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The Banach lattice $C[0,1]$ is super d-rigid

by

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To Professor A. Pełczyński on the occasion of his 70th birthday

Abstract. The following properties of $C[0,1]$ are proved here. *Let $T : C[0,1] \to Y$ be a disjointness preserving bijection onto an arbitrary vector lattice Y. Then the inverse operator T^{-1} is also disjointness preserving, the operator T is regular, and the vector lattice Y is order isomorphic to $C[0,1]$. In particular if Y is a normed lattice, then T is also automatically norm continuous.* A major step needed for proving these properties is provided by Theorem 3.1 asserting that T satisfies some technical condition that is crucial in the study of operators preserving disjointness.

1. Introduction. The primary goal of this paper is to prove that the classical Banach lattice $C[0,1]$ satisfies the following two remarkable properties [1].

(1) For each disjointness preserving bijection T from $C[0,1]$ onto an *arbitrary* vector lattice Y the inverse operator $T^{-1} : Y \to C[0,1]$ is also disjointness preserving.

(2) Each operator T appearing in (1) is regular.

Later, in the comments following Definition 3.2, it will be explained that in fact property (2) implies (1). We refer to (1) by saying that $C[0,1]$ is *d-rigid*, and to (2) by saying that $C[0,1]$ is *super d-rigid*. If we replace $C[0,1]$ by a vector lattice X, then absolutely similarly we arrive at the definition of a (super) d-rigid vector lattice X; see Definition 3.2.

None of the classical Banach lattices $L_p[0,1]$, $p \geq 1$, is d-rigid; see [1, Theorem 13.14]. On the other hand, each of the discrete Banach lattices, in particular each ℓ_p, is super d-rigid. The latter fact, however, is in some sense trivial for the following reason: in each of the discrete vector lattices

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[1] The interval $[0,1]$ can, of course, be replaced by any interval $[a,b]$.

any two non-disjoint elements have non-zero proportional components (2), and under this condition the proof of the super d-rigidity becomes relatively simple (see Theorem 11.6 in [1] for details). The Banach lattice $C[0,1]$ is a first non-trivial example of a super d-rigid vector lattice.

From properties (1) and (2) it follows (see the discussion after Definition 3.2) that *every disjointness preserving bijection T from $C[0,1]$ onto an arbitrary vector lattice Y is automatically (r_u)-continuous, and that Y is order isomorphic to $C[0,1]$. Furthermore, if Y is a normed vector lattice, then T is also automatically norm continuous.*

The automatic regularity and/or continuity of operators is not a new topic in the framework of operators preserving disjointness. Without trying to be complete, we mention here only a few articles containing results on automatic continuity of such operators: [1, 4, 5, 7–10]. Most of these results have been proven for broad classes of domain vector lattices, but at the price of some restrictions on the image Y. We emphasize that as we will show, the domain $C[0,1]$ does not require any restrictions on Y.

Our proof that $C[0,1]$ is d-rigid depends heavily on Theorem 3.1, our first main result, asserting that *each disjointness preserving bijection $T : C[0,1] \to Y$ satisfies condition* $(\pitchfork)$ (see Definition 2.1 below). It is worth pointing out that in spite of the fact that there are no topological assumptions on the operator T in Theorem 3.1, the proof utilizes the completeness of the space and depends heavily on rather involved functional-analytical considerations. Combining Theorem 3.1 with Theorem 4.1 in [2] stating that *if $T : C[0,1] \to Y$ is a disjointness preserving bijection satisfying* $(\pitchfork)$, *then the operator T^{-1} is also disjointness preserving*, we deduce immediately in Theorem 3.3 that $C[0,1]$ is d-rigid. The remaining (more sophisticated) part of Theorem 3.3 establishes that $C[0,1]$ is super d-rigid.

The last section, Section 4, contains some further generalizations. For example, Corollary 4.4 describes a rather large class of d-rigid vector sublattices of $C[0,1]$. It should be pointed out, however, that this class is not as large as the class we erroneously proclaimed in [2, Theorem 4.5].

All necessary terminology and notations related to operators preserving disjointness can be found in [1, 2]. The general terminology regarding operators and vector lattices is standard and follows [6]. All vector lattices under consideration are assumed to be Archimedean.

2. Some lemmas. We begin by recalling the definition of property $(\pitchfork)$ that was introduced in [1].

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(2) In the case of the discrete vector lattices this condition is obvious, but there are also non-discrete Banach lattices satisfying this condition; for instance, $C(\beta\mathbb{N} \setminus \mathbb{N})$, see [1, p. 85].

2.1. Definition. A disjointness preserving operator $T : X \to Y$ between vector lattices satisfies *condition* $(\pitchfork)$ if for each $x \in X$ and for each band U in X the following implication holds:

$$Tx \perp TU \;\Rightarrow\; x \perp U.$$

We express the fact that T satisfies condition $(\pitchfork)$ by writing $T \in (\pitchfork)$.

The next lemma is a special case of Proposition 3.2 in [2].

2.2. Lemma. *For a disjointness preserving bijection $T : X \to Y$ the following two statements are equivalent.*

1) $T \in (\pitchfork)$.

2) *For each band U in X we have $\{TU\}^{dd} = TU$, that is, TU is a band in Y.*

A function $f \in C[a,b]$ is said to be *locally constant* at a point $t \in [a,b]$ if there exists an open neighborhood $V = (\alpha,\beta)$ of t such that f is constant on V, that is, $f(t') = f(t)$ for each $t' \in V$. We refer to the open (3) interval $V = (\alpha,\beta)$ as an *interval of constancy* of f. The union of all intervals of constancy of f will be denoted by $\mathrm{const}(f)$. Equivalently, $\mathrm{const}(f)$ is the open set of all those points in $[a,b]$ at which f is locally constant. If the set $\mathrm{const}(f)$ is dense in $[a,b]$, then f is known as a (continuous) *essentially constant* function.

The standard uniform norm on the space of continuous functions is denoted by $\|\cdot\|$ and, as usual, for each $f \in C[a,b]$ its support set, $\mathrm{supp}(f)$, is the closure of the set $\{t \in [a,b] : f(t) \neq 0\}$.

2.3. Lemma. *Consider a bounded interval $[a,b]$ in $\mathbb{R}$. Then for each $\varepsilon > 0$ and each $t_0 \in (a,b)$ there exist non-negative functions $F, G, H \in C[a,b]$ with the following properties*:

1) $F(a) = G(a) = H(a) = F(b) = G(b) = H(b) = 0$.

2) $\|F\| = \|G\| = \|H\| = F(t_0) = G(t_0) = H(t_0) = 1$.

3) $\|F - G\| + \|G - H\| \le \varepsilon$.

4) *Each of the functions F, G, H is essentially constant.*

5) *For any two of the functions F, G, H the union of their intervals of constancy is the whole interval $[a,b]$.*

Proof. Without loss of generality we can assume that $a = 0$, $b = 1$, and $t_0 = 1/2$. Let C be a Cantor set (i.e., a closed, nowhere dense subset of $[0,1]$ without isolated points) of measure zero and let f be a (continuous, increasing from 0 to 1) Cantor function associated with C, i.e., f is constant on each open interval complementary to C.

(3) If $\alpha = a$ or $\beta = b$, then, of course, we are talking about $[a,\beta)$ or $(\alpha,b]$, respectively. We adhere to the same agreement about the endpoints throughout the paper.

From a familiar description of a Cantor set C and the uniform continuity of f it follows easily that there exist a finite number of pairwise disjoint open intervals (a_k, b_k), $k = 1, \dots, n$, such that their closures are also pairwise disjoint, $C \subset \bigcup_{k=1}^{n}(a_k, b_k)$, and the oscillation of f on each interval (a_k, b_k) is less than $\varepsilon/3$.

Therefore we can easily find pairwise disjoint open intervals (a'_k, b'_k) such that $[a_k, b_k] \subset (a'_k, b'_k)$ and the oscillation of f on (a'_k, b'_k) is less than $\varepsilon/2$.

Next, we will construct a function $g \in C[0,1]$ satisfying the following four conditions:

(i) $g \equiv f$ on $[0,1] \setminus \bigcup_{k=1}^{n}(a'_k, b'_k)$.

(ii) $g \equiv f((a_k + b_k)/2)$ on $[a_k, b_k]$.

(iii) On each of the intervals $[a'_k, a_k]$ we define g to be a monotone essentially constant (Cantor like) function taking the values $f(a'_k)$ and $f((a_k + b_k)/2)$ at the endpoints, respectively.

(iv) On each of the intervals $[b_k, b'_k]$ we define g to be a monotone essentially constant (Cantor like) function taking the values $f((a_k + b_k)/2)$ and $f(b'_k)$ at the endpoints, respectively.

This definition guarantees that g is a continuous essentially constant function. Clearly $\|f - g\| \le \varepsilon/2$, $g(1) = 1$, and the union of the intervals of constancy of f and g is $[0,1]$.

Let B be the complement to the union of the intervals of constancy of g. Then $C \cup B$ is a nowhere dense closed subset of $[0,1]$ without isolated points, $C \cup B$ has measure zero and we can repeat the arguments above to produce a third continuous essentially constant function h such that the union of the intervals of constancy of h and f (and of h and g) is $[0,1]$.

Next, let us extend the functions f, g, and h from $[0,1]$ to $[0,2]$ by symmetry about the point 1, that is, for each $t \in [1,2]$ we simply let $f(t) = f(2-t)$, $g(t) = g(2-t)$, and $h(t) = h(2-t)$.

Finally, for $t \in [0,1]$ we define $F(t) = f(2t)$, $G(t) = g(2t)$, and $H(t) = h(2t)$. A straightforward verification shows that these functions are as required. ∎

2.4. Lemma. *Assume that a disjointness preserving bijection* $T : C[0,1] \to Y$ *onto a vector lattice does not satisfy* $(\pitchfork)$. *Then for any two sequences* $\{\varepsilon_n\}$ *and* $\{A_n\}$ *of positive scalars satisfying* $\varepsilon_n \searrow 0$ *and* $A_n \nearrow \infty$ *there exist pairwise disjoint intervals* $(a_n, b_n) \subset (0,1)$, *points* $t_n \in (a_n, b_n)$, *and non-negative functions* $f_n, g_n, h_n \in C[0,1]$ *such that*

1) $\operatorname{supp}(|f_n| + |g_n| + |h_n|) \subset (a_n, b_n)$.

2) $\max(\|f_n - g_n\|, \|f_n - h_n\|, \|g_n - h_n\|) \le \varepsilon_n$.

3) $f_n(t_n) = g_n(t_n) = h_n(t_n) = A_n$.

4) $\max(\|f_n\|, \|g_n\|, \|h_n\|) \le 2A_n$.

5) *For each n the elements Tf_n, Tg_n, and Th_n are pairwise disjoint in Y.*

Proof. Since the operator T does not satisfy condition ($\pitchfork$), a simple argument shows that we can find a function $u \in C[0,1]$ and a closed interval $[c,d] \subset [0,1]$ such that $u > 0$ on the whole interval $[c,d]$ and $Tu \perp Tv$ for each $v \in C[0,1]$ with $\operatorname{supp}(v) \subseteq [c,d]$. Reducing the size of the interval $[c,d]$ if necessary, we can assume additionally that $\max_{t\in[c,d]} u(t) \le 2\min_{t\in[c,d]} u(t)$. Fix any $\delta > 0$. By Lemma 2.3 there exist essentially constant functions $F, G, H \in C[0,1]$ with support in (c,d) and such that $\mathbf{0} \le F, G, H \le \mathbf{1}$, $\|F - G\| + \|G - H\| < \delta$, for any two of these functions the union of their intervals of constancy is $[0,1]$, and $F(t_0) = G(t_0) = H(t_0) = 1$, where $t_0 = (c+d)/2$.

Let $f = Fu$, $g = Gu$ and $h = Hu$. Our lemma will be proved if we establish that the elements Tf, Tg, and Th are pairwise disjoint in Y.

We will verify that $Tf \perp Tg$. Let x be an arbitrary function in $C[0,1]$. Since the intervals of constancy of F and G cover $[0,1]$, there exists a finite subcover consisting of these intervals. Therefore, using a partition of unity subordinate to this finite cover, we can find functions $x_i \in C[0,1]$ such that $x = x_1 + \ldots + x_m$ and the support of each x_i is contained in an interval of constancy of either F or G. In the first case we have $F \equiv c$ on $\operatorname{supp} x_i$ and so $cu - f \perp x_i$, implying that $T(cu - f) \perp Tx_i$. This guarantees that $Tf \perp Tx_i$ because $Tu \perp Tf$ in view of our condition on u and on the interval $[c,d]$. In the second case, we obtain $Tg \perp Tx_i$, and thus $|Tf| \wedge |Tg| \perp Tx_i$. This is true for each i and consequently $|Tf| \wedge |Tg| \perp Tx$. This guarantees that $Tf \perp Tg$ because $x \in C[0,1]$ is arbitrary and $T(C[0,1]) = Y$. Similarly one can verify that $Tf \perp Th$ and $Tg \perp Th$.

Finally, substituting for (c,d) a sequence of disjoint intervals (a_n, b_n) and letting $f_n = A_n f$, $g_n = A_n g$, and $h_n = A_n h$ we complete the proof. ∎

For each $a \in (0,1)$ we define the following two bands L_a and R_a in $C[0,1]$:

$$L_a = \{f \in C[0,1] : f \equiv 0 \text{ on } [a,1]\},$$
$$R_a = \{f \in C[0,1] : f \equiv 0 \text{ on } [0,a]\}.$$

2.5. Lemma. *If $T : C[0,1] \to Y$ is a disjointness preserving bijection onto a vector lattice, then for each $a \in (0,1)$ either TL_a or TR_a is necessarily a band in Y.*

Proof. Because $L_a \perp R_a$ we have $TL_a \perp TR_a$, and $\operatorname{codim}(L_a \oplus R_a) = 1$ implies $\operatorname{codim}(TL_a \oplus TR_a) = 1$. Therefore if, say, $TL_a \subsetneq \{TL_a\}^{dd}$, then it must be true that $TR_a = \{TR_a\}^{dd}$. ∎

Our next lemma provides more delicate information.

2.6. Lemma. *If $T : C[0,1] \to Y$ is a disjointness preserving bijection, then for each subinterval $[c,d]$ of $[0,1]$ there exists a non-empty open subinterval $(p,q) \subseteq [c,d]$ such that either $TL_t = \{TL_t\}^{dd}$ for each $t \in (p,q)$ or $TR_t = \{TR_t\}^{dd}$ for each $t \in (p,q)$.*

Proof. If $TL_t = \{TL_t\}^{dd}$ for each $t \in (c,d)$, then there is nothing to prove. So suppose that for some $a \in (c,d)$ we have $TL_a \neq \{TL_a\}^{dd}$. Fix $u \in C[0,1]$ such that $Tu \notin TL_a$ but $Tu \in \{TL_a\}^{dd}$. In particular, $Tu \perp TR_a$. Observe that necessarily $u(a) \neq 0$. Indeed, if $u(a) = 0$, then $u = u_1 \oplus u_2$ with $u_1 \in L_a$ and $u_2 \in R_a$. Since $Tu \perp TR_a$ and $Tu_1 \perp Tu_2$ it would follow that $Tu_2 = 0$, whence $u_2 = 0$, and consequently $u = u_1 \in L_a$, contradicting our assumption that $Tu \notin TL_a$. Without loss of generality we can assume that $u(a) > 0$.

Fix a small $\delta > 0$ such that $u(t) > 0$ for each $t \in (a, a+\delta)$. For each such t the band R_t is smaller than R_a and so $Tu \perp TR_t$. At the same time, the band L_t is larger than L_a and so $Tu \in \{TL_t\}^{dd}$. Also, $u \notin L_t$ since $u(t) \neq 0$. Hence $TL_t \neq \{TL_t\}^{dd}$. Therefore, by Lemma 2.5, we have $TR_t = \{TR_t\}^{dd}$ for each $t \in (a, a+\delta)$.

Similarly, if for some $a \in (c,d)$ we have $TR_a \neq \{TR_a\}^{dd}$, then there exists some $\delta > 0$ such that $TL_t = \{TL_t\}^{dd}$ for each $t \in (a-\delta, a)$. ∎

The next lemma follows immediately from Lemma 2.6.

2.7. Lemma. *Under the conditions of Lemma* 2.4 *we can choose intervals (a_n, b_n) in such a way that either*

(i) $b_n < a_{n+1}$ *and* $TR_t = \{TR_t\}^{dd}$ *for any* $t \in (a_1, \sup_n b_n)$, *or*
(ii) $b_{n+1} < a_n$ *and* $TL_t = \{TL_t\}^{dd}$ *for any* $t \in (\inf_n a_n, b_1)$.

Observe that the second case in Lemma 2.7 can always be reduced to the first one. Indeed, consider the order isomorphism $S : C[0,1] \to C[0,1]$ defined for $f \in C[0,1]$ by $Sf(t) = f(1-t)$ and notice that the operators T^{-1} and $(TS)^{-1}$ either both preserve disjointness or both do not. Therefore in what follows *we will always assume that case* (i) *holds.*

2.8. Definition. Let $T : C[0,1] \to Y$ be a disjointness preserving bijection onto a vector lattice. For every $f \in C[0,1]$ the elements $T^{-1}((Tf)^+)$ and $T^{-1}((Tf)^-)$ in $C[0,1]$ will be denoted by f'_T and f''_T, respectively,

Clearly $f'_T - f''_T = f$ and $f'_T + f''_T = T^{-1}|Tf|$. Also, it follows easily that if $Tf \perp Tg$ for some $f, g \in C[0,1]$, then $(f+g)'_T = f'_T + g'_T$, $(f+g)''_T = f''_T + g''_T$ and $(f-g)'_T = f'_T + g''_T$.

2.9. Lemma. *Assume that a disjointness preserving bijection $T : C[0,1] \to Y$, where Y is an arbitrary vector lattice, does not satisfy* (⫛), *and let intervals (a_n, b_n) in $(0,1)$ satisfy case* (i) *of Lemma* 2.7. *Assume also that we have a sequence of functions $f_n \in C[0,1]$ with* $\operatorname{supp}(f_n) \subseteq (a_n, b_n)$ *and*

$\|f_n\| \searrow 0$ *so that the series* $u = \sum_{n=1}^{\infty} f_n$ *converges in* $C[0,1]$. *Then for each* $t \in (0, a_{n+1})$ *we have*

$$u'_T(t) = \sum_{k=1}^{n} (f_k)'_T(t), \qquad u''_T(t) = \sum_{k=1}^{n} (f_k)''_T(t).$$

Proof. Fix any n and any point $t \in (0, a_{n+1})$. We have $u = v + w$, where $v = \sum_{k=1}^{n} f_k$ and $w = \sum_{k=n+1}^{\infty} f_k$. Clearly $v \perp w$, whence $Tv \perp Tw$ and so $u'_T = v'_T + w'_T$.

Since $\{TR_t\}^{dd} = TR_t$ and $Tw \in TR_t$ it follows that $(Tw)^+$ also belongs to TR_t and therefore $w'_T \in R_t$. Hence $w'_T(t) = 0$. It remains to notice that

$$v'_T(t) = \sum_{k=1}^{n} (f_k)'_T(t),$$

because the elements $Tf_1, \ldots, Tf_n$ are pairwise disjoint in Y. The proof for $u''_T(t)$ is identical. ∎

3. Main results. We are now ready to prove our main result. Recall that if $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences and $\beta_n > 0$, then the notation $\alpha_n = O(\beta_n)$ means that the sequence $\{\alpha_n/\beta_n\}$ is bounded. The notation $\alpha_n \asymp \beta_n$ means that $c \le \alpha_n/\beta_n \le C$ for some constants c and C satisfying $0 < c \le C$.

3.1. Theorem. *Let $T : C[0,1] \to Y$ be a disjointness preserving bijection onto an arbitrary vector lattice. Then T satisfies condition* ($\pitchfork$).

Proof. Assume, contrary to our claim, that $T \notin (\pitchfork)$. In view of Lemma 2.6 and the comment after Lemma 2.7 there exists a non-empty open interval $(p,q) \subset (0,1)$ such that

$\langle 1 \rangle$ $\quad TR_t = \{TR_t\}^{dd}$ for each $t \in (p,q)$.

Fix a sequence of pairwise disjoint non-empty intervals $(a_n, b_n) \subset (p,q)$ satisfying

$\langle 2 \rangle$ $\quad b_n < a_{n+1}$ for each n.

The midpoint of (a_n, b_n) will be denoted by t_n.

Next for each $n \in \mathbb{N}$ we will construct inductively some constants γ_n and A_n, some functions $f_n, g_n, h_n \in C[0,1]$ with

$\langle 3 \rangle$ $\quad \operatorname{supp}(|f_n| + |g_n| + |h_n|) \subseteq (a_n, b_n)$,

and also some auxiliary constant $\delta_n > 0$ and a function $e_n \in C[0,1]$ with the properties indicated below.

Consider first $n = 1$. Let $\gamma_1 = 1$ and $A_1 = \gamma_1^2$. By Lemma 2.4 there are functions f_1, g_1, h_1 in $C[0,1]$ with supports in (a_1, b_1) and such that

$\langle 4_1 \rangle$ Tf_1, Tg_1, and Th_1 are pairwise disjoint in Y,

$\langle 5_1 \rangle$ $\|f_1 - g_1\| + \|f_1 - h_1\| \le 1/2$,

$\langle 6_1 \rangle$ $f_1(t_1) = g_1(t_1) = h_1(t_1) = A_1$,

$\langle 7_1 \rangle$ $\max(\|f_1\|, \|g_1\|, \|h_1\|) \le 2A_1$.

Consider the six functions $|(f_1)'_T|$, $|(f_1)''_T|$, $|(g_1)'_T|$, $|(g_1)''_T|$, $|(h_1)'_T|$, $|(h_1)''_T|$ in $C[0,1]$. In view of $\langle 1 \rangle$ each of them is zero on $[0, a_1]$. The continuity of these functions coupled with condition $\langle 2 \rangle$ guarantees the existence of a small scalar $\delta_1 > 0$ such that

$\langle 8_1 \rangle$ $b_1 + \delta_1 < a_2 - \delta_1$,

$\langle 9_1 \rangle$ the oscillation of each of the six functions on $[b_1, b_1 + \delta_1]$ is less than $1/2$.

Finally, we denote by e_1 a function in $C[0,1]$ satisfying:

$\langle 10_1 \rangle$ $\mathbf{0} \le e_1 \le \mathbf{1}$, e_1 is 1 on $[a_1, b_1]$ and is 0 off $(a_1 - \delta_1, b_1 + \delta_1)$.

For the induction hypothesis assume that for each $i \le n-1$ we have already defined constants γ_i, A_i and constructed functions $f_i, g_i, h_i \in C[0,1]$ such that their supports lie in (a_i, b_i) and

$\langle 4_i \rangle$ Tf_i, Tg_i, and Th_i are pairwise disjoint in Y,

$\langle 5_i \rangle$ $\|f_i - g_i\| + \|f_i - h_i\| \le 1/(i+1)$,

$\langle 6_i \rangle$ $f_i(t_i) = g_i(t_i) = h_i(t_i) = A_i$, and

$\langle 7_i \rangle$ $\max(\|f_i\|, \|g_i\|, \|h_i\|) \le 2A_i$.

The auxiliary constants $\delta_i > 0$ satisfy

$\langle 8_i \rangle$ $b_i + \delta_i < a_{i+1} - \delta_i$,

and are so small that

$\langle 9_i \rangle$ the oscillation of each of $|(f_i)'_T|$, $|(f_i)''_T|$, $|(g_i)'_T|$, $|(g_i)''_T|$, $|(h_i)'_T|$, $|(h_i)''_T|$ on $[b_i, b_i + \delta_i]$ is less than $1/(i+1)$.

Finally, for each $i \le n-1$ we have also fixed a function $e_i \in C[0,1]$ such that

$\langle 10_i \rangle$ $\mathbf{0} \le e_i \le \mathbf{1}$, e_i is 1 on $[a_i, b_i]$ and e_i is 0 off $(a_i - \delta_i, b_i + \delta_i)$.

We are ready to describe the induction step for n. Let

$$\gamma_n = 2\sum_{i=1}^{n-1} \frac{\gamma_i}{f_i(t_i)}\big(\|((f_i)'_T e_i)'_T\| + \|((f_i)'_T e_i)''_T\| + \|((h_i)'_T e_i)'_T\| + \|((h_i)'_T e_i)''_T\|\big)$$

and

$$A_n = \gamma_n^2\Big[1 + \sum_{i=1}^{n-1}(\|(f_i)'_T\| + \|(f_i)''_T\| + \|(g_i)'_T\| + \|(g_i)''_T\| + \|(h_i)'_T\| + \|(h_i)''_T\|) + \sum_{i=1}^{n-1}(\|((f_i)'_T e_i)'_T\| + \|((f_i)'_T e_i)''_T\| + \|((h_i)'_T e_i)'_T\| + \|((h_i)'_T e_i)''_T\|)\Big].$$

By Lemma 2.4, there are functions $f_n, g_n, h_n \in C[0,1]$ with supports in (a_n, b_n) and such that

$\langle 4_n\rangle$ Tf_n, Tg_n, and Th_n are pairwise disjoint in Y,

$\langle 5_n\rangle$ $\|f_n - g_n\| + \|f_n - h_n\| \le 1/(n+1)$,

$\langle 6_n\rangle$ $f_n(t_n) = g_n(t_n) = h_n(t_n) = A_n$,

$\langle 7_n\rangle$ $\max(\|f_n\|, \|g_n\|, \|h_n\|) \le 2A_n$.

We proceed with a delicate thing as to how to define $\delta_n > 0$. To this end, consider the continuous functions $|(f_n)'_T|$, $|(f_n)''_T|$, $|(g_n)'_T|$, $|(g_n)''_T|$, $|(h_n)'_T|$, and $|(h_n)''_T|$. In view of condition $\langle 1\rangle$ each of them is zero to the left of a_n. The continuity implies that we can find a scalar $\delta_n \in (0, \delta_{n-1})$ such that

$\langle 8_n\rangle$ $b_n + \delta_n < a_{n+1} - \delta_n$,

$\langle 9_n\rangle$ the oscillation of each of these six functions on $[b_n, b_n + \delta_n]$ is less than $1/(n+1)$.

Finally, we fix a function $e_n \in C[0,1]$ such that

$\langle 10_n\rangle$ $\mathbf{0} \le e_n \le \mathbf{1}$, e_n is 1 on $[a_n, b_n]$ and e_n is 0 off $(a_n - \delta_n, b_n + \delta_n)$.

This concludes the induction.

Consider next the following three series:

$$u = \sum_{n=1}^{\infty}(f_n - g_n), \qquad v = \sum_{n=1}^{\infty}(f_n - h_n), \qquad w = \sum_{n=1}^{\infty}(g_n - h_n).$$

In view of $\langle 5_n\rangle$ each of these series converges in $C[0,1]$, and so the functions u, v, and w do exist in $C[0,1]$. Also we will need the functions u'_T, u''_T, v'_T, v''_T, w'_T, and w''_T. Let C be a constant that is greater than or equal to the norm of each of these six functions.

In view of Lemma 2.9, for each $t \in [a_n, b_n]$ we have

$$u'_T(t) = \sum_{i=1}^{n}(f_i - g_i)'_T(t), \qquad v'_T(t) = \sum_{i=1}^{n}(f_i - h_i)'_T(t),$$
$$w'_T(t) = \sum_{i=1}^{n}(g_i - h_i)'_T(t).$$

The first equality implies that

$$(f_n - g_n)'_T(t) = u'_T(t) - \sum_{i=1}^{n-1}(f_i - g_i)'_T(t),$$

and hence

$$|(f_n - g_n)'_T(t)| \le C + \sum_{i=1}^{n-1}|(f_i - g_i)'_T(t)|.$$

Similarly

$$|(f_n - g_n)''_T(t)| \le C + \sum_{i=1}^{n-1}|(f_i - g_i)''_T(t)|.$$

Since $Tf_i \perp Tg_i$ for each i, we know that $(f_i - g_i)'_T(t) = (f_i)'_T(t) + (g_i)''_T(t)$ and $(f_i - g_i)''_T(t) = (f_i)''_T(t) + (g_i)'_T(t)$, and therefore the previous two inequalities can be rewritten as

$$|(f_n)'_T(t) + (g_n)''_T(t)| \le C + \sum_{i=1}^{n-1}|(f_i)'_T(t) + (g_i)''_T(t)|, \tag{1}$$

$$|(f_n)''_T(t) + (g_n)'_T(t)| \le C + \sum_{i=1}^{n-1}|(f_i)''_T(t) + (g_i)'_T(t)|. \tag{2}$$

Similar estimates are true for the pair $|(f_n)'_T(t) + (h_n)''_T(t)|$, $|(f_n)''_T(t) + (h_n)'_T(t)|$, and for the pair $|(g_n)'_T(t) + (h_n)''_T(t)|$, $|(g_n)''_T(t) + (h_n)'_T(t)|$.

To simplify what follows, let us introduce the following constant:

$$\begin{aligned} M_n = \max_{t\in[a_n,b_n]}&[|(f_n)'_T(t) + (g_n)''_T(t)| + |(f_n)''_T(t) + (g_n)'_T(t)| \\ &+ |(f_n)'_T(t) + (h_n)''_T(t)| + |(f_n)''_T(t) + (h_n)'_T(t)| \\ &+ |(g_n)'_T(t) + (h_n)''_T(t)| + |(g_n)''_T(t) + (h_n)'_T(t)|]. \end{aligned}$$

Using estimates (1), (2) above, their four analogues for $|(f_n)'_T(t) + (h_n)''_T(t)|$, $|(f_n)''_T(t) + (h_n)'_T(t)|$, $|(g_n)'_T(t) + (h_n)''_T(t)|$, and $|(g_n)''_T(t) + (h_n)'_T(t)|$, as well as the definition of the constant A_n, we obtain

$$\begin{aligned} M_n \le 6C + \max_{t\in[a_n,b_n]} \sum_{i=1}^{n-1}&[|(f_i)'_T(t) + (g_i)''_T(t)| + |(f_i)''_T(t) + (g_i)'_T(t)| \\ &+ |(f_i)'_T(t) + (h_i)''_T(t)| + |(f_i)''_T(t) + (h_i)'_T(t)| \\ &+ |(g_i)'_T(t) + (h_i)''_T(t)| + |(g_i)''_T(t) + (h_i)'_T(t)|] \\ \le 6C + 2A_n/\gamma_n^2 = 6C + 2f_n(t_n)/\gamma_n^2.& \end{aligned}$$

In other words, we have

$$M_n = O(f_n(t_n)/\gamma_n^2).$$

Using the obvious identity

$$(f_n)'_T + (f_n)''_T = ((f_n)'_T + (g_n)''_T) - ((g_n)''_T + (h_n)'_T) + ((f_n)''_T + (h_n)'_T),$$

we immediately see that

$$\max_{t\in[a_n,b_n]} |(f_n)'_T(t) + (f_n)''_T(t)| = O(f_n(t_n)/\gamma_n^2). \tag{3}$$

At the same time, for each $t \in [a_n, b_n]$ (in fact, for each $t \in [0,1]$) we have

$$(f_n)'_T(t) - (f_n)''_T(t) = f_n(t). \tag{4}$$

Estimates (3) and (4) imply easily that

$$\max_{t\in[a_n,b_n]} |(f_n)'_T(t) - f_n(t)/2| = O(f_n(t_n)/\gamma_n^2). \tag{5}$$

By symmetry we also have

$$\max_{t\in[a_n,b_n]} |(h_n)'_T(t) - h_n(t)/2| = O(h_n(t_n)/\gamma_n^2). \tag{6}$$

Recalling that $h_n(t_n) = f_n(t_n)$ and that $\|f_n - h_n\| \to 0$, we can rewrite (6) as

$$\max_{t\in[a_n,b_n]} |(h_n)'_T(t) - f_n(t)/2| = O(f_n(t_n)/\gamma_n^2). \tag{7}$$

From (5) and (7) it follows that

$$\max_{t\in[a_n,b_n]} |(f_n)'_T(t) + (h_n)'_T(t) - f_n(t)| = O(f_n(t_n)/\gamma_n^2)$$

and so, in particular,

$$|(f_n)'_T(t_n) + (h_n)'_T(t_n) - f_n(t_n)| = O(f_n(t_n)/\gamma_n^2).$$

This implies immediately that

$$(f_n)'_T(t_n) + (h_n)'_T(t_n) \asymp f_n(t_n). \tag{8}$$

From (5) and (7) it also follows that

$$\max_{t\in[a_n,b_n]} |(f_n)'_T(t) - (h_n)'_T(t)| = O(f_n(t_n)/\gamma_n^2).$$

Moreover, in view of $\langle 9_n \rangle$ we have

$$\max_{t\in[a_n-\delta_n,b_n+\delta_n]} |(f_n)'_T(t) - (h_n)'_T(t)| = O(f_n(t_n)/\gamma_n^2).$$

This implies that the disjoint sequence $\left\{\frac{\gamma_n}{f_n(t_n)}((f_n)'_T - (h_n)'_T)e_n\right\}$ converges in norm to zero, and therefore the function

$$x = \sum_{i=1}^{\infty} \frac{\gamma_i}{f_i(t_i)} ((f_i)'_T - (h_i)'_T)e_i$$

exists in $C[0,1]$. Let

$$\widehat{x} = T^{-1}(|Tx|), \qquad \widehat{x}_n = T^{-1}\biggl(\biggl|T\biggl(\sum_{i=1}^{n} \frac{\gamma_i}{f_i(t_i)}\,((f_i)'_T - (h_i)'_T)e_i\biggr)\biggr|\biggr).$$

We will arrive at a contradiction by showing that the function $\widehat{x} \in C[0,1]$ is unbounded. From condition $\langle 1\rangle$ it follows at once that

$$\widehat{x}(t_n) = \widehat{x}_n(t_n), \tag{9}$$

and from the pairwise disjointness of the terms in the last sum above it follows that

$$\widehat{x}_n = T^{-1}\biggl(\sum_{i=1}^{n}\biggl|T\biggl(\frac{\gamma_i}{f_i(t_i)}\,((f_i)'_T - (h_i)'_T)e_i\biggr)\biggr|\biggr).$$

Consequently,

$$\begin{aligned}
|\widehat{x}_n(t_n)| &\ge \frac{\gamma_n}{f_n(t_n)}\,|T^{-1}(|T((f_n)'_T - (h_n)'_T)e_n)|)(t_n)| \\
&\quad - \sum_{i=1}^{n-1}\biggl|T^{-1}\biggl(\biggl|T\biggl(\frac{\gamma_i}{f_i(t_i)}\,((f_i)'_T - (h_i)'_T)e_i\biggr)\biggr|\biggr)(t_n)\biggr| \\
&\ge \frac{\gamma_n}{f_n(t_n)}\,|T^{-1}(|T((f_n)'_T - (h_n)'_T)e_n)|)(t_n)| \\
&\quad - \sum_{i=1}^{n-1}\frac{\gamma_i}{f_i(t_i)}\,[\|((f_i)'_T e_i)'_T\| + \|((f_i)'_T e_i)''_T\| \\
&\quad + \|((h_i)'_T e_i)'_T\| + \|((h_i)'_T e_i)''_T\|].
\end{aligned}$$

Since the last sum is simply $\gamma_n/2$, we have

$$|\widehat{x}_n(t_n)| \ge \frac{\gamma_n}{f_n(t_n)}\,|T^{-1}(|T((f_n)'_T - (h_n)'_T)e_n)|)(t_n)| - \gamma_n/2. \tag{10}$$

We claim that

$$|T^{-1}(|T(((f_n)'_T - (h_n)'_T)e_n)|)(t_n)| = |T^{-1}(|T((f_n)'_T - (h_n)'_T)|)(t_n)|.$$

To prove this, set for brevity $\alpha := ((f_n)'_T - (h_n)'_T)e_n$ and $\beta := (f_n)'_T - (h_n)'_T$, and note that the functions α and β coincide on $[0, b_n]$ and so their difference $\alpha - \beta$ is in R_t for any $t \le b_n$. Since, by $\langle 1\rangle$, TR_t is a band, we have $|T(\alpha-\beta)| \in TR_t$, implying that $T^{-1}(|T(\alpha-\beta)|)$ also belongs to R_t and so, in particular, $T^{-1}(|T(\alpha-\beta)|)(t_n) = 0$. It remains to notice that $|T^{-1}(|T\alpha|) - T^{-1}(|T\beta|)| \le |T^{-1}(|T(\alpha-\beta)|)|$, which guarantees that $T^{-1}(|T\alpha|)(t_n) = T^{-1}(|T\beta|)(t_n)$.

Next observe that $T((f_n)'_T - (h_n)'_T) = T(f_n)'_T - T(h_n)'_T = (Tf_n)^+ - (Th_n)^+$ and hence, since the last two terms are disjoint, $|T((f_n)'_T - (h_n)'_T)| =$

$(Tf_n)^+ + (Th_n)^+$. Taking into consideration (8) now yields

$$|T^{-1}(|T((f_n)'_T - (h_n)'_T)|)(t_n)| = |T^{-1}((Tf_n)^+ + (Th_n)^+)(t_n)| \\ = (f_n)'_T(t_n) + (h_n)'_T(t_n) \asymp f_n(t_n).$$

In other words, the first term in (10) is equivalent to γ_n, that is,

$$\frac{\gamma_n}{f_n(t_n)} |T^{-1}(|T((f_n)'_T - (h_n)'_T)e_n)|)(t_n)| \asymp \gamma_n.$$

Using this and returning back to inequality (10), we see that $\widehat{x}_n(t_n) \geq c\gamma_n/2$ for some constant $c > 0$ that is independent of n. But then, in view of (9), we get $\widehat{x}(t_n) \geq c\gamma_n/2$, which is impossible since the function $\widehat{x}$ must be bounded. ∎

We are ready to prove our second main result. It establishes important order properties of $C[0,1]$. We precede it with a formal definition of (super) d-rigidity already discussed in the introduction. The notion of d-rigidity was introduced in [2], and its strengthening, super d-rigidity, is considered here for the first time.

3.2. Definition. A vector lattice X is said to be *d-rigid* if for each disjointness preserving bijection T from X onto an arbitrary vector lattice Y the inverse operator $T^{-1} : Y \to X$ is also disjointness preserving. If, additionally, each such operator T is regular, then X is said to be *super d-rigid*.

It should be noticed that according to Theorem 4.12 of [1] the regularity of a disjointness preserving bijection $T : X \to Y$ guarantees that T^{-1} is disjointness preserving, and thus the latter condition in Definition 3.2 implies the former. Moreover, T^{-1} is also regular and the vector lattices X and Y are necessarily order isomorphic.

It is interesting to notice that the super d-rigidity of a vector lattice X implies that the Boolean algebra $\mathcal{B}(X)$ of all bands in X completely determines the order structure of X in the following sense: If $T : X \to Y$ is a bijection onto an arbitrary vector lattice Y such that the mapping $B \mapsto T(B)$, $B \in \mathcal{B}(X)$, defines a Boolean isomorphism from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$, then Y is order isomorphic to X.

3.3. Theorem. *The vector lattice $C[0,1]$ is super d-rigid, i.e., if Y is an arbitrary vector lattice and $T : C[0,1] \to Y$ is a disjointness preserving bijection, then the inverse operator T^{-1} is also disjointness preserving and the operator T is necessarily regular.*

Proof. Let $T : C[0,1] \to Y$ be a disjointness preserving bijection onto an arbitrary vector lattice.

The d-rigidity of $C[0,1]$ follows easily from our previous theorem and a theorem in [2]. Indeed, by Theorem 3.1 the operator T satisfies condition ($\pitchfork$).

Under this condition the desired conclusion that T^{-1} preserves disjointness is proved in [2, Theorem 4.1].

It is harder to prove that the operator T is also regular, and so $C[0,1]$ is super d-rigid. As before, for each $t \in (0,1)$ we consider the bands L_t and R_t introduced prior to Lemma 2.5, and let $U_t = T(L_t)$ and $V_t = T(R_t)$ be their images in Y. Clearly U_t and V_t are disjoint bands in Y.

Notice that if some $y \in Y$ belongs to $U_t \oplus V_t$ for each $t \in (0,1)$, then necessarily $y = 0$. Indeed, let $x = T^{-1}y \in C[0,1]$. Because T^{-1} preserves disjointness, it follows that $x \in L_t \oplus R_t$, i.e., $x(t) = 0$ for each $t \in (0,1)$. Thus $x = 0$ and so $y = Tx = 0$.

Since the constantly one function $\mathbf{1}$ does not have non-trivial components in $C[0,1]$ and since T^{-1} preserves disjointness, the element $T\mathbf{1}$ does not have non-trivial components in Y either, and therefore either $|T\mathbf{1}| = T\mathbf{1}$ or $|T\mathbf{1}| = -T\mathbf{1}$. Replacing (if necessary) T by $-T$ we can always assume that the former case holds, i.e., $|T\mathbf{1}| = T\mathbf{1}$. Clearly $T\mathbf{1}$ is a weak unit in Y.

Assume contrary to our claim that the operator T is not regular. Then by the McPolin–Wickstead theorem (see [11] or Theorem 5.1 in [1]) there exists a sequence $\{x_n\}$ of non-negative functions in $C[0,1]$ such that $\|x_n\| \to 0$ and $|Tx_n| \geq y$ for all $n \in \mathbb{N}$ and some $0 < y \in Y$. We will assume that $\|x_n\| \leq 1$ for each n.

Next we will show that without loss of generality we can assume additionally that

$$Tx_n \geq 0 \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

To this end, let $x'_n = T^{-1}|Tx_n|$, whence $Tx'_n = |Tx_n|$. As $|Tx_n| = (Tx_n)^+ + (Tx_n)^-$ and T^{-1} preserves disjointness, we see that $\|x'_n\| = \|x_n\|$ and so $\|x'_n\| \to 0$. Consider finally $x''_n = x'_n + \|x'_n\|\mathbf{1}$. Clearly $x''_n \geq 0$ and $\|x''_n\| \to 0$. It remains to notice that

$$Tx''_n = Tx'_n + \|x'_n\|T\mathbf{1} = |Tx_n| + \|x'_n\|T\mathbf{1} \geq |Tx_n| \geq y.$$

Therefore, replacing if necessary the initial sequence $\{x_n\}$ by the sequence $\{x''_n\}$, we can indeed assume that $\{x_n\}$ satisfies additionally condition (11).

Because $T\mathbf{1}$ is a weak unit in Y we have $y \wedge T\mathbf{1} \neq 0$. Consequently, there is some $t \in (0,1)$ such that

$$y \wedge T\mathbf{1} \notin U_t \oplus V_t. \tag{12}$$

Fix such a t. Obviously $x_n - x_n(t)\mathbf{1} \in L_t \oplus R_t$ and so

$$Tx_n - x_n(t)T\mathbf{1} \in U_t \oplus V_t. \tag{13}$$

We will prove next that it follows from (13) that

$$Tx_n \wedge T\mathbf{1} - x_n(t)T\mathbf{1} \in U_t \oplus V_t. \tag{14}$$

To this end, consider in Y the principal ideal $Y(v)$ generated by the element $v(= v_n) = Tx_n + T\mathbf{1}$. By the Krein–Kakutani theorem there exists a compact Hausdorff space K such that $Y(v)$ can be identified with an order dense vector sublattice of $C(K)$ in such a way that v is identified with χ_K.

Since the elements Tx_n and $T\mathbf{1}$ belong to $Y(v)$ (and hence to $C(K)$), the elements $f := Tx_n - x_n(t)T\mathbf{1}$ and $g := Tx_n \wedge T\mathbf{1} - x_n(t)T\mathbf{1}$ also belong to $Y(v)$.

Clearly $U_t \cap Y(v)$ and $V_t \cap Y(v)$ are the bands in $Y(v)$. Therefore (since $Y(v)$ is order dense in $C(K)$) there exist two unique bands U_t' and V_t' in $C(K)$ that correspond to the bands U_t and V_t, respectively. To establish (14), that is, that $g \in U_t' \oplus V_t'$, it suffices to show that $g(k) = 0$ provided $f(k) = 0$ for $k \in K$. So, assume that $f(k) = (Tx_n)(k) - x_n(t)(T\mathbf{1})(k) = 0$ at some $k \in K$. That is, $(Tx_n)(k) = x_n(t)(T\mathbf{1})(k)$ and hence, since $0 \le x_n(t) \le \|x_n\| \le 1$, it follows that $(Tx_n)(k) \le (T\mathbf{1})(k)$. Therefore, $(Tx_n)(k) \wedge (T\mathbf{1})(k) = (Tx_n)(k)$, and hence

$$g(k) = (Tx_n \wedge T\mathbf{1})(k) - x_n(t)(T\mathbf{1})(k) = Tx_n(k) - x_n(t)(T\mathbf{1})(k) = f(k) = 0,$$

as claimed. This proves (14).

Since $x_n(t)T\mathbf{1} \to 0$ with the regulator of convergence $T\mathbf{1} \in U_t \oplus V_t$ and since $0 < y \wedge T\mathbf{1} \le Tx_n \wedge T\mathbf{1}$, it follows from (14) that $y \wedge T\mathbf{1} \in U_t \oplus V_t$. This contradicts (12). The proof is complete. ∎

We single out some useful consequences of the previous theorem.

3.4. Corollary. *Let $T : C[0,1] \to Y$ be a disjointness preserving bijection onto an arbitrary vector lattice. Then Y is order isomorphic to $C[0,1]$.*

Proof. By Theorem 3.3 the disjointness preserving operator T is regular. Then Theorem 4.12 in [1] guarantees that $C[0,1]$ is order isomorphic to Y. ∎

3.5. Corollary. *Let $T : C[0,1] \to Y$ be a disjointness preserving bijection onto an arbitrary vector lattice. Then T is automatically (r_u)-continuous.*

3.6. Corollary. *Let $T : C[0,1] \to Y$ be a disjointness preserving bijection onto an arbitrary normed vector lattice. Then Y is necessarily norm complete and T is norm continuous.*

4. Some generalizations and remarks. An inspection of the proof of Theorem 3.1 shows that its statement remains true for a large class of vector sublattices of $C[0,1]$. To describe this class precisely, we need to introduce two definitions.

4.1. Definition. A unital subalgebra $\mathcal{A}$ of $C[0,1]$ is said to be *EC-rich* if for any interval $(a,b) \subset (0,1)$ the algebra $\mathcal{A}$ contains an essentially constant function f such that $\mathbf{0} \le f \le \mathbf{1}$, $f \equiv 0$ on $[0,a]$, and $f \equiv 1$ on $[b,1]$.

4.2. Definition. A vector sublattice X of $C[0,1]$ is *c_0-complete* if for every disjoint sequence $\{x_n\}$ in X satisfying $\|x_n\| \to 0$ the element $\sum_{n=1}^{\infty} x_n$ belongs to X.

A vector sublattice X of $C[0,1]$ is *weakly c_0-complete* if there exists a sequence $\{\varepsilon_n\}$ of positive scalars such that $\varepsilon_n \searrow 0$ and for any disjoint sequence $\{x_n\}$ in X satisfying $\|x_n\| \leq \gamma\varepsilon_n$ for some constant $\gamma > 0$ the element $\sum_{n=1}^{\infty} x_n$ belongs to X.

The proof of the next result repeats, practically verbatim, that of Theorem 3.1.

4.3. Theorem. *Let X be an order dense vector sublattice of $C[0,1]$ satisfying the following two conditions.*

1) *X is weakly c_0-complete.*
2) *$\mathcal{A}X \subseteq X$ for some EC-rich subalgebra $\mathcal{A}$ of $C[0,1]$.*

Then each disjointness preserving bijection $T : X \to Y$ satisfies condition (⫛).

Using this theorem we can now describe a large class of d-rigid vector lattices. We would like to repeat that, as mentioned in the introduction, this class is not as large as the class we erroneously proclaimed in [2, Theorem 4.5].

4.4. Corollary. *Let X be an order dense vector sublattice of $C[0,1]$ satisfying the following two conditions*:

1) *X is weakly c_0-complete.*
2) *$\mathcal{A}X \subseteq X$ for some EC-rich subalgebra $\mathcal{A}$ of $C[0,1]$.*

Then the vector lattice X is d-rigid.

Proof. Let $T : X \to Y$ be a disjointness preserving bijection onto a vector lattice Y. By the previous theorem T satisfies (⫛). Assume, contrary to our claim, that T^{-1} does not preserve disjointness. Then, using Lemma 5.3 of [2], we can find positive elements $u, v \in X$ such that

(i) $Tu \perp Tv$,

(ii) for each $\varepsilon > 0$ there exist linear combinations s_ε and t_ε of u and v, respectively, such that $|s_\varepsilon - v| \leq \varepsilon u$ and $|t_\varepsilon - u| \leq \varepsilon v$.

A straightforward argument shows that for continuous functions u and v on $[0,1]$ condition (ii) implies that the functions u and v must be proportional. However, this contradicts condition (i). Consequently, the inverse T^{-1} preserves disjointness, and so X is d-rigid. ∎

4.5. Remark. (i) Theorem 4.3 does not hold if no additional assumptions on the order dense vector sublattice X of $C[0,1]$ are imposed. Namely, in the absence of both assumptions in Theorem 4.3 there even exists a *band* *preserving* bijection $T : X \to Y$, where X is an order dense vector sublattice of $C[0,1]$ and Y is a vector sublattice of the universal completion X^u of X, such that the inverse operator $T^{-1} : Y \to X$ does not preserve disjointness (see [3, Example 4.7]).

(ii) It follows from [3, Theorem 4.2] that Condition 2 in Theorem 4.3 guarantees that for any invertible band preserving operator $T : X \to X^u$ the inverse operator $T^{-1} : TX \to X$ is also band preserving. Nevertheless, this condition alone does not guarantee that X is d-rigid. A corresponding counterexample will be presented elsewhere. Note incidentally that our counterexample also settles in the negative one more open problem regarding the possibility of range-domain exchange in the Huijsmans–de Pagter–Koldunov Theorem (see Section 9 in [1]).

(iii) We do not know if Condition 1 in Theorem 4.3 is enough to guarantee that T^{-1} is disjointness preserving. It seems that it might, especially in view of Theorem 4.6 below.

The situation becomes much simpler if one assumes additionally that the range vector lattice Y is (r_u)-complete.

4.6. Theorem. *Let X be a weakly c_0-complete order dense vector sublattice of $C[0,1]$ and let Y be an (r_u)-complete vector lattice. If $T : X \to Y$ is a disjointness preserving bijection, then the inverse operator $T^{-1} : Y \to X$ preserves disjointness.*

Proof. Assume contrary to our claim that T^{-1} does not preserve disjointness. Using Lemma 5.3 of [2] we can conclude that T does not satisfy $(\pitchfork)$ either. (Otherwise, as in the proof of Corollary 4.4, it would follow that T^{-1} were disjointess preserving.) Therefore, exactly as at the very beginning of the proof of Lemma 2.4, there exists an element $f \in X$ and an interval $(a,b) \subset (0,1)$ such that

$$f > 0 \text{ on } [a,b] \text{ and } Tf \perp Tg \text{ for any } g \in X \text{ with } \operatorname{supp}(g) \subseteq [a,b].$$

As said before, we can always assume that $TR_t = \{TR_t\}^{dd}$ for any $t \in (a,b)$. Let (a_n,b_n) be disjoint intervals in (a,b) such that $b_n < a_{n+1}$.

Fix n for a moment. Because X is order dense in $C[0,1]$, for any $m \in \mathbb{N}$ we can find m functions $f_{n,1},\dots,f_{n,m} \in X$ and a point $t_n \in (a_n,b_n)$ satisfying the following conditions:

- $\operatorname{supp}(f_{n,1}) \subset (a_n,b_n)$ and $0 \le f_{n,1} \le f$,
- $0 < f_{n,i+1} \le f_{n,i}$ and $\operatorname{supp}(f_{n,i+1}) \subseteq \{t : f_{n,i}(t) = f(t)\}$, $i = 1,\dots, m-1$,
- $f_{n,1}(t_n) = \ldots = f_{n,m}(t_n) = f(t_n)$.

We claim that $Tf_{n,1},\dots,Tf_{n,m}$ are pairwise disjoint in Y. Indeed, pick any $i,j \in \mathbb{N}$ satisfying $1 \le i < j \le m$ and consider $Tf_{n,i}$ and $Tf_{n,j}$.

Obviously $f - f_{n,i} \perp f_{n,m}$ and so $T(f - f_{n,i}) \perp Tf_{n,j}$ as T preserves disjointness. But $T(f-f_{n,i}) = Tf - Tf_{n,i}$ and $Tf \perp Tf_{n,i}$, since $\operatorname{supp}(f_{n,i}) \subseteq [a,b]$; therefore it follows that $Tf_{n,i} \perp Tf_{n,i}$.

Let $\{\varepsilon_n\}$ be a null sequence of positive scalars guaranteed by Definition 4.2, and fix an increasing sequence $m_n \in \mathbb{N}$ such that

$$\frac{m_n \varepsilon_n}{n} \to \infty \quad \text{as } n \to \infty.$$

Next, for each n, we produce the functions $f_{n,1}, \ldots, f_{n,m_n} \in X$, as explained above. Let $g_n = \sum_{i=1}^{m_n} (-1)^{i+1} f_{n,i}$. Then $Tg_n = \sum_{i=1}^{m_n} (-1)^{i+1} Tf_{n,i}$, and hence $|Tg_n| = \sum_{i=1}^{m_n} |Tf_{n,i}|$, since $Tf_{n,1}, \ldots, Tf_{n,m_n}$ are pairwise disjoint.

The functions $g_1, \ldots, g_n, \ldots$ are pairwise disjoint in X and clearly $\|g_n\| \le \|f\|$. Therefore, since X is weakly c_0-complete, the element

$$u = \sum_{n=1}^{\infty} \varepsilon_n g_n$$

exists in X. Therefore $Tu \in Y$, and so $v = |Tu|$ also belongs to Y.

Finally, consider the element

$$y = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{n} \Big(\sum_{i=1}^{m_n} Tf_{n,i} \Big),$$

which exists in Y since the vector lattice Y is (r_u)-complete (we omit the simple verification that the series defining y is indeed (r_u)-Cauchy with v as its regulator).

To obtain a contradiction, we will show that the function $T^{-1}y \in X$ is unbounded. To get this, one should keep in mind that $TR_t = \{TR_t\}^{dd}$ for each $t \in (a,b)$, and then a direct calculation shows that

$$(T^{-1}y)(t_n) = \frac{\varepsilon_n}{n} m_n f(t_n) \ge c \frac{\varepsilon_n}{n} m_n \to \infty \quad \text{as } n \to \infty,$$

where $c = \min\{f(t) : t \in [a,b]\} > 0$ in view of our assumption that f is strictly positive on $[a,b]$. ∎

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