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Spaces with maximal projection constants

by

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Dedicated to Olek Pełczyński on the occasion of his 70th birthday with thanks for all his questions

Abstract. We show that n-dimensional spaces with maximal projection constants exist not only as subspaces of l_∞ but also as subspaces of l_1. They are characterized by a rigid set of vector conditions. Nevertheless, we show that, in general, there are many non-isometric spaces with maximal projection constants. Several examples are discussed in detail.

1. Spaces with maximal projection constants. In this paper we study the question of non-uniqueness of finite-dimensional spaces with maximal projection constant and their imbeddings into l_∞ and l_1. Given a (closed) subspace X of a Banach space Z, the *relative projection constant of X in Z* is

$$\lambda(X,Z) := \inf\{\|P\| \mid P : Z \to X \text{ is a linear projection onto } X\},$$

and the (*absolute*) *projection constant of X* is

$$\lambda(X) := \sup\{\lambda(X,Z) \mid Z \text{ a Banach space containing } X \text{ as a subspace}\}.$$

The scalar field $\mathbb{K}$ will be either the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Any separable Banach space can be imbedded into l_∞; for any such imbedding $\lambda(X) = \lambda(X, l_\infty)$, l_∞ is the natural superspace. For finite-dimensional spaces, $\lambda(X) \le \sqrt{\dim X}$ by Kadets–Snobar [KS].

In fact, more is known: Let $1 < n < N < \infty$ and

$$f_{\mathbb{K}}(n,N) := \sup\{\lambda(X,Z) \mid X \subseteq Z,\ \dim X = n,\ \dim Z = N\},$$
$$g_{\mathbb{K}}(n) := \sup\{\lambda(X) \mid \dim X = n\}.$$

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Further, let us introduce

$$F(n,N) := \sqrt{n}\,([\sqrt{n}+\sqrt{(N-1)(N-n)}]/N),$$
$$G_{\mathbb{R}}(n) := [2+(n-1)\sqrt{n+2}]/(n+1),$$
$$G_{\mathbb{C}}(n) := [1+(n-1)\sqrt{n+1}]/(n).$$

Then by [KLL] and [KT2] for all $n < N$ one has

$$f_{\mathbb{K}}(n,N) \le F(n,N), \tag{1.1}$$
$$g_{\mathbb{K}}(n) \le G_{\mathbb{K}}(n). \tag{1.2}$$

We note that $F(n,N), G_{\mathbb{K}}(n) < \sqrt{n}$ and, in fact,

$$G_{\mathbb{R}}(n) = F(n, n(n+1)/2), \qquad G_{\mathbb{C}}(n) = F(n, n^2). \tag{1.3}$$

An n-dimensional subspace $X_n \subseteq l_\infty^N$ can be given by a basis $(f_j)_{j=1}^n$ where $f_j = (f_{js})_{s=1}^N \in \mathbb{K}^N$. Writing the coordinates of $\widetilde{x} = \sum_{j=1}^n x_j f_j \in X_n$ as $x = (x_j)_{j=1}^n$, we have

$$\|\widetilde{x}\| = \Big\| \sum_{j=1}^n x_j f_j \Big\|_\infty = \sup_{1\le s\le N} |\langle x, x_s\rangle| =: |||x||| \tag{1.4}$$

where $x_s = (f_{js})_{j=1}^n \in \mathbb{K}^n$ and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{K}^n$. We identify in the following $(\mathbb{K}^n, |||\cdot|||)$ with $X_n \subseteq l_\infty^N$ and write $|||x|||$ instead of $\|\widetilde{x}\|$, both spaces being isometric.

A very rigid set of conditions needs to be imposed on the vectors $x_s \in \mathbb{K}^n$ $(s = 1,\dots,N)$ to have equality in (1.1) or (1.2), i.e. $\lambda(X_n, l_\infty^N) = F(n,N)$ or $\lambda(X) = G_{\mathbb{K}}(n)$: the vectors have to form a tight spherical 4-design (see [KT2]). In spite of this being a very strong assumption, we will show that—in general—there are many non-isometric spaces with maximal projection constant, some of them even being realized as subspaces of l_1. In a recent paper, Chalmers and Lewicki [CL] show that among the *symmetric sequence spaces* with maximal projection constants there are symmetric subspaces of l_1. This result motivated a part of the current paper.

We use standard Banach space terminology (see e.g. [TJ]). In particular, l_p^N denotes $\mathbb{K}^N$ with the p-norm if $N \in \mathbb{N}$ and $1 \le p \le \infty$. Given a measure μ on $\{1,\dots,N\}$, $l_p^N(\mu)$ denotes $\mathbb{K}^N$, equipped with the norm $\|x\| = (\sum_{s=1}^N |x_s|^p \mu_s)^{1/p}$, $x = (x_s)_{s=1}^N \in \mathbb{K}^N$. The *Banach–Mazur distance* between two n-dimensional normed spaces X and Y is given by

$$d(X,Y) = \inf\{\|T\|\,\|T^{-1}\| \mid T : X \to Y \text{ is a linear isomorphism}\}.$$

Our main results are:

THEOREM 1. *Let $n \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$ and $X_n \subseteq l_\infty^N$ be a space such that X_n has maximal projection constant among all n-dimensional spaces.*

Thus $\lambda(X_n) = g_{\mathbb{K}}(n)$. *Then there is* $Y_n \subseteq l_1^N$ *also having maximal projection constant*

$$\lambda(Y_n) = \lambda(Y_n, l_1^N) = \lambda(X_n) = g_{\mathbb{K}}(n). \tag{1.5}$$

A corresponding fact holds for relative projection constants: *if* $X_n \subset l_\infty^N$ *satisfies* $\lambda(X_n, l_\infty^N) = f_{\mathbb{K}}(n, N)$ *for* $N \in \mathbb{N}$, *there is* $Y_n \subseteq l_1^N$ *with*

$$\lambda(Y_n, l_1^N) = \lambda(X_n, l_\infty^N) = f_{\mathbb{K}}(n, N). \tag{1.6}$$

In the real two-dimensional case, $\lambda(X_2) = g(2) = f(2,3) = 4/3$ is uniquely attained by the space X_2 having the regular hexagon as its unit ball, and $Y_2 = X_2$ holds isometrically. Both spaces are represented in $\mathbb{R}^3$ with the $\|\cdot\|_\infty$- or $\|\cdot\|_1$-norm by the hyperplane $H = \{x = (x_j)_{j=1}^3 \in \mathbb{R}^3 \mid \sum_{j=1}^3 x_j = 0\}$. In general, however, spaces $X_n \subseteq l_\infty^N$ and $Y_n \subseteq l_1^N$ with maximal projection constant are not isometric. This already occurs for $n = 3$ in the real and $n = 2$ in the complex case.

PROPOSITION 2. *Let* D *be the dodecahedron in* $\mathbb{R}^3$ *and* I *be the icosahedron having as its vertices the midpoints of the faces of* D. *Let* $K = D \cap \phi I$ *where* $\phi = (1+\sqrt{5})/2$. *Let* X_3 *and* Y_3, *respectively*, *be the* 3*-dimensional spaces having* D *and* K *as their unit balls. Then* $X_3 \subseteq l_\infty^6$, $Y_3 \subseteq l_1^6$ *and both have maximal projection constant*

$$\lambda(X_3) = \lambda(Y_3) = \lambda(Y_3, l_1^6) = G_{\mathbb{R}}(3) = F(3,6) = \phi.$$

A similar example exists for $\mathbb{K} = \mathbb{C}$ *and* $n = 2$.

Clearly Y_3 and X_3 are non-isometric, Y_3 having 12 regular pentagons and 20 regular triangles as its faces. There are infinitely many non-isometric spaces with unit balls between K and D having maximal projection constant ϕ.

Known examples of n-dimensional spaces with maximal projection constant are often realized as subspaces of l_∞^N where $N \sim n^2$. In this kind of situation we can always find many non-isometric spaces with extremal projection constant, n being sufficiently large.

THEOREM 3. *Let* $n > 2$ *and* $8 \le N \le e^{\sqrt{n}/(8e)}$. *If* $X_n \subseteq l_\infty^N$ *is an* n*-dimensional space with maximal projection constant*, $\lambda(X_n) = g_{\mathbb{K}}(n)$, *there are infinitely many mutually non-isometric* n*-dimensional spaces* Y_n *with*

$$\lambda(X_n) = \lambda(Y_n) = g_{\mathbb{K}}(n).$$

These spaces are constructed by probabilistic methods. There is, however, an asymptotic sequence of spaces defined more explicitly exhibiting a similar property to the one in Proposition 2.

PROPOSITION 4. *Let* $n = p^m$ *be an odd prime power and set* $N = n^2 - n + 1$. *Then there exist complex* n*-dimensional subspaces of* $\mathbb{C}^N$, *which* *we call X_n and Y_n when considered as subspaces of l_∞^N and l_1^N, respectively, satisfying*:

(a) *X_n and Y_n have extremal relative projection constant* $\lambda(X_n, l_\infty^N) = \lambda(Y_n, l_1^N) = f_{\mathbb{C}}(n,N) = F(n,N)$,

(b) *X_n and Y_n are non-isometric. In fact, the Banach–Mazur distance to l_2^n satisfies* $d(X_n, l_2^n) = \sqrt{n}$, $d(Y_n, l_2^n) \le \sqrt{2}$.

We note that the *absolute* projection constants of X_n are *almost* the maximal possible ones since $\lambda(X_n) \le G(n)$ always holds and $G(n) - F(n,N) \le 1/(2n^{3/2})$, $F(n,N) \ge \sqrt{n} - 1/(2\sqrt{n})$.

2. Characterization of extremal cases. The proofs of our theorems rely on the following duality result:

Proposition 5. *Let $N \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$, $n < N$. Then*

$$\begin{aligned} (2.1)\qquad &\sup\{\lambda(Z_n, l_\infty^N) \mid Z_n \text{ is an } n\text{-dimensional subspace of } l_\infty^N\} \\ &= n \sup\Big\{ \sum_{s,t=1}^{N} \mu_s \mu_t \,\Big|\, |\langle x_s, x_t\rangle| \Big\} =: \Lambda \end{aligned}$$

where the second supremum is taken over all discrete probability measures $\mu = (\mu_s)_{s=1}^N$ on $\{1,\ldots,N\}$ or $\mathbb{N}$, $\|\mu\|_1 = 1$, and over all sets of vectors $x_s \in S^{n-1}(\mathbb{K})$, $s = 1,\ldots,N$, such that

$$\mathrm{Id}_n = n \sum_{t=1}^{N} \mu_t \langle \cdot, x_t\rangle x_t \qquad (\textit{on } \mathbb{K}^n). \tag{2.2}$$

Both suprema are, in fact, maxima. Given extremal elements (x_t, μ_t) attaining Λ, let $S := \operatorname{supp}\mu$ and $M := |S| \le N$. Then an n-dimensional space $X_n \subseteq l_\infty^M$ with maximal projection constant Λ is given by its norm

$$\|x\| := \sup_{s\in S} |\langle x, x_s\rangle|, \qquad x \in \mathbb{K}^n.$$

The dual unit ball of X_n is the absolutely convex hull of the vectors $(x_s)_{s\in S}$. Further, $\Lambda = \sum_{t=1}^N \mu_t |\langle x_s, x_t\rangle|$ is independent of $s \in S$, and the formula $u = (\mathrm{sgn}(\langle x_s, x_t\rangle)\mu_t)_{s,t\in S}$ defines a map on l_∞^M with $u|_{X_n} = (\Lambda/n)\,\mathrm{Id}_{X_n}$.

Proposition 5 is essentially a consequence of proofs in [KT2] except for some lemma which was formulated there under an additional but unnecessary condition. To formulate the improved version, for $n \in \mathbb{N}$ and $N \in \mathbb{N} \cup \{\infty\}$ set $T = \{1,\ldots,N\}$ and

$$\varphi(n,T) = \sup \sum_{s,t\in T} \Big| \sum_{j=1}^{n} f_j(s)\overline{f_j(t)} \Big| \mu_s \mu_t$$

where the supremum is extended over all probability measures $\mu = (\mu_s)_{s=1}^N$ on T and all orthonormal systems $(f_j)_{j=1}^n$ of length n in $l_2^N(\mu)$.

LEMMA 6. *Assume that* $\mu^\circ = (\mu_s^\circ)_{s=1}^N$ *and* $f_1^\circ, \ldots, f_n^\circ$ *attain the supremum*

$$\varphi(n,T) = \sum_{s,t\in T} \Big| \sum_{j=1}^n f_j^\circ(s)\overline{f_j^\circ(t)}\Big| \mu_s^\circ \mu_t^\circ.$$

Then for all $l, m = 1, \ldots, n$, *there exists a sequence* $1 \le l_0, \ldots, l_k \le n$ *such that* $l_0 = l$, $l_k = m$ *and for any* $1 \le r \le k$ *we have*

$$\operatorname{supp} f_{l_{r-1}}^\circ \cap \operatorname{supp} f_{l_r}^\circ \cap \operatorname{supp} \mu^\circ \ne \emptyset.$$

Proof. For $0 < \tau \le 1$, let $\mathcal{M}_\tau$ denote the set of all discrete measures on T such that $\mu(T) = \tau$. By $\varphi(n,T,\tau)$ we denote the supremum analogous to $\varphi(n,T)$ except that $\mu \in \mathcal{M}_\tau$, the orthonormalization of the f_j's being taken with respect to μ/τ. Then $\varphi(n,T) = \varphi(n,T,1)$ and $\varphi(n,T,\tau) = \tau^2\varphi(n,T,1)$. Further $\varphi(n_1,T_1,1) \le \varphi(n,T,1)$ if $n_1 \le n$ and $T_1 \subseteq T$. Assume that μ° and $f_1^\circ, \ldots, f_n^\circ$ attain the supremum $\varphi(n,T)$. Let $J_1 \subseteq \{1,\ldots,n\}$ be a maximal set with the following property: $J_1 = \{j_1, \ldots, j_\varrho\}$ and for every $1 < r \le \varrho$ we have

$$\operatorname{supp} f_{j_r}^\circ \cap \bigcup_{l=1}^{r-1} \operatorname{supp} f_{j_l}^\circ \cap \operatorname{supp} \mu^\circ \ne \emptyset. \tag{2.3}$$

Let $J_2 = \{1,\ldots,n\} \setminus J_1$. Moreover, put $T_1 = \bigcup_{j\in J_1} \operatorname{supp} f_j^\circ \cap \operatorname{supp} \mu^\circ$ and $T_2 = T - T_1$. Then

$$f_j^\circ(s)\mu_s = 0 \quad \text{if } (s,j) \in (T_2 \times J_1) \cup (T_1 \times J_2). \tag{2.4}$$

Indeed, for $j \in J_1$ and $s \in T_2$ this follows from the definition of T_2. For $j \in J_2$ and $s \in T_1$ this is a consequence of the maximality of J_1 since otherwise $J_1 \cup \{j\}$ would satisfy (2.3).

The definition of J_1 and an easy induction show that $m \in J_1$ if and only if there exists a finite sequence joining j_1 and m, i.e. a sequence $l_0, \ldots, l_k$ in J_1 with $l_0 = j_1$ and $l_k = m$ such that

$$\operatorname{supp} f_{l_r} \cap \operatorname{supp} f_{l_{r-1}} \cap \operatorname{supp} \mu^\circ \ne \emptyset \quad \text{for all } 1 \le r \le k.$$

If $l, m \in J_1$ are arbitrary, a similar sequence satisfying the conclusion of the lemma is obtained by concatenating sequences joining l with j_1 and j_1 with m.

Finally, we show that the maximality assumption defining J_1 implies that $J_1 = \{1,\ldots,n\}$. Let $n_i := |J_i|$ and $\tau_i := \sum_{s\in T_i} \mu_s^\circ$ for $i = 1,2$. Thus $n = n_1 + n_2$ and $\tau_1 + \tau_2 = 1$. For $I \subseteq J := \{1,\ldots,n\}$ and $U \subseteq T$ define

$$\phi(I,U) = \sum_{s,t\in U} \Big| \sum_{i\in I} f_i^\circ(s)\overline{f_i^\circ(t)}\Big| \mu_s^\circ \mu_t^\circ.$$

Then, by (2.4),

$$\begin{aligned}\varphi(n,T) &= \phi(J,T) = \phi(J_1,T_1) + \phi(J_2,T_2)\\ &\le \varphi(n_1,T_1,\tau_1) + \varphi(n_2,T_2,\tau_2) = \tau_1^2\varphi(n_1,T_1) + \tau_2^2\varphi(n_2,T_2)\\ &\le (\tau_1^2+\tau_2^2)\varphi(n,T).\end{aligned}$$

Thus $\tau_1^2+\tau_2^2 \ge 1$ and hence $\tau_1 = 1$, $\tau_2 = 0$ since $\tau_1+\tau_2 = 1$, and $\tau_1 > 0$ since $J_1 \neq \emptyset$ and $T_1 \neq \emptyset$. This implies that $T_1 = \operatorname{supp}\mu^\circ$ and, by the maximality of J_1, that $J_1 = J$. ∎

We will need the nuclear norm ν on spaces of finite rank operators between Banach spaces and the fact that the trace of a finite rank operator $T \in \mathcal{L}(X)$ can be estimated by $|\operatorname{tr}(s)| \le \nu(s)$; cf. e.g. [TJ].

Proof of Proposition 5. We indicate how the statements in Proposition 5 follow from the results and proofs in [KT2] and Lemma 6.

Let $T = \{1,\dots,N\}$ if $N \in \mathbb{N}$ or $T = \mathbb{N}$ if $N = \infty$. Let $X_n \subseteq l_\infty^N$ be an n-dimensional subspace. By Proposition 2.2 of [KT2], the left side of (2.1) is bounded by $\varphi(n,T)$ since $\lambda(X_n, l_\infty^N) \le \varphi(n,T)$ is proved there using a duality argument. The supremum in $\varphi(n,T)$ is attained (see Section 4 of [KT2]), say by a probability measure $\mu^\circ = (\mu_s^\circ)_{s=1}^N$ and a μ-orthonormal system $f_1^\circ,\dots,f_n^\circ \in \mathbb{K}^N$. Let $f^\circ := (\sum_{j=1}^n |f_j|^2)^{1/2} \in \mathbb{K}^N$ denote the square function. It is shown in Proposition 3.1 of [KT2] by use of Lagrange multipliers that the square function f° is constant μ-a.e.; then from the orthonormality, $f^\circ(s) = \sqrt{n}$ if $\mu_s^\circ \neq 0$. (If $\mu_s^\circ = 0$, nothing can be said about $f^\circ(s)$; in general $f^\circ(s)$ may be non-zero, contrary to what is stated in [KT2].) The proof there relies on an analogue of Lemma 6 derived there under an additional assumption. The crucial point where this is needed is (3.28) of [KT2]. The notation used there is $z_{sk} := f_k^\circ(s)\sqrt{\mu_s}$. The Lagrange equations of the first kind yield an eigenvalue equation for the map

$$u := \Big(\operatorname{sgn}\Big(\sum_{k=1}^n f_k^\circ(s)\overline{f_k^\circ(t)}\Big)\mu_t^\circ\Big)_{s,t\in T} : \mathbb{K}^N \to \mathbb{K}^n \tag{2.5}$$

of the form $\mu^\circ(s)(uf_k^\circ(s) - \alpha_k f_k^\circ(s)) = 0$ for $k = 1,\dots,n,\ s \in T$ (which is a reformulation of (3.16) in [KT2]). Thus with $S := \operatorname{supp}\mu^\circ$, $(uf_k^\circ)(s) = \alpha_k f_k^\circ(s)$ for $s \in S$. By (3.27) and the next two lines of [KT2], for each $1 \le l,m \le n$ and $s \in T$ one has

$$0 = (\alpha_m - \alpha_l) z_{sm} z_{sl} = (\alpha_m - \alpha_l) f_m^\circ(s) f_l^\circ(s)\mu_s^\circ. \tag{2.6}$$

By Lemma 6 there exists a sequence $l = l_0, l_1, \dots, l_k = m$ with

$$\operatorname{supp} f_{l_{r-1}}^\circ \cap \operatorname{supp} f_{l_r}^\circ \cap \operatorname{supp}\mu^\circ \neq \emptyset$$

for all $1 \le r \le k$. Thus (2.6) implies that $\alpha_l = \alpha_{l_0} = \alpha_{l_1} = \ldots = \alpha_{l_k} = \alpha_m$. Hence all values α_k coincide, $\alpha_1 = \ldots = \alpha_n =: \alpha$; this is (3.28) of [KT2].

The second Lagrange equation (3.15) in [KT2] means that

$$\sum_{t\in T}\Big|\sum_{k=1}^{n} f_k^\circ(s)\overline{f_k^\circ(t)}\Big|\mu_t^\circ = \varphi(n,T) = \Lambda \tag{2.7}$$

is constant in $s \in S$. Multiplying $uf_k^\circ = \alpha f_k^\circ$ pointwise by $\overline{f_k^\circ}$ and summing over $k = 1,\dots,n$, one deduces from (2.5) and (2.7), using $z \operatorname{sgn} z = |z|$ for $z = \sum_{k=1}^n f_k^\circ(s)\overline{f_k^\circ(t)}$, that for $s \in S$,

$$\begin{aligned}\alpha f^\circ(s)^2 &= \alpha\sum_{k=1}^{n} |f_k^\circ(s)|^2 = \alpha\sum_{k=1}^{n} uf_k^\circ(s)\cdot\overline{f_k^\circ(s)}\\ &= \sum_{t\in T}\Big|\sum_{k=1}^{n} f_k^\circ(s)\overline{f_k^\circ(t)}\Big|\mu_t^\circ = \Lambda.\end{aligned}$$

Hence the square function f° is constant μ°-a.e., $f^\circ(s) = \sqrt{\Lambda/\alpha}$ for $s \in S$. Since the f_k's were orthonormal, $f^\circ(s) = \sqrt{n}$ for $s \in S$. Hence $\alpha = \Lambda/n$ and $uf_k = \Lambda/n \cdot f_k$ for $k = 1,\dots,n$. Introducing $x_t = n^{-1/2}(f_k^\circ(t))_{k=1}^n \in \mathbb{K}^n$, we have $x_t \in S^{n-1}(\mathbb{K})$ and for any $s \in S$,

$$\varphi(n,T) = n\sum_{t\in T}|\langle x_s, x_t\rangle|\mu_t^\circ = n\sum_{s,t\in T}|\langle x_s, x_t\rangle|\mu_t^\circ\mu_s^\circ.$$

The vectors x_s satisfy (2.2) since the f_k° are μ-orthonormal. This proves "$\le$" in (2.1).

As for the reverse inequality, the Lagrange multiplier approach outlined above (with details in [KT2]) yields a sequence of points $x_s \in S^{n-1}(\mathbb{K})$ and a probability measure μ with (2.2) and

$$\Lambda = n\sum_{s,t\in T}|\langle x_s, x_t\rangle|\mu_s\mu_t$$

and an operator u similar to (2.5), with $S := \operatorname{supp}\mu$, $M = |S| \le N$,

$$u = (\operatorname{sgn}(\langle x_s, x_t\rangle)\mu_t)_{s,t\in S}\cdot \mathbb{K}^M \to \mathbb{K}^M$$

with $(uf_k)(s) = (\Lambda/n)f_k(s)$ for $k = 1,\dots,n$, $s \in S$. Consider the space X_n spanned by the vectors $(f_k(s))_{s\in S} \subseteq l_\infty^M$ in l_∞^M. Then $u|_{X_n} = (\Lambda/n)\operatorname{Id}_{X_n}$, and for any projection $P : l_\infty^N \to X_n$,

$$\Lambda = \operatorname{tr}(u|_{X_n}) \le \nu(u|_{X_n}) \le \|P\|\nu(u) = \|P\|$$

since $\nu(u) = \sum_{t=1}^N \mu_t = 1$. Hence $\Lambda \le \lambda(X_n, l_\infty^N)$, which proves "$\ge$" in (2.1). ∎

Proof of Theorem 1. Assume that $X_n \subseteq l_\infty^N$ has maximal projection constant among n-dimensional subspaces of N-dimensional superspaces. By Proposition 5, we find points $x_s \in S^{n-1}$ and $\mu_s \ge 0$, $s = 1,\dots,N$, with

$\sum_{s=1}^{N} \mu_s = 1$ satisfying (2.2) such that

$$\lambda(X_n) = \lambda(X_n, l_\infty^N) = \Lambda = \sum_{t=1}^{N} \mu_t |\langle x_s, x_t \rangle|, \qquad s \in S := \operatorname{supp} \mu.$$

Let $M := |S| \le N$. By Proposition 5, too, suppose $u = (\operatorname{sgn}(\langle x_s, x_t \rangle)\mu_t)_{s,t\in S} : l_\infty^M \to l_\infty^M$ maps $\widetilde{X}_n$ into itself where $\widetilde{X}_n = \operatorname{span}(f_1, \ldots, f_n)$, $f_j = (x_{sj})_{s\in S} \in \mathbb{K}^M$ for $j = 1, \ldots, n$, and in fact $u|_{\widetilde{X}_n} = (\Lambda/n)\operatorname{Id}_{\widetilde{X}_n}$. (If $\operatorname{supp}\mu = \{1, \ldots, N\}$, we may take $X_n = \widetilde{X}_n$.) Hence $\operatorname{tr}(u|_{\widetilde{X}_n}) = \Lambda$ and $\nu(u) = \sum_{t\in S} \sup_s |u_{st}| = \sum_{t\in S} \mu_s = 1$. For any projection $Q : l_\infty^M \to l_\infty^M$ onto $\widetilde{X}_n$,

$$\Lambda = \operatorname{tr}(u|_{\widetilde{X}_n}) \le \nu(u|_{\widetilde{X}_n}) \le \|Q\|\nu(u) = \|Q\|,$$

with the $l_2^M(\mu)$-orthogonal projection P attaining $\|P\| = \Lambda$.

Consider now $\widetilde{X}_n$ as a subspace of $l_1^M(\mu)$, i.e. $\mathbb{K}^M$ equipped with the norm given by $\|x\|_{1,\mu} = \sum_{t\in S} \mu_t |\langle x, x_t\rangle|$, and denote this space by Y_n. Let $D_\mu : l_1^M(\mu) \to l_1^M$ be the diagonal map $(y_s) \mapsto (\mu_s y_s)$; it is an isometry. The map

$$l_1^M(\mu) \xrightarrow{D_\mu} l_1^M \xrightarrow{u^*} l_1^M \xrightarrow{D_\mu^{-1}} l_1^M(\mu)$$

has as its matrix representation $(\mu_s^{-1} u_{ts} \mu_t)_{s,t\in S} = (u_{st})_{s,t\in S} = u$ since $u_{st} = \operatorname{sgn}(\langle x_s, x_t\rangle)\mu_t$. Since $\nu(u^*)_{l_1^M} = \nu(u)_{l_\infty^M} = 1$ and D_μ is an isometry, $\nu(u : l_1^M(\mu) \to l_1^M(\mu)) = 1$. Hence as above, for any projection $Q : l_1^M(\mu) \to l_1^M(\mu)$ onto Y_n,

$$\Lambda = \operatorname{tr}(u|_{Y_n}) \le \nu(u|_{Y_n}) \le \|Q\|\nu(u) = \|Q\|.$$

Thus $\lambda(Y_n, l_1^M(\mu)) \ge \Lambda = \lambda(X_n, l_\infty^N)$. But $\lambda(X_n)$ was maximal among n-dimensional subspaces of N- (hence also for M-) dimensional superspaces. Hence $\lambda(Y_n, l_1^M(\mu)) \le \Lambda$ holds as well. Taking $Z_n = D_\mu(Y_n) \subseteq l_1^M$, we can also realize such a space as a subspace of l_1^M, $\lambda(Z_n, l_1^M) = \lambda(\widetilde{X}_n, l_\infty^M) = \lambda(X_n) = \Lambda$. ∎

In general, X_n will not be isometric to Y_n or Z_n except for $\mathbb{K} = \mathbb{R}$, $n = 2$ when these spaces have the regular hexagon as their unit ball. Now let us consider the three-dimensional real case.

Proof of Proposition 2. For $\mathbb{K} = \mathbb{R}$, $n = 3$ we have $G(3) = (1 + \sqrt{5})/2 =: \Phi$. The space $X_3 \subseteq l_\infty^6$ having as its unit ball the dodecahedron attains this bound, $\lambda(X_3) = \Phi$; cf. [KT2]. The dual unit ball, the icosahedron, is the convex hull of its six equiangular diagonals $x_1, \ldots, x_6 \in S^2 \subset \mathbb{R}^3$ given by the vectors

$$c\begin{pmatrix} \phi \\ \pm 1 \\ 0 \end{pmatrix}, c\begin{pmatrix} 0 \\ \phi \\ \pm 1 \end{pmatrix}, c\begin{pmatrix} \pm 1 \\ 0 \\ \phi \end{pmatrix}, \qquad c := \frac{1}{\sqrt{\phi + 2}},$$

with $|\langle x_s, x_t\rangle| = 1/\sqrt{5}$ for $1 \le s,t \le 6$. Thus $\|x\|_{X_3} = \sup_{1\le s\le 6} |\langle x, x_s\rangle|$. Take $\mu_s = 1/6$ for $s = 1,\dots,6$. Then $3\cdot\sum_{t=1}^6 \mu_t|\langle x_s, x_t\rangle| = \phi$; this attains the sup in (2.1), (2.2) being satisfied. The map $x \mapsto (\langle x, x_s\rangle)_{s=1}^6$ realizes the isometric imbedding $X_3 \hookrightarrow l_\infty^6$. The "extremal" map $u : l_\infty^N \to l_\infty^N$ used in the proof of Theorem 1 is in this case

$$u = (\operatorname{sgn}\langle x_s, x_t\rangle)\mu_t)_{s,t=1}^6 : l_\infty^6 \to l_\infty^6.$$

By the same proof of Theorem 1, the same linear space, but considered as a subspace of l_1^6, denoted by $Y_3 \subseteq l_1^6$, has the same projection constant: $\lambda(Y_3) = \lambda(Y_3, l_1^6) = \phi$. This holds since u is symmetric and $D_\mu = \frac{1}{6}\,\mathrm{Id}_6$. However, the unit ball of Y_3 is not the dodecahedron; it has 12 regular pentagons and 20 regular triangles as its faces and thus is not isometric to X_3: The norm on Y_3 is given by $\|x\|_{Y_3} = \frac{1}{6}\sum_{s=1}^6 |\langle x, x_s\rangle|$; writing it as

$$\|x\|_{Y_3} = \frac{1}{6}\sup_{\varepsilon_s=\pm1}\Big|\Big\langle x, \sum_{s=1}^6 \varepsilon_s x_s\Big\rangle\Big|,$$

one finds that only 16 combinations $(\varepsilon_1,\dots,\varepsilon_6)$ of signs are needed to represent the norm and thus Y_3 is isometrically imbeddable into l_∞^{16}. The choice $\varepsilon_s = \operatorname{sgn}(\langle x_s, x_t\rangle) = 6u_{st}$ for fixed t yields

$$\frac{1}{6}\sum_{s=1}^6 \varepsilon_s x_s = \frac{\phi}{3}x_s \qquad (s = 1,\dots,6).$$

Thus, eliminating the factor $\phi/3$, the points $x_1,\dots,x_6$ are again needed to imbed Y_3 into l_∞; in addition, one needs the 10 vectors (also after multiplying by $3/\phi$)

$$c\begin{pmatrix}\phi\\0\\\pm\phi^{-1}\end{pmatrix},\ c\begin{pmatrix}\pm\phi^{-1}\\\phi\\0\end{pmatrix},\ c\begin{pmatrix}0\\\pm\phi^{-1}\\\phi\end{pmatrix},\ c\begin{pmatrix}1\\\pm1\\\pm1\end{pmatrix},\qquad c := \frac{1}{\sqrt{\phi+2}}.$$

If these 16 vectors are called $x_1,\dots,x_{16}$, one has

$$\frac{3}{\phi}\,\|x\|_{Y_3} = \sup_{1\le s\le 16}|\langle x, x_s\rangle|, \qquad Y_3 \hookrightarrow l_\infty^{16} \text{ isometrically.}$$

We remark that $x_7,\dots,x_{16}$ are one half of the vertices of the regular dodecahedron and thus the unit ball of Y_3 is the intersection of the dodecahedron and a multiple of the icosahedron yielding the above-mentioned face structure. In fact, if the vertices of the icosahedron are chosen to be the midpoints of the faces of the dodecahedron, one has to multiply this icosahedron by ϕ and intersect it with the dodecahedron to get the unit ball of Y_3.

In the case of complex 2-dimensional spaces, there are four equiangular vectors

$$\begin{pmatrix}1\\0\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \sqrt{2}\,w^j\end{pmatrix}, \qquad j=0,1,2,\ w=\exp(2\pi i/3),$$

in $\mathbb{C}^2$. If we call them $z_1,\ldots,z_4\in\mathbb{C}^2$, then the space with norm

$$\|z\|_{X_2}=\sup_{1\le s\le 4}|\langle z,z_s\rangle|$$

has maximal projection constant among 2-dimensional complex spaces with $\lambda(X_2)=(1+\sqrt{3})/2$, $X_2\subseteq l_\infty^4$ (cf. [KT2]). As a subspace Y_2 of l_1^4, it also has $\lambda(Y_2)=\lambda(Y_2,l_1^4)=(1+\sqrt{3})/2$, but is not isometric to X_2 since the Banach–Mazur distances to Hilbert space satisfy

$$d(X_2,l_2^2)=\sqrt{3/2}\ne d(Y_2,l_2^2)=(1+\sqrt{3})/\sqrt{6}.$$

The distance ellipsoid here is the standard euclidean ball in $\mathbb{C}^2$ by symmetry reasons, the values of $\sqrt{3/2}$ and $(1+\sqrt{3})/\sqrt{6}$ are obtained by calculating the maximum and minimum of $\|z\|_2$ subject to $\|z\|_{X_2}=1$ or $\|z\|_{Y_2}=1$; the quotient of these maxima and minima then gives the above distance values. The maxima and minima are attained at 4 points each, up to factors $e^{i\theta}$. We would like to thank Prof. A. Pełczyński for some stimulating discussions on this topic. ∎

Remark. The fact that the space Y_3, imbedded into l_∞^{16}, allows no projection of norm $<\phi$, can also be checked by a map $\widetilde{u}:l_\infty^{16}\to l_\infty^{16}$ similar to u for $X_3\subseteq l_\infty^6$. One may just take

$$\widetilde{u}=(\operatorname{sgn}\langle x_s,x_t\rangle\mu_t)_{s,t=1}^{16}:l_\infty^{16}\to l_\infty^{16}$$

where $(x_s)_{s=1}^{16}\subset\mathbb{R}^3$ are the 16 vectors given in the previous proof and where $\mu_t=0$ for all $t=7,\ldots,16$. Again $\nu(\widetilde{u})=1$ and $\widetilde{u}|_{Y_3}=(\phi/3)\,\mathrm{Id}_{Y_3}$. Thus $\widetilde{u}$ has 10 columns of zeros; for the rows $s=7,\ldots,16$ Proposition 5 gives no information on the square function $\|x_s\|_2^2$ as compared to $\|x_s\|_2^2=1$ for $s=1,\ldots,6$. In fact, $\|x_s\|_2^2=3/(\phi+2)<1$ for $s=7,\ldots,16$.

Proof of Theorem 3. Let $X_n\subseteq l_\infty^N$ be an n-dimensional space with maximal projection constant for n-dimensional spaces, $\lambda(X_n)=g(n)$, and where $N\le e^{\sqrt{n}/(8e)}$ holds. By Proposition 5, we conclude that there are *unit* vectors $(x_s)_{s=1}^N$ in l_2^n, $x_s\in S^{n-1}$, and a probability measure $\mu=(\mu_s)_{s=1}^N$ on $\{1,\ldots,N\}$ such that:

- Using (1.4) we identify X_n with the space $\mathbb{K}^n$, equipped with the norm $\|x\|=\sup_{1\le s\le N}|\langle x,x_s\rangle|$. Then the dual unit ball is the absolutely convex hull of the vectors x_s.
- $\Lambda:=\lambda(X_n)=n\sum_{t=1}^N\mu_t|\langle x_s,x_t\rangle|$ for all $s=1,\ldots,N$.
- $\mathrm{Id}_n=n\sum_{t=1}^N\mu_t\langle\cdot,x_t\rangle x_t$ on l_2^n.
- $u|_{X_n}=(\Lambda/n)\,\mathrm{Id}_n$, where $u=(\operatorname{sgn}(\langle x_s,x_t\rangle)\mu_t)_{s,t=1}^N:l_\infty^N\to l_\infty^N$.

We will assume for simplicity and without loss of generality that all μ_s are > 0. Thus X_n in l_∞^N is spanned by the vectors $f_j := (x_{sj})_{s=1}^N$ which are, up to the factor $\sqrt{n}$, orthonormal vectors in $l_2^N(\mu)$. Since $\mathrm{tr}(u|_{X_n}) = \Lambda$ and the nuclear norm of u in l_∞^N is 1, $\nu(u) = \sum_{t=1}^N \mu_t = 1$, any projection $P : l_\infty^N \to X_n$ must have norm $\geq \Lambda$,

$$\Lambda = \mathrm{tr}(u|_{X_n}) \leq \nu(u|_{X_n}) \leq \|P\|\nu(u) = \|P\|.$$

We will construct a vector x_{N+1} which is not in the absolutely convex hull of $x_1, \dots, x_N$ such that the map $x \mapsto (\langle x, x_s\rangle)_{s=1}^{N+1}$ yields another n-dimensional extremal space $Y_n \subseteq l_\infty^{N+1}$, $\lambda(X_n) = \lambda(Y_n)$, which is not isometric to X_n since the unit ball of Y_n has more faces than the one of X_n.

Let $\alpha := 1/(2\sqrt{\log N})$. For any $y \in S^{n-1}$ and $t \in \{1, \dots, N\}$ let

$$\varepsilon_t(y) := \sqrt{n}\,\alpha\langle y, x_t\rangle.$$

Then

$$z(y) := \frac{n}{\Lambda}\sum_{t=1}^N \varepsilon_t(y)\mu_t x_t = \frac{\sqrt{n}}{\Lambda}\,\alpha n \sum_{t=1}^N \mu_t\langle y, x_t\rangle x_t = \frac{\sqrt{n}}{\Lambda}\,\alpha y,$$

so $\|z(y)\|_2 = \sqrt{n}\,\alpha/\Lambda$. We estimate the average norm of $z(y)$ in X_n. For this, let m be the normalized Lebesgue measure on S^{n-1}. Take $p = 2$. Then

$$\begin{aligned}\int_{S^{n-1}} \|z(y)\|_{X_n}\,dm(y) &= \int_{S^{n-1}} \sup_{1\leq s\leq N} |\langle z(y), x_s\rangle|\,dm(y)\\ &\leq \int_{S^{n-1}} \Big(\sum_{s=1}^N |\langle z(y), x_s\rangle|^p\Big)^{1/p} dm(y) \leq \Big(\sum_{s=1}^N \int_{S^{n-1}} |\langle z(y), x_s\rangle|^p\,dm(y)\Big)^{1/p}\\ &\leq N^{1/p}\|z(y)\|_2\Big(\int_{S^{n-1}} |y_1|^p\,dm(y)\Big)^{1/p} \leq N^{1/p}\,\frac{\sqrt{n}}{\Lambda}\,\alpha\sqrt{\frac{p}{n}}.\end{aligned}$$

Here we used the generalized triangle inequality and the rotation invariance of m. The moments $(\int_{S^{n-1}} |y_1|^p\,dm(y))^{1/p}$ are explicitly known in terms of Gamma functions: they can be estimated by $\sqrt{p/n}$ for $p > 2$. Choosing $p = \log N$, by definition of α we get

$$\int_{S^{n-1}} \|z(y)\|_{X_n}\,dm(y) \leq \frac{e}{\Lambda}.$$

By Chebyshev's inequality,

$$m\Big\{y \in S^{n-1} \;\Big|\; \|z(y)\|_{X_n} < 2\,\frac{e}{\Lambda}\Big\} > \frac{1}{2}. \tag{2.8}$$

Since $2e/\Lambda < n/\Lambda^2\alpha^2$ by assumption on N, for these vectors $z(y)$ one has

$$\|z(y)\|_{X_n} < \frac{2e}{\Lambda} < \frac{n}{\Lambda^2}\,\alpha^2 = \|z(y)\|_2^2 \leq \|z(y)\|_{X_n}\|z(y)\|_{X_n^*}.$$

Hence $\|z(y)\|_{X_n^*} > 1$, which means that $z(y)$ is not in the absolutely convex hull of the vectors $x_1, \ldots, x_N$. On the other hand, we want to guarantee that the values $\varepsilon_t(y)$ can be bounded by 1 uniformly in t. Integration by polar coordinates yields the following well known tail estimate for linear functionals (t being fixed):

$$m\{y \in S^{n-1}(\mathbb{K}) \mid |\langle y, x_t\rangle| > \beta\}$$
$$= \begin{cases} \int_\beta^1 (1-u^2)^{(n-3)/2}\, du \Big/ \int_0^1 (1-u^2)^{(n-3)/2}\, du \le e^{-n\beta^2/2} & (\mathbb{K}=\mathbb{R}), \\ \int_\beta^1 u(1-u^2)^{n-2}\, du \Big/ \int_0^1 u(1-u^2)^{n-2}\, du & \\ \qquad = (1-\beta^2)^{n-1} \le e^{-n\beta^2/2} & (\mathbb{K}=\mathbb{C}) \end{cases}$$

for $n \ge 2$, $0 < \beta \le 1$. (For integration in the complex case, $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$.) Choosing $\beta = 1/(\alpha\sqrt{n})$, we get

$$m\{y \in S^{n-1} \mid |\langle y, x_t\rangle| > 1/(\alpha\sqrt{n})\} \le e^{-\alpha^{-2}/2} \le 1/N^2 < 1/(2N).$$

Letting t vary from 1 to N, one finds for the complement

$$(2.9) \quad m\{y \in S^{n-1} \mid |\langle y, x_t\rangle| \le 1/(\alpha\sqrt{n}) \text{ for all } t = 1, \ldots, N\} > 1/2.$$

By (2.8) and (2.9) we can find a vector $\overline{y} \in S^{n-1}$ such that

- $\|z(\overline{y})\|_{X_n} < 2e/\Lambda$, implying $\|z(\overline{y})\|_{X_n^*} > 1$.
- $|\varepsilon_t(\overline{y})| = \sqrt{n}\,\alpha|\langle \overline{y}, x_t\rangle| \le 1$ for all $t = 1, \ldots, N$.

Put $x_{N+1} = z(\overline{y})$. Then $\|x_{N+1}\|_2 = \sqrt{n}\,\alpha/\Lambda \le 2\alpha < 1$ (Λ is close to $\sqrt{n}$) and $x_{N+1} \notin$ absolutely convex hull of $(x_1, \ldots, x_N)$. Define Y_n by $\|x\|_{Y_n} := \sup_{1\le s\le N+1} |\langle x, x_s\rangle|$, $x \in \mathbb{K}^n$, and let $\mu_{N+1} = 0$. Then $\|x\|_{Y_n} \ge \|x\|_{X_n}$ and there are points $x \in \mathbb{K}^n$ with $\|x\|_{Y_n} > \|x\|_{X_n}$. The unit ball of Y_n thus has more faces than X_n, and Y_n is not isometric to X_n. Let $\widetilde{f}_j = (x_{sj})_{s=1}^{N+1}$. Then $Y_n = \mathrm{span}(\widetilde{f}_1, \ldots, \widetilde{f}_n) \subseteq l_\infty^{N+1}$; these vectors are homothetic to an orthonormal basis in Y_n as a subspace of $l_2^{N+1}(\mu)$. We define an extension $\widetilde{u} : l_\infty^{N+1} \to l_\infty^{N+1}$ of the map $u : l_\infty^N \to l_\infty^N$ by putting $\widetilde{u}_{st} = u_{st}$ if $1 \le s, t \le N$, $\widetilde{u}_{s,N+1} = 0$ for $s = 1, \ldots, N+1$ and $\widetilde{u}_{N+1,t} = \varepsilon_t(\overline{y})\mu_t$ for $t = 1, \ldots, N$. Then $(\widetilde{u}\widetilde{f}_j)_s = (uf_j)_s = (\Lambda/n)(f_j)_s = (\Lambda/n)(\widetilde{f}_j)_s$ for $s = 1, \ldots, N$, and by definition of $z(\overline{y})$,

$$(\widetilde{u}\widetilde{f}_j)_{N+1} = \sum_{t+1}^{N} \widetilde{u}_{N+1,t}(\widetilde{f}_j)_t = \sum_{t=1}^{N} \varepsilon_t(\overline{y})\mu_t x_{tj}$$
$$= \frac{\Lambda}{n}\, z(\overline{y})_j = \frac{\Lambda}{n}\, x_{N+1,j} = \frac{\Lambda}{n}\, (\widetilde{f}_j)_{N+1}.$$

We thus found that $\widetilde{u}|_{Y_n} = (\Lambda/n)\,\mathrm{Id}_{Y_n}$. The nuclear norm of $\widetilde{u} : l_\infty^{N+1} \to l_\infty^{N+1}$

is (in view of $\mu_{n+1} = 0$)

$$\nu(\widetilde{u}) = \sum_{t=1}^{N+1} \sup_{1\le s\le N+1} |\widetilde{u}_{st}| = \sum_{t=1}^{N} \mu_t = 1.$$

If $P : l_\infty^{N+1} \to Y_n$ is any projection,

$$\Lambda = \operatorname{tr}(\widetilde{u}|_{Y_n}) \le \nu(\widetilde{u}|_{Y_n}) \le \|P\|\nu(\widetilde{u}) = \|P\|.$$

Thus $\Lambda(Y_n) \ge \Lambda = \lambda(X_n)$. However, $\lambda(X_n)$ was maximal among all n-dimensional spaces. Hence $\lambda(Y_n) = \lambda(X_n) = \Lambda$ and Y_n is not isometric to X_n. Obviously, the construction yields infinitely many non-isometric spaces with maximal projection constant Λ. ■

Proof of Proposition 4. (a) As subspaces of l_∞^N, these spaces have already been considered in [KT1]. For $n = p^m + 1$ there exist numbers $d_1, \ldots, d_n \in \{0, \ldots, N-1\}$ such that the differences $d_i - d_j$ modulo N are all different and yield all $n(n-1) = N-1$ integers between 1 and $N-1$; see [HR]. Define $x_s := n^{-1/2}(\exp((2\pi i/N)d_j s))_{j=1}^n \in S^{n-1}(\mathbb{C})$ for $s = 1, \ldots, N$ and let $f_j = (x_{sj})_{s=1}^N \in \mathbb{C}^n$. Then

$$\langle f_j, f_k\rangle = \sum_{s=1}^{N} \exp\biggl(\frac{2\pi i}{N}(d_j - d_k)s\biggr)/n = (N/n)\delta_{jk}$$

and hence

$$\operatorname{Id}_n = \frac{n}{N}\sum_{s=1}^{N}\langle\cdot, x_s\rangle x_s \quad (\text{on } \mathbb{C}^n).$$

The vectors $(x_s)_{s=1}^N$ are equiangular as the evaluation of $|\langle x_s, x_t\rangle|^2$ shows (see [KT1]). One finds

$$|\langle x_s, x_t\rangle| = \sqrt{n-1}/n \quad \text{for } 1 \le s \ne t \le N.$$

Let $Z_n = \operatorname{span}(f_1, \ldots, f_n) \subseteq \mathbb{C}^N$. As a subspace of l_2^N, $(\sqrt{n/N}\, f_j)_{j=1}^n$ is an orthonormal basis in Z_n, $\dim Z_n = n$, and $P := (n/N)(\langle x_s, x_t\rangle)_{s,t=1}^n$ is a projection onto Z_n (the orthogonal projection in l_2^N).

Let X_n and Y_n denote the linear space Z_n considered as a subspace of l_∞^N and l_1^N, respectively. Then

$$\lambda(Y_n) \le \|P\| = \frac{n}{N}\sup_t \sum_{s=1}^{N}|p_{st}| = \frac{n}{N}\biggl(1 + (N-1)\frac{\sqrt{n-1}}{n}\biggr) = F(n, N).$$

The last equality is verified by calculation (F is given in Section 1). Similarly $\lambda(X_n) \le F(n, N)$ since P is hermitean.

Let

$$u := (\langle x_s, x_t\rangle)_{s,t=1}^N - \biggl(1 - \frac{\sqrt{n-1}}{n}\biggr)\operatorname{Id}_N : \mathbb{C}^N \to \mathbb{C}^N.$$

Then for all $1 \le s,t \le N$,

$$|u_{st}| = \frac{\sqrt{n-1}}{n} \quad \text{and} \quad u|_{Z_n} = \left(\frac{N}{n} - 1 + \frac{\sqrt{n-1}}{n} \right) \mathrm{Id}_{Z_n} .$$

Hence the trace of u on Z_n is $\mathrm{tr}(u|_{Z_n}) = (N - n + \sqrt{n-1})$. On the other hand, the nuclear norm of the hermitean map u, considered in either l_1^N or l_∞^N, is

$$\nu(u) = \sum_{s=1}^{N} \sup_t |u_{st}| = N \frac{\sqrt{n-1}}{n}.$$

This implies, for any projection $Q : l_1^N \to Y_n$,

$$N - n + \sqrt{n-1} = \mathrm{tr}(u|_{Y_n}) \le \nu(u|_{Y_n}) \le \|Q\| \nu(u) = \|Q\| N \frac{\sqrt{n-1}}{n}.$$

Thus

$$\lambda(Y_n) \ge \frac{N - n + \sqrt{n-1}}{N} \, \frac{n}{\sqrt{n-1}} = F(n,N),$$

the last equality again being the result of a calculation. Similarly $\lambda(X_n) \ge F(n,N)$. We showed that $\lambda(X_n) = \lambda(Y_n) = F(n,N)$, which is the maximal possible value (see [KLL]).

(b) We show that the Banach–Mazur distance of X_n to l_2^n is $\sqrt{n}$. (By John's theorem, this is extremal since for any n-dimensional space Z_n one has $d(Z_n, l_2^n) \le \sqrt{n}$.) The proof is similar to the one in [KT1].

Let $\beta_j := \exp((2\pi i/N) d_j)$ for $j = 1, \dots, n$ and $I : X_n \to X_n$ be defined by $\sum_{j=1}^n a_j f_j \mapsto \sum_{j=1}^n \beta_j a_j f_j$. Clearly I is an isometry and $I^N = \mathrm{Id}$. Any inner product $[\cdot,\cdot]$ on X_n which is invariant under I is diagonal in the basis (f_j) of X_n. In fact, if

$$[x,y] = \sum_{k,l=1}^{n} t_{kl} a_k \overline{b}_l, \qquad x = \sum_{k=1}^{n} a_k f_k, \ y = \sum_{l=1}^{n} b_l f_l,$$

$[Ix, Iy] = [x,y]$ for all $x,y \in X_n$ implies $t_{kl} = t_{kl} \beta_k \overline{\beta}_l$. For $k \ne l$, clearly $\beta_k \overline{\beta}_l \ne 1$, hence $t_{kl} = 0$ for $k \ne l$.

Now let $(\cdot,\cdot)$ be an inner product on X_n which determines the Banach–Mazur distance $d = d(X_n, l_2^n)$, normalized so that

$$(1/d^2)\|x\|^2 \le (x,x) \le \|x\|^2, \qquad x \in X_n.$$

Define the inner product $[\cdot,\cdot]$ by

$$[x,y] := \frac{1}{N} \sum_{s=0}^{N-1} (I^s x, I^s y), \qquad x,y \in X_n.$$

Then $[Ix, Iy] = [x,y]$, and also

$$(1/d^2)\|x\|^2 \le [x,x] \le \|x\|^2, \qquad x \in X_n.$$

By the preceding remark, there are $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$[x, y] = \sum_{k=1}^{n} \lambda_k a_k \overline{b}_k, \qquad x = \sum_{n=1}^{n} a_k f_k,\ y = \sum_{k=1}^{n} b_k f_k.$$

Since $[x, x] \leq \|x\|^2$, $\lambda_k \leq \|f_k\|^2 = 1/n$ for all $k = 1, \ldots, n$. Let $z := \sum_{k=1}^n f_k$. Then $[z, z] = \sum_{k=1}^n \lambda_k$ and, taking $s = 0$, we find

$$\|z\| = \sup_{0 \leq s < N} \frac{1}{\sqrt{n}} \left| \sum_{k=1}^{n} \exp\left(\frac{2\pi i}{N}\ d_j s\right) \right| = \sqrt{n}.$$

So

$$d \geq \sup_{x \neq 0} \|x\| / [x, x]^{1/2} \geq \frac{\sqrt{n}}{(\sum_{k=1}^n \lambda_k)^{1/2}} \geq \sqrt{n}.$$

Since by John's theorem, $d \leq \sqrt{n}$, we find $d = \sqrt{n}$.

As a subspace of l_1^N, the distance of the space Y_n to l_2^n, however, is uniformly bounded by $\sqrt{2}$ as we now show. Thus X_n and Y_n cannot be isometric. In fact, for large n, they have very different distances to l_2^n. For $1 \leq p < \infty$ and $x \in \mathbb{C}^n$, let

$$|||x|||_p = \left(\frac{1}{N} \sum_{s=1}^{N} |\langle x, x_s \rangle|^p\right)^{1/p}.$$

We calculate the 4-norm:

$$\sum_{s=1}^{N} |\langle x, x_s \rangle|^4$$
$$= \frac{1}{n^2} \sum_{j_1, j_2, j_3, j_4 = 1}^{n} x_{j_1} \overline{x}_{j_2} x_{j_3} \overline{x}_{j_4} \sum_{s=1}^{N} \exp\left(\frac{2\pi i}{N} (d_{j_1} - d_{j_2} + d_{j_3} - d_{j_4}) s\right).$$

Since $(d_i - d_j)(N)$ for $i \neq j$ runs over all numbers from 1 to $N-1$ exactly once, $d_{j_1} - d_{j_2} + d_{j_3} - d_{j_4}$ is 0 modulo n if and only if either ($j_1 = j_2$ and $j_3 = j_4$) or ($j_1 = j_4$ and $j_2 = j_3$).

In this case, the inner sum is N, else it is 0. Hence

$$\begin{aligned} |||x|||_4 &:= \left(\frac{1}{N} \sum_{s=1}^{N} |\langle x, x_s \rangle|^4\right)^{1/4} \\ &= \frac{1}{\sqrt{n}} \Big(2 \sum_{j \neq k = 1}^{n} |x_j|^2 |x_k|^2 + \sum_{j=1}^{n} |x_j|^4\Big)^{1/4} \\ &\leq \frac{\sqrt[4]{2}}{\sqrt{n}} \Big(\sum_{j=1}^{n} |x_j|^2\Big)^{1/2} = \sqrt[4]{2}\, |||x|||_2 \end{aligned}$$

where the last equality is easy. A standard interpolation argument now yields the distance estimate of Y_n to l_2^n: By Hölder's inequality

$$\|x\|_2 \le \|x\|_1^{1/3} \, \|x\|_4^{2/3} \le (\sqrt[4]{2})^{2/3} \|x\|_1^{1/3} \, \|x\|_2^{2/3},$$

and thus $\|x\|_2 \le \sqrt{2}\,\|x\|_1$. Since trivially $\|x\|_1 \le \|x\|_2$ holds, we find that $d(Y_n, l_2^n) \le \sqrt{2}$. ∎

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