B(x_z, (R_2 - r)/2) \subset B_{z,r}$,

$$\begin{aligned}
\mathbb{P}^x_{z,r}(\tau_{U_r} < \tau_{D_r}) &\le \mathbb{P}^x_{z,r}(\tau_{B_{z,r}} < \tau_{D_r}) \\
&= \frac{1}{K_r(x,z)} \mathbb{E}^x[K_r(W_{\tau_{B_{z,r}}}, z);\, \tau_{B_{z,r}} < \tau_{D_r}] \\
&\le C_1^2 \cdot \frac{|x-z|^d}{\delta_{D_r}(x)} \mathbb{E}^x\left[\frac{\delta_{D_r}(W_{\tau_{B_{z,r}}})}{|W_{\tau_{B_{z,r}}} - z|^d};\, \tau_{B_{z,r}} < \tau_{D_r}\right].
\end{aligned}$$

Since $2R_2 < R_1$, the last term is less than

$$\begin{aligned}
C_1^2 \cdot \frac{2(2(R_2 - r))^d}{R_2 - r} \cdot \frac{R_1 - r}{(R_1 - r)^d} &= C_1^2 2^{d+1} \cdot \left(\frac{R_2 - r}{R_1 - r}\right)^{d-1} \\
&\le C_1^2 2^{d+1} \cdot \left(\frac{R_2}{R_1}\right)^{d-1} < 1,
\end{aligned}$$

where the last inequality follows from Lemma 3.3 too. ■

LEMMA 3.6. *There exists a constant $c = c(b, D)$ such that for every $r \in [0, R_3]$ we have*

$$\inf_{(x,z)\in\partial D_{R_2}\times\partial D_r} \mathbb{P}^x_{z,r}(\tau_{U_r} = \tau_{D_r}) \ge c.$$

Proof. Let $\rho = R_3/12$. Since F_0 is compact, by Vitali's covering lemma, from the family of balls $\{B(y,\rho)\}_{y\in F_0}$ we may choose a disjoint, finite subcover $\{B(y_i,\rho)\}_{i\in I}$ such that $F_0 \subset \bigcup_{i\in I} B(y_i, 3\rho)$. Fix $r \in [0, R_3]$, $z \in \partial D_r$ and $x \in \partial D_{R_2}$. Choose $x_z \in \partial D_{R_2}$ such that

$$\delta_{D_r}(x_z) = |z - x_z| = R_2 - r.$$

Since F_0 is also connected, we may choose a set of balls

$$\{B(y_1, 3\rho), \ldots, B(y_m, 3\rho)\} \subset \{B(y_i, 3\rho)\}_{i\in I}$$

such that $x \in B(y_1, 3\rho)$, $x_z \in B(y_m, 3\rho)$ and

$$B(y_k, 3\rho) \cap B(y_{k+1}, 3\rho) \ne \emptyset$$

for every $k = 1, \ldots, m-1$. Let

$$\tau_k = \inf\{t \ge 0 : W_t \notin B(y_k, 4\rho)\}, \qquad T_k = \inf\Big\{t \ge 0 : W_t \notin \bigcup_{i=1}^{k} B(y_i, 4\rho)\Big\},$$

and for $k = 1, \ldots, m-1$ set

$$A_k = B(y_{k+1}, 3\rho) \cap \partial B(y_k, 4\rho).$$

Let $S_k \in \partial B(y_k, 4\rho)$ be the closest point to y_{k+1}. Then $|S_k - y_{k+1}| \le 2\rho$ and

$$B(S_k, \rho) \cap \partial B(y_k, 4\rho) \subset B(y_{k+1}, 3\rho).$$

Furthermore,

$$\sigma(A_k) \ge \sigma(B(S_k, \rho) \cap \partial B(y_k, 4\rho)) = c_1 = c_1(\rho, d) > 0,$$

where σ is the Hausdorff surface measure on $\partial B(y_k, 4\rho)$. By Lemma 3.3,

$$\bigcup_{k=1}^{m} B(y_k, 8\rho) \subseteq U_r \subseteq D_r,$$

and by Lemma 3.2 (with $4\rho, 3\rho$ instead of ρ, ρ'), there exists a constant $c_2 = c_2(D)$ such that for every $k \in \{1, \ldots, m-1\}$ and $y \in B(y_k, 3\rho)$,

$$\mathbb{P}^y_{z,r}(W_{\tau_k} \in A_k) \ge c_2 \rho \sigma(A_k) \ge c_1 c_2 \rho.$$

In particular, $\mathbb{P}^x_{z,r}(W_{\tau_1} \in A_1) \ge c_1 c_2 \rho$, and by the strong Markov property, for every $k \in \{2, \ldots, m-1\}$ we have

$$\begin{aligned}
\mathbb{P}^x_{z,r}(W_{T_k} \in A_k) &= \mathbb{E}^x_{z,r}[\mathbb{P}^{W_{T_{k-1}}}_{z,r}(W_{T_k} \in A_k)] \\
&\ge \mathbb{E}^x_{z,r}[\mathbb{P}^{W_{T_{k-1}}}_{z,r}(W_{\tau_k} \in A_k);\, W_{T_{k-1}} \in A_{k-1}] \ge c_1 c_2 \rho \mathbb{P}^x_{z,r}(W_{T_{k-1}} \in A_{k-1}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{P}^x_{z,r}(W_{T_{m-1}} \in B(x_z, 6\rho)) &\ge \mathbb{P}^x_{z,r}(W_{T_{m-1}} \in B(y_m, 3\rho)) \\
&\ge \mathbb{P}^x_{z,r}(W_{T_{m-1}} \in A_{m-1}) \ge (c_1 c_2 \rho)^{m-1}.
\end{aligned}$$

Observe that

$$\frac{R_2 - r}{2} \ge \frac{R_2 - R_3}{2} \ge \frac{R_2}{4} \ge \frac{R_3}{2} = 6\rho,$$

so $B(x_z, 6\rho) \subseteq B(x_z, (R_2 - r)/2)$. By Lemma 3.5, there exists a constant $c_3 < 1$ depending only on D and b such that for every $y \in B(x_z, 6\rho)$,

$$\mathbb{P}^y_{z,r}(\tau_{U_r} = \tau_{D_r}) = 1 - \mathbb{P}^y_{z,r}(\tau_{U_r} < \tau_{D_r}) \ge 1 - c_3.$$

Hence,

$$\begin{aligned}
\mathbb{P}^x_{z,r}(\tau_{U_r} = \tau_{D_r}) &= \mathbb{E}^x_{z,r}[\mathbb{P}^{W_{T_{m-1}}}_{z,r}(\tau_{U_r} = \tau_{D_r})] \\
&\ge \mathbb{E}^x_{z,r}[\mathbb{P}^{W_{T_{m-1}}}_{z,r}(\tau_{U_r} = \tau_{D_r});\, W_{T_{m-1}} \in B(x_z, 6\rho)] \ge (1 - c_3)(c_1 c_2 \rho)^{m-1}. \ \blacksquare
\end{aligned}$$

LEMMA 3.7. *There exist a constant $c = c(b, D)$ such that for every $r \in [0, R_3]$, $z \in \partial D_r$ and $x \in \partial D_{R_2}$ we have*

$$c^{-1} \leq \mathbb{E}^x_{z,r}[N(\tau_{D_r});\, \tau_{U_r} = \tau_{D_r}] \leq c.$$

Proof. The lower bound of the estimate follows from Lemma 3.6, the Jensen inequality for the conditional expectation and (13), exactly as in the proof of [CrZ, Lemma 3.3]. The upper bound follows immediately from (14). ∎

We are now ready to prove Theorem 3.1. Basically we follow the proof of [CrZ, Theorem 3.1], but to exhibit the uniformity of the estimate we present the details.

Proof of Theorem 3.1. Fix $r \in [0, R_3]$. Define a sequence of stopping times as follows:

$$T_0 = 0, \quad T_{2n-1} = T_{2n-2} + \tau_{D_{R_2}} \circ \theta_{T_{2n-2}},$$
$$T_{2n} = T_{2n-1} + \tau_{U_r} \circ \theta_{T_{2n-1}}, \quad n \geq 1.$$

Since $\mathbb{P}^x_{z,r}(\tau_{D_r} < \infty) = 1$ for every $x \in D_r$, we see that a.s. there exists n such that $T_{2n} = \tau_{D_r}$. By the strong Markov property, for all $x \in D_r$,

$$\begin{aligned}\mathbb{E}^x_{z,r} N(\tau_{D_r}) &= \sum_{n=1}^{\infty} \mathbb{E}^x_{z,r}[N(\tau_{D_r});\, T_{2n-2} < \tau_{D_r},\, T_{2n} = \tau_{D_r}] \\ &= \sum_{n=1}^{\infty} \mathbb{E}^x_{z,r}\{N(T_{2n-1})\mathbb{E}^{W_{T_{2n-1}}}_{z,r}[N(\tau_{D_r});\, \tau_{U_r} = \tau_{D_r}];\, T_{2n-2} < \tau_{D_r}\}.\end{aligned}$$

On $\{T_{2n-2} < \tau_{D_r}\}$ we have $W_{T_{2n-1}} \in \partial D_{R_2}$; by Lemma 3.7, for some constant $c_1 = c_1(D, b)$ and all $x \in D_r$ we have

$$\begin{aligned}c_1^{-1} \sum_{n=1}^{\infty} \mathbb{E}^x_{z,r}[N(T_{2n-1});\, T_{2n-2} < \tau_{D_r}] &\leq \mathbb{E}^x_{z,r} N(\tau_{D_r}) \\ &\leq c_1 \sum_{n=1}^{\infty} \mathbb{E}^x_{z,r}[N(T_{2n-1});\, T_{2n-2} < \tau_{D_r}].\end{aligned}$$

By the definition of $\mathbb{P}^x_{z,r}$,

$$\begin{aligned}&\mathbb{E}^x_{z,r}[N(T_{2n-1});\, T_{2n-2} < \tau_{D_r}] \\ &\qquad = \frac{1}{K_r(x, z)} \mathbb{E}^x_r[K_r(W_{T_{2n-1}}, z) N(T_{2n-1});\, T_{2n-2} < \tau_{D_r}].\end{aligned}$$

Since D is bounded and

$$\mathrm{dist}(\overline{D}_{R_2}, \partial D_r) = R_2 - r \geq 2R_3 - r \geq R_3,$$

by (12) there exists a constant $c_2 = c_2(D,b)$ such that

$$c_2^{-1} \le \frac{K_r(y,z)}{K_r(x,z)} \le c_2$$

for all $x \in \overline{D}_{R_2}$, $y \in \partial D_{R_2}$ and $z \in \partial D_r$. We will now focus on $x \in \overline{D}_{R_2}$. We conclude that

$$\begin{aligned} c_2^{-1}\mathbb{E}_r^x[N(T_{2n-1});\, T_{2n-2} < \tau_{D_r}] &\le \mathbb{E}_{z,r}^x[N(T_{2n-1});\, T_{2n-2} < \tau_{D_r}] \\ &\le c_2\mathbb{E}_r^x[N(T_{2n-1});\, T_{2n-2} < \tau_{D_r}], \end{aligned}$$

and so

$$\begin{aligned} (c_1c_2)^{-1}\sum_{n=1}^{\infty}\mathbb{E}_r^x[N(T_{2n-1});\, T_{2n-2} < \tau_{D_r}] &\le \mathbb{E}_{z,r}^x N(\tau_{D_r}) \\ &\le c_1c_2\sum_{n=1}^{\infty}\mathbb{E}_r^x[N(T_{2n-1});\, T_{2n-2} < \tau_{D_r}]. \end{aligned}$$

This gives

$$\sup_{z\in\partial D_r}\mathbb{E}_{z,r}^x N(\tau_{D_r}) \le (c_1c_2)^2 \inf_{z\in\partial D_r}\mathbb{E}_{z,r}^x N(\tau_{D_r}).$$

Furthermore, by (10) we have

$$\inf_{z\in\partial D_r}\mathbb{E}_{z,r}^x N(\tau_{D_r}) \le 1 \le \sup_{z\in\partial D_r}\mathbb{E}_{z,r}^x N(\tau_{D_r}).$$

Thus, for every $z \in \partial D_r$ and $x \in \overline{D}_{R_2}$,

$$(c_1c_2)^{-2} \le \mathbb{E}_{z,r}^x N(\tau_{D_r}) \le (c_1c_2)^2.$$

If $x \in D_r \setminus \overline{D}_{R_2}$, then $x \in U_r$. By the strong Markov property

$$\begin{aligned} \mathbb{E}_{z,r}^x N(\tau_{D_r}) &= \mathbb{E}_{z,r}^x[N(\tau_{U_r});\, \tau_{U_r} = \tau_{D_r}] \\ &\quad + \mathbb{E}_{z,r}^x[N(\tau_{U_r})\mathbb{E}_{z,r}^{W_{\tau_{U_r}}} N(\tau_{D_r});\, \tau_{U_r} < \tau_{D_r}]. \end{aligned}$$

Since $W_{\tau_{U_r}} \in D_{R_2}$ on $\{\tau_{U_r} < \tau_{D_r}\}$, we have

$$\begin{aligned} \mathbb{E}_{z,r}^x N(\tau_{D_r}) &\le \mathbb{E}_{z,r}^x[N(\tau_{U_r});\, \tau_{U_r} = \tau_{D_r}] + (c_1c_2)^2\mathbb{E}_{z,r}^x[N(\tau_{U_r});\, \tau_{U_r} < \tau_{D_r}] \\ &\le (c_1c_2)^2\mathbb{E}_{z,r}^x N(\tau_{U_r}) < 2(c_1c_2)^2, \end{aligned}$$

where the last inequality follows from (14). On the other hand,

$$\begin{aligned} \mathbb{E}_{z,r}^x N(\tau_{D_r}) &\ge \mathbb{E}_{z,r}^x[N(\tau_{U_r});\, \tau_{U_r} = \tau_{D_r}] + (c_1c_2)^{-2}\mathbb{E}_{z,r}^x[N(\tau_{U_r});\, \tau_{U_r} < \tau_{D_r}] \\ &\ge (c_1c_2)^{-2}\mathbb{E}_{z,r}^x N(\tau_{U_r}). \end{aligned}$$

Using Jensen's inequality and (13) we obtain

$$\mathbb{E}_{z,r}^x N(\tau_{D_r}) \ge (c_1c_2)^{-2}\exp\left[-\mathbb{E}_{z,r}^x\left|M(\tau_{U_r}) - \frac{1}{2}\langle M\rangle_{\tau_{U_r}}\right|\right] \ge (c_1c_2)^{-2}e^{-1/8},$$

and the proof of the theorem is complete. ∎

Corollary 3.8. *There exists a constant C_2 depending only on D and b such that for all $r \in [0, R_3]$, $x \in D_r$ and $z \in \partial D_r$ we have*

$$C_2^{-1} \frac{\delta_{D_r}(x)}{|x-z|^d} \le P_r^L(x,z) \le C_2 \frac{\delta_{D_r}(x)}{|x-z|^d}. \tag{15}$$

Proof. This estimate is an immediate consequence of (4), (9) and Theorem 3.1. ∎

Remark 3.9. The estimate in Theorem 3.1 can also be proved for $z \in D_r$ using similar techniques and following the methods of Cranston, Fabes and Zhao [CrFZ] (see also [ChuZ, Extended Conditional Gauge Theorem]), but the proof is more complicated in this case. By (8) we then obtain the same estimate for the Green function G_r^L as for G_r in (3). This result can also be deduced from [KiS1], where the authors have shown the comparability of the Green functions of $\frac{1}{2}\Delta$ and $\frac{1}{2}\Delta + \mu(\cdot)\cdot\nabla$ as a consequence of heat kernel estimates. Here μ is a vector-valued signed measure, which can be slightly more singular than the drift in [CrZ] and in the present paper. However, our approach to this subject is different and may be of independent interest.

4. Hardy spaces. In this section we will define the L-harmonic Hardy spaces $h_L^p(D)$ and we will prove Theorem 1.1. We denote by $\mathcal{M}(\partial D)$ the set of finite complex Borel measures on ∂D and for $\mu \in \mathcal{M}(\partial D)$ let $\|\mu\|$ be the total variation norm of μ. For $1 \le p < \infty$ and a Borel function f on ∂D let

$$\|f\|_p = \Big(\int_{\partial D} |f(x)|^p \, d\sigma(x) \Big)^{1/p},$$

and let $\|f\|_\infty$ denote the essential supremum norm on ∂D with respect to σ. For $f \in L^1(\partial D, \sigma)$ and $\mu \in \mathcal{M}(\partial D)$, the *Poisson integrals* of f and μ are defined by

$$P_L[f](x) = \int_{\partial D} P_0^L(x,y) f(y)\, d\sigma(y), \qquad P_L[\mu](x) = \int_{\partial D} P_0^L(x,y)\, d\mu(y).$$

Using (7) and the strong Markov property, one can easily prove that $P_L[f]$ and $P_L[\mu]$ are L-harmonic on D.

For $1 \le p \le \infty$ we define the *Hardy space* $h_L^p(D)$ to be the class of functions u L-harmonic on D for which

$$\|u\|_{h^p} = \sup_{0<r<R_3} \|u \circ \pi_r\|_p < \infty,$$

where π_r is the projection defined after Lemma 2.1 and R_3 was introduced after Lemma 3.4. Since $\sigma(\partial D) < \infty$, we have $h_L^p(D) \subset h_L^1(D)$ for all $p > 1$. We also note that $h_L^\infty(D)$ consists of the functions L-harmonic and bounded on D, and

$$\|u\|_{h^\infty} = \sup_{x\in D} |u(x)|.$$

From (6) and (9) we conclude that the Martin kernel of L for D_r satisfies

$$K_r^L(x,z) = \frac{\mathbb{E}_{z,r}^x N(\tau_{D_r})}{\mathbb{E}_{z,r}^{x_0} N(\tau_{D_r})} K_r(x,z) = \frac{P_r^L(x,z)}{P_r^L(x_0,z)}. \tag{16}$$

As stated in [CrZ], every positive L-harmonic function u on D has the unique Martin representation

$$u(x) = \int_{\partial D} K_0^L(x,y)\, d\mu(y), \tag{17}$$

for some positive measure $\mu \in \mathcal{M}(\partial D)$. Hence we conclude that $u = P_L[\widetilde{\mu}]$, where $d\widetilde{\mu} = d\mu / P_r^L(x_0, \cdot)$.

We will now discuss the corresponding version of the relative Fatou theorem. In the classical case, this theorem was proved in [Do1] for positive harmonic functions on the ball in $\mathbb{R}^d$, and in [Wu] on Lipschitz domains. Analogous results for the fractional Laplacian were obtained in [BD] for $C^{1,1}$ domains and in [MR2] for Lipschitz domains. For each $z \in \partial D$ and $\alpha > 0$ we define the "cone" of aperture α and vertex z by

$$\Gamma_\alpha(z) = \{x \in D : |x - z| < (1+\alpha)\delta_D(x)\}.$$

We say that a function u on D has a *nontangential limit* l at $z \in \partial D$ if, for every $\alpha > 0$, $u(x) \to l$ as $x \to z$ within $\Gamma_\alpha(z)$.

PROPOSITION 4.1 (Relative Fatou theorem). *Let u, v be two positive L-harmonic functions on D, and let μ, $\nu \in \mathcal{M}(\partial D)$ be positive measures such that $u = P_L[\mu]$ and $v = P_L[\nu]$. Then u/v has a finite nontangential limit at ν-almost every point of ∂D. This limit is ν-almost everywhere equal to the Radon–Nikodym derivative of the absolutely continuous component of μ with respect to ν.*

Proof. Following [Wu] it is enough to show that for every $z \in \partial D$ there is a constant c_1 depending on z, α and b such that

$$\limsup_{\Gamma_\alpha(z) \ni x \to z} \frac{u(x)}{v(x)} \le c_1 \lim_{\rho \to 0} \frac{\mu(\nabla(z,\rho))}{\nu(\nabla(z,\rho))}, \tag{18}$$

where $\nabla(z,\rho) = \partial D \cap B(z,\rho)$. Denote $u'(x) = \int_{\partial D} P_0(x,z)\, d\mu(z)$, $v'(x) = \int_{\partial D} P_0(x,z)\, d\nu(z)$, where $P_0(x,z)$ is the Poisson kernel of $\frac{1}{2}\Delta$ for D. Then by (11) we have

$$c_2^{-1} \frac{u'(x)}{v'(x)} \le \frac{u(x)}{v(x)} \le c_2 \frac{u'(x)}{v'(x)},$$

where c_2 depends only on D and b. As stated in [Wu], the inequality (18) holds with u'/v' instead of u/v, and thus the proposition is proved. ∎

As we mentioned in the Introduction, in the proof of Theorem 1.1 we will use the methods introduced in [MR1]. We start with the following estimate of the Poisson kernel.

Lemma 4.2. *There exists a constant $c = c(D,b)$ such that for all $r \in (0,R_3]$, $s \in [0,r)$ and $z, w \in \partial D_s$ we have*

$$c^{-1} P_s^L(\pi_{s,r}(w), z) \le P_s^L(\pi_{s,r}(z), w) \le c P_s^L(\pi_{s,r}(w), z).$$

Proof. By (15), it is enough to show that

$$c^{-1} \le \frac{|\pi_{s,r}(z) - w|}{|\pi_{s,r}(w) - z|} \le c.$$

Since for every $r \in [0,R_3]$, D_r satisfies the ball condition with the same constant R_0 (Lemma 2.1), the proof is the same as in the case of the analogous estimate of the Martin kernel in [MR1, Lemma 6]. ∎

Lemma 4.3. *Let $u = P_L[\mu]$ for some positive $\mu \in \mathcal{M}(\partial D)$. There exists a constant $c = c(D,b)$ such that for all $r, s \in (0,R_3]$ we have*

$$c^{-1}\|u \circ \pi_r\|_1 \le \|\mu\| \le c\|u \circ \pi_s\|_1.$$

Proof. By Lemma 4.2, for $r \in (0,R_3]$ we have

$$\begin{aligned}\|u \circ \pi_r\|_1 &= \int_{\partial D}\int_{\partial D} P_0^L(\pi_r(x), y)\, d\mu(y)\, d\sigma(x) \\ &\le c \int_{\partial D}\int_{\partial D} P_0^L(\pi_r(y), x)\, d\sigma(x)\, d\mu(y) = c\|\mu\|.\end{aligned}$$

On the other hand,

$$\|u \circ \pi_s\|_1 \ge c^{-1} \int_{\partial D}\int_{\partial D} P_0^L(\pi_s(y), x)\, d\sigma(x)\, d\mu(y) = c^{-1}\|\mu\|$$

for any $s \in (0,R_3]$. ∎

Let u be L-harmonic on D. For $r \in (0,R_3]$ we define

$$u_r(x) = \begin{cases} \int_{\partial D_r} P_r^L(x,y)|u(y)|\, d\sigma_r(y), & x \in D_r, \\ |u(x)|, & x \in D \setminus D_r. \end{cases}$$

It is clear that u_r is nonnegative on D and L-harmonic on D_r. For every $x \in D$ and r we have $|u(x)| \le u_r(x)$. Furthermore, for $s \in (0,r)$ and $x \in D_r$,

$$\begin{aligned} u_r(x) &= \int_{\partial D_r} P_r^L(x,y)|u(y)|\, d\sigma_r(y) \\ &\le \int_{\partial D_r} P_r^L(x,y)\Big(\int_{\partial D_s} P_s^L(y,z)|u(z)|\, d\sigma_s(z)\Big)\, d\sigma_r(y) \\ &= \int_{\partial D_s} P_s^L(x,z)|u(z)|\, d\sigma_s(z) = u_s(x).\end{aligned}$$

Hence, the limit

$$u^*(x) = \lim_{r \searrow 0} u_r(x)$$

exists and $|u(x)| \le u^*(x)$ for every $x \in D$. If u^* is finite, then monotone convergence implies that it is L-harmonic on D.

LEMMA 4.4. *Let u be L-harmonic on D. Then $u = P_L[\mu]$ for some $\mu \in \mathcal{M}(\partial D)$ if and only if there exists a positive L-harmonic function v on D such that $|u| \le v$. In fact, if $u = P_L[\mu]$, then $u^* = P_L[|\mu|]$.*

Proof. Using (7), (16), (17) and Proposition 4.1 we follow the proof of analogous properties in [MR1, Lemma 1 and Theorem 1]. ∎

A consequence of Lemma 4.4 is the following weaker version of Theorem 1.1.

LEMMA 4.5. *The following conditions are equivalent:*

(i) $u \in h^1_L(D)$.
(ii) $u^*(x) = \lim_{r\searrow 0} u_r(x)$ *is finite for every* x.
(iii) $u = P_L[\mu]$ *for some* $\mu \in \mathcal{M}(\partial D)$.

Proof. (i)⇒(ii). If $u \in h^1_L(D)$ then by the estimate (15) and Lemma 2.2 we have

$$\begin{aligned} u^*(x) &= \lim_{r\searrow 0} u_r(x) = \lim_{r\searrow 0} \int_{\partial D_r} P^L_r(x,y)|u(y)|\,d\sigma_r(y) \\ &\le \limsup_{r\searrow 0} \int_{\partial D_r} C_2 \frac{\delta_{D_r}(x)}{|x-y|^d}|u(y)|\,d\sigma_r(y) \le \frac{C_2 c}{(\delta_D(x))^{d-1}}\|u\|_{h^1}, \end{aligned}$$

where the constant c depends only on D.

(ii)⇒(iii). If u^* is finite, then it is L-harmonic and positive on D (or $u \equiv 0$). Since $|u(x)| \le u^*(x)$ on D, $u = P_L[\mu]$ by Lemma 4.4, for some $\mu \in \mathcal{M}(\partial D)$.

(iii)⇒(i). If $u = P_L[\mu]$, then by Lemma 4.4, $u^* = P_L[|\mu|]$. From Lemma 4.3 it follows that $u^* \in h^1_L(D)$, and thus $u \in h^1_L(D)$. ∎

LEMMA 4.6. *If $u \in h^1_L(D)$, then $u^* \in h^1_L(D)$. In fact, there is a constant $c = c(b, D)$ such that $\|u^*\|_{h^1} \le c\|u\|_{h^1}$.*

Proof. By the monotone convergence theorem, for $r \in (0, R_3]$ we have

$$\begin{aligned} \|u^* \circ \pi_r\|_1 &= \int_{\partial D} u^*(\pi_r(x))\,d\sigma(x) \\ &= \int_{\partial D} \lim_{s\searrow 0} u_s(\pi_r(x))\,d\sigma(x) = \lim_{s\searrow 0} \int_{\partial D} u_s(\pi_r(x))\,d\sigma(x). \end{aligned}$$

By (1) we have

$$\begin{aligned}\|u^*\circ\pi_r\|_1 &= \lim_{s\searrow 0}\int\limits_{\partial D}\Big(\int\limits_{\partial D_s} P_s^L(\pi_r(x),y)|u(y)|\,d\sigma_s(y)\Big)\,d\sigma(x)\\ &= \lim_{s\searrow 0}\int\limits_{\partial D_s}\Big(\int\limits_{\partial D} P_s^L(\pi_{s,r}(\pi_s(x)),y)\,d\sigma(x)\Big)|u(y)|\,d\sigma_s(y).\end{aligned}$$

By Lemma 4.2, for a constant $c_1 = c_1(b,D)$, the last term is less than

$$\begin{aligned} c_1\limsup_{s\searrow 0}&\int\limits_{\partial D_s}\Big(\int\limits_{\partial D} P_s^L(\pi_{s,r}(y),\pi_s(x))\,d\sigma(x)\Big)|u(y)|\,d\sigma_s(y)\\ &\le c_1c_2\limsup_{s\searrow 0}\int\limits_{\partial D_s}\Big(\int\limits_{\partial D_s} P_s^L(\pi_{s,r}(y),z)\,d\sigma_s(z)\Big)|u(y)|\,d\sigma_s(y)\\ &= c_1c_2\limsup_{s\searrow 0}\int\limits_{\partial D_s}|u(y)|\,d\sigma_s(y)\le c_1c_2^2\|u\|_{h^1},\end{aligned}$$

where the constant c_2, by Lemma 2.2, depends only on D. ∎

Proof of Theorem 1.1. By Lemma 4.5, if $u\in h^1_L(D)$ then $u = P_L[\mu]$ for some $\mu\in\mathcal{M}(\partial D)$ and, by Lemma 4.4, $u^* = P_L[|\mu|]$. Lemmas 4.3 and 4.6 give

$$\|\mu\|\le c_1\|u^*\|_{h^1}\le c_1c_2\|u\|_{h^1}\le c_1c_2\|u^*\|_{h^1}\le c_1^2c_2\|\mu\|.$$

This proves the first part of the theorem.

Now let $p>1$. If $u = P_L[f]$ for some $f\in L^p(\partial D,\sigma)$, then by Jensen's inequality, for $p<\infty$ we have

$$|u(x)|^p\le\int\limits_{\partial D}P_0^L(x,y)|f(y)|^p\,d\sigma(y) = P_L[|f|^p](x).$$

By Lemma 4.3 we obtain

$$\|u\|_{h^p}^p\le\|P_L[|f|^p]\|_{h^1}\le c_1\|f\|_p^p.$$

If $u = P_L[f]$ for some $f\in L^\infty(\partial D,\sigma)$, then

$$|u|\le P_L[|f|]\le\|f\|_\infty,$$

and thus $\|u\|_{h^\infty}\le\|f\|_\infty$.

Conversely, suppose that $u\in h^p_L(D)$. Then $u\in h^1_L(D)$ and $u=P_L[\mu]$ for some $\mu\in\mathcal{M}(\partial D)$. We may decompose μ into $d\mu = f\,d\sigma + d\mu_s$, where μ_s is singular with respect to σ. By Proposition 4.1, for σ-almost every $z\in\partial D$, $\lim_{r\to 0}u(\pi_r(z)) = f(z)$. By the Fatou lemma, for $p<\infty$,

$$\int\limits_{\partial D}|f(z)|^p\,d\sigma(z)\le\liminf_{r\to 0}\int\limits_{\partial D}|u(\pi_r(z))|^p\,d\sigma(z)\le\|u\|_{h^p}^p.$$

If $p=\infty$, then for σ-almost every $z\in\partial D$,

$$|f(z)| = \lim_{r\to 0}|u(\pi_r(z))|\le\limsup_{r\to 0}\|u\circ\pi_r\|_\infty\le\|u\|_{h^\infty}.$$

Therefore $f \in L^p(\partial D, \sigma)$ and $P_L[f] \in h_L^p(D)$. This implies that also $v = P_L[\mu_s] \in h_L^p(D)$. We will show that $\|\mu_s\| = 0$. The condition $v \in h_L^p(D)$ for $p > 1$ implies that the family $v(\pi_r(\cdot))$, $r > 0$, is uniformly integrable. By Proposition 4.1, $\lim_{r\to 0} v(\pi_r(z)) = 0$ σ-a.e. on ∂D, and using Egorov's theorem we obtain

$$\lim_{r\to 0} \int_{\partial D} |v(\pi_r(z))|\, d\sigma(z) = 0.$$

Finally, by Lemma 4.3 we have

$$\|\mu_s\| \le c_1 \liminf_{r\to 0} \int_{\partial D} |v(\pi_r(z))|\, d\sigma(z) = 0,$$

so the proof is complete. ∎

5. Schrödinger operators. In this section we extend the h^p-theory to q-harmonic functions, where $q \in K_d^{\mathrm{loc}}(D)$, i.e., $|q|$ is uniformly integrable with respect to the measures $\mu(x, dy) = |x - y|^{2-d} dy$, $x \in \overline{D}$. As in the previous sections, we only consider $d \ge 3$. We will say that a continuous function u on D is *q-harmonic* on D if $\frac{1}{2}\Delta u + qu \equiv 0$ in the weak sense. For standard properties of q-harmonic functions see [ChuZ] and [CrFZ]. Let, as before, W_t be the Brownian motion killed on exiting D_r, where $r \in [0, R_0]$ and R_0 is defined after Lemma 2.3. For $t \ge 0$ let $e_q(t) = \exp\{\int_0^t q(W_s)\, ds\}$. The *gauge* and *conditional gauge* for D_r are defined, respectively, as

$$\begin{aligned} g_r(x) &= \mathbb{E}_r^x e_q(\tau_{D_r}), && x \in D_r, \\ g_r(x, z) &= \mathbb{E}_{z,r}^x e_q(\tau_{D_r}), && x \in D_r,\ z \in \partial D_r. \end{aligned}$$

Fix $r \in [0, R_0]$. Suppose that $g_r(x_0, z_0) < \infty$ for some $(x_0, z_0) \in D_r \times \partial D_r$, or, equivalently, that (D_r, q) is *gaugeable* (i.e. $g_r(x)$ is bounded in D_r, see [ChuZ]). Then by the Conditional Gauge Theorem, there exists a constant c depending on D_r and q such that

$$c^{-1} \le g_r(x, z) \le c.$$

By the Feynman–Kac formula, the q-Poisson kernel for D_r is given by

$$P_r^q(x, z) = g_r(x, z) P_r(x, z), \qquad x \in D_r,\ z \in \partial D_r. \tag{19}$$

The main tool of this section is the following uniform version of the Conditional Gauge Theorem.

THEOREM 5.1. *Suppose that (D, q) is gaugeable. Then there exist constants c, r_0 depending only on D and q such that for all $r \in [0, r_0]$, $x \in D_r$ and $z \in \partial D_r$ we have*

$$c^{-1} \le g_r(x, z) \le c.$$

Proof. Exactly as in [ChuZ, Lemma 7.1] it follows that for any open subset C_r of D_r and every $(x,z) \in D_r \times \partial D_r$ we have

$$\mathbb{E}^x_{z,r} \int_0^{\tau_{C_r}} |q(W_s)|\, ds = \int_{C_r} \frac{G_r(x,y)|q(y)|K_r(y,z)}{K_r(x,z)}\, dy.$$

Invoking the assumptions on q and using Lemma 2.4 we conclude that there exists $\delta = \delta(q, D) > 0$ such that if $m(C_r) < \delta$ then

$$\sup_{(x,z)\in D_r\times\partial D_r} \mathbb{E}^x_{z,r} \int_0^{\tau_{C_r}} |q(W_s)|\, ds < \frac{1}{2} \tag{20}$$

for every $r \in [0, R_0]$. By Khas'minskii's lemma ([ChuZ, Lemma 3.7]) we have

$$\sup_{(x,z)\in D_r\times\partial D_r} \mathbb{E}^x_{z,r} e_{|q|}(\tau_{C_r}) < 2. \tag{21}$$

Let r_1, r_2, r_3 be the constants and F_0, U_r the sets from Lemma 3.3 for the given δ. Then for every $r \in [0, r_3]$ we have $\partial D_r \subset \partial U_r$, $m(U_r) < \delta$, and in the same way as in Lemma 3.6 we find that there exists a constant $c_1 = c_1(D, q)$ such that for every $r \in [0, r_3]$,

$$\inf_{(x,z)\in\partial D_{r_2}\times\partial D_r} \mathbb{P}^x_{z,r}(\tau_{U_r} = \tau_{D_r}) \ge c_1.$$

Consequently, in the same manner as in Lemma 3.7 (using (20), (21) instead of (13), (14)) we conclude that there is $c_2 = c_2(D, q)$ such that for every $r \in [0, r_3]$, $z \in \partial D_r$ and $x \in \partial D_{r_2}$ we have

$$c_2^{-1} \le \mathbb{E}^x_{z,r}[e_q(\tau_{D_r});\ \tau_{U_r} = \tau_{D_r}] \le c_2,$$

and following the proof of Theorem 3.1 we obtain

$$\sup_{z\in\partial D_r} g_r(x,z) \le c_3 \inf_{z\in\partial D_r} g_r(x,z)$$

for every $r \in [0, r_3]$ and $x \in \overline{D}_{r_2}$, where $c_3 = c_3(D, q)$. By [ChuZ, Proposition 5.12] we have

$$g_r(x) = \int_{\partial D_r} g_r(x,z) P_r(x,z)\, d\sigma_r(z)$$

and so

$$\inf_{z\in\partial D_r} g_r(x,z) \le g_r(x) \le \sup_{z\in\partial D_r} g_r(x,z).$$

Since (D, q) is gaugeable, by the Conditional Gauge Theorem

$$c_4^{-1} \le g_0(x) \le c_4,$$

where $c_4 = c_4(D, q)$. On the other hand, by the strong Markov property

$$g_0(x) = \mathbb{E}^x_0 e_q(\tau_D) = \mathbb{E}^x_0[e_q(\tau_{D_r}) g_0(X_{\tau_{D_r}})].$$

Hence

$$c_4^{-1} g_r(x) \le g_0(x) \le c_4 g_r(x),$$

which implies that

$$c_4^{-2} \le g_r(x) \le c_4^2$$

for every $r \in [0, R_0]$ and $x \in D_r$. Therefore

$$\frac{1}{c_3 c_4^2} \le g_r(x,z) \le c_3 c_4^2$$

for every $r \in [0, r_3]$ and $(x,z) \in \overline{D}_{r_2} \times \partial D_r$. For $x \in D \setminus \overline{D}_{r_2}$ we follow the proof of Theorem 3.1, replacing $N(\tau_{D_r})$ with $e_q(\tau_{D_r})$. ∎

An immediate consequence of (4), (19) and Theorem 5.1 is the following estimate.

COROLLARY 5.2. *Suppose that (D,q) is gaugeable. There exist constants c, r_0 depending only on D and q such that for all $r \in [0, r_0]$, $x \in D_r$ and $z \in \partial D_r$ we have*

$$c^{-1} \frac{\delta_{D_r}(x)}{|x-z|^d} \le P_r^q(x,z) \le c \frac{\delta_{D_r}(x)}{|x-z|^d}.$$

Just as in Section 4, we now define the q-harmonic Hardy spaces $h_q^p(D)$. Using the basic facts showed in [ChuZ] and [CrFZ] for q-harmonic functions and the methods from Section 4 we obtain the following theorem, which is the main result of this section.

THEOREM 5.3. *Suppose that (D,q) is gaugeable and let u be q-harmonic on D. Then*

(i) *$u \in h_q^1(D)$ if and only if $u = P_q[\mu]$ with a unique $\mu \in \mathcal{M}(\partial D)$. Furthermore, there exists a constant c_1 depending on D and q such that, for every $\mu \in \mathcal{M}(\partial D)$,*

$$c_1^{-1} \|\mu\| \le \|P_q[\mu]\|_{h^1} \le c_1 \|\mu\|.$$

(ii) *$u \in h_q^p(D)$ for a given $p \in (1, \infty]$ if and only if $u = P_q[f]$ with a unique $f \in L^p(\partial D, \sigma)$. Furthermore, there exists a constant c_2 depending on D and q such that, for every $f \in L^p(\partial D, \sigma)$,*

$$c_2^{-1} \|f\|_p \le \|P_q[f]\|_{h^p} \le c_2 \|f\|_p.$$

REMARK 5.4. Let $M = L - \mu$, where $\mu \in K_d^{\mathrm{loc}}(D)$ is a signed Borel measure, and for r sufficiently small let $\mathbb{E}_{z,r}^x e_\mu(\tau_{D_r})$ be the conditional gauge for D_r with respect to the L-diffusion X_t (see [Ch]). Suppose that for some $(x_0, z_0) \in D_r \times \partial D_r$ we have $\mathbb{E}_{z_0,r}^{x_0} e_\mu(\tau_{D_r}) < \infty$. Then by [CrZ, Theorem 3.15],

$$P_r^M(x,z) = \mathbb{E}_{z,r}^x e_\mu(\tau_{D_r}) P_r^L(x,z), \quad x \in D_r,\ z \in \partial D_r,$$

where $P_r^M(x,dz)$ is the Poisson kernel of M for D_r. Furthermore, there exists a constant c depending on D_r, b and μ such that

$$c^{-1} \le \mathbb{E}_{z,r}^x e_\mu(\tau_{D_r}) \le c.$$

Using similar methods to those in Sections 3 and 5 we can prove that c does not depend on small r if (D,μ) is gaugeable with respect to X_t (see [Ch]); however, in this case we also need the uniform estimate of the Green functions G_r^L (see Remark 3.9). In view of [CrZ, Theorem 3.15], as in Section 4 one obtains analogous properties of M-harmonic Hardy spaces.

Remark 5.5. In the case of the operator $\mathcal{L}+\mu\cdot\nabla+\nu$ with μ,ν satisfying the conditions of [KiS2], and under the assumption of the gaugeability of (D,ν), the estimates obtained for the Green function are the same as for the Laplacian. Analyzing the proofs one can verify that the comparison constant depends only on the $C^{1,1}$-characteristics, the diameter of the domain and the Kato norms of μ and ν. By the argument given in [An] or [ChWZ], this also implies a uniform estimate for the harmonic measure, so one can extend the h^p-theory in this direction as well. The same concerns the operators discussed in [HS], [IR] and [R].

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