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# Gelfand numbers and metric entropy of convex hulls in Hilbert spaces

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*Dedicated to Aleksander Pełczyński on his 70th birthday*

**Abstract.** For a precompact subset $K$ of a Hilbert space we prove the following inequalities:

$$n^{1/2} c_n(\mathrm{cov}(K)) \le c_K\Big(1 + \sum_{k=1}^{n} k^{-1/2} e_k(K)\Big), \quad n \in \mathbb{N},$$

and

$$k^{1/2} c_{k+n}(\mathrm{cov}(K)) \le c\bigg[\log^{1/2}(n+1)\varepsilon_n(K) + \sum_{j=n+1}^{\infty} \frac{\varepsilon_j(K)}{j \log^{1/2}(j+1)}\bigg],$$

$k, n \in \mathbb{N}$, where $c_n(\mathrm{cov}(K))$ is the $n$th Gelfand number of the absolutely convex hull of $K$ and $\varepsilon_k(K)$ and $e_k(K)$ denote the $k$th entropy and $k$th dyadic entropy number of $K$, respectively. The inequalities are, essentially, a reformulation of the corresponding inequalities given in [CKP] which yield asymptotically optimal estimates of the Gelfand numbers $c_n(\mathrm{cov}(K))$ provided that the entropy numbers $\varepsilon_n(K)$ are slowly decreasing. For example, we get optimal estimates in the non-critical case where $\varepsilon_n(K) \preceq \log^{-\alpha}(n+1)$, $\alpha \ne 1/2$, $0 < \alpha < \infty$, as well as in the critical case where $\alpha = 1/2$. For $\alpha = 1/2$ we show the asymptotically optimal estimate $c_n(\mathrm{cov}(K)) \preceq n^{-1/2}\log(n+1)$, which refines the corresponding result of Gao [Ga] obtained for entropy numbers. Furthermore, we establish inequalities similar to that of Creutzig and Steinwart [CrSt] in the critical as well as non-critical cases. Finally, we give an alternative proof of a result by Li and Linde [LL] for Gelfand and entropy numbers of the absolutely convex hull of $K$ when $K$ has the shape $K = \{t_1, t_2, \ldots\}$, where $\|t_n\| \le \sigma_n$, $\sigma_n \downarrow 0$. In particular, for $\sigma_n \le \log^{-1/2}(n+1)$, which corresponds to the critical case, we get a better asymptotic behaviour of Gelfand numbers, $c_n(\mathrm{cov}(K)) \preceq n^{-1/2}$.

**1. Introduction and basic tools.** The main aim of this paper is to complement existing results concerning the behaviour of Gelfand numbers of absolutely convex hulls $\mathrm{cov}(K)$ of precompact subsets $K$ of a Hilbert

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space $H$ when the entropy numbers of $K$ decay slowly. This enables us to refine results of Gao [Ga], Creutzig and Steinwart [CrSt] in the so-called critical case. The upper estimates, which we establish in the critical case, are already implicitly contained in [CKP]. By using an ingenious example of Gao we prove that in the critical case the estimates are asymptotically optimal. Moreover, we show that there is a difference between the behaviour of Gelfand numbers of absolutely convex hulls $\mathrm{cov}(K)$ generated by sets $K \subset H$ consisting of "many" and those with "few" extremal points. For more information and references about the behaviour of the metric entropy of convex hulls we refer to the papers [C2], [CKP] and [St].

For our purposes we use the following notation and quantities. Let $(M,d)$ be a metric space and $B(s,\varepsilon) := \{t \in M \mid d(s,t) \le \varepsilon\}$ be the closed $\varepsilon$-ball in $M$ with centre $s$. For a bounded set $K \subset M$ and $\varepsilon > 0$ the *covering number* of $K$ is defined by

$$N(K;\varepsilon) := \inf\Big\{n \;\Big|\; \exists s_1,\dots,s_n \in M : K \subset \bigcup_{i=1}^{n} B(s_i,\varepsilon)\Big\}.$$

We denote the *entropy numbers* of $K$ by

$$\varepsilon_n(K) := \inf\{\varepsilon > 0 \mid N(K;\varepsilon) \le n\}$$

and its *dyadic entropy numbers* by

$$e_n(K) := \varepsilon_{2^{n-1}}(K), \qquad n \in \mathbb{N}.$$

Moreover, the entropy numbers of a (bounded linear) operator $T : E \to F$ from a Banach space $E$ into a Banach space $F$ are defined by

$$\varepsilon_n(T) := \varepsilon_n(T(B_E))$$

and its dyadic entropy numbers by

$$e_n(T) := \varepsilon_{2^{n-1}}(T), \qquad n \in \mathbb{N},$$

where $B_E$ is the closed unit ball of $E$. Furthermore, the $n$th *approximation*, *Gelfand* and *Kolmogorov numbers* of $T$ are defined by

$$\begin{aligned} a_n(T) &:= \inf\{\|T - A\| \mid \mathrm{rank}(A) < n\},\\ c_n(T) &:= \inf\{\|T|_M\| \mid M \subset E,\ \mathrm{codim}(M) < n\},\\ d_n(T) &:= \inf\{\|Q_N^F T\| \mid N \subset E,\ \dim N < n\}, \end{aligned}$$

respectively, where $Q_N^F : F \to F/N$ is the quotient map. We have $c_n(T) = d_n(T')$, where $T'$ is the dual operator of $T$.

If $l_1(K)$ denotes the Banach space of all summable families of real numbers $(\xi_t)_{t\in K}$ over the index set $K$ with the norm given by

$$\|(\xi_t)\| = \sum_{t\in K} |\xi_t|,$$

then the entropy numbers and Gelfand numbers of the absolutely convex hull $\mathrm{cov}(K)$ of a bounded set $K \subset E$ of a Banach space $E$ are expressed in terms of entropy and Gelfand numbers of operators: $e_n(\mathrm{cov}(K)) = e_n(T)$ and $c_n(\mathrm{cov}(K)) = c_n(T)$, where $T : l_1(K) \to E$ is the operator defined on the canonical basis $(e_t)_{t\in K}$ of $l_1(K)$ by $Te_t := t$.

Finally, we recall the $l$-norm of an operator $T : E \to F$ (or $\pi_\gamma$ summing norm in [LP]). Let $l_2^n$ be the $n$-dimensional Euclidean space and $S : l_2^n \to F$ an operator; then the *$l$-norm* of $S$ is defined by

$$l(S) := \Big( \int\limits_{\mathbb{R}^n} \|Sx\|^2 \, d\gamma_n(x) \Big)^{1/2},$$

where $\gamma_n$ is the canonical Gaussian probability measure of $\mathbb{R}^n$; and for an operator $T : E \to F$ we define

$$l(T) := \sup\{ l(TA) \mid \|A : l_2^n \to E\| \le 1,\ n \in \mathbb{N}\}.$$

If $A : E_0 \to E$ and $B : F \to F_0$ are operators between Banach spaces, then $l$ has the ideal property (cf. [LP]),

$$l(BTA) \le \|B\| l(T) \|A\|.$$

Now we give diverse tools for estimating Gelfand and entropy numbers of absolutely convex hulls in Hilbert spaces. We start with two general inequalities of [CKP].

Theorem A. *There is a universal constant $c > 0$ such that for each precompact subset $K$ of the unit ball $B_H$ of a Hilbert space $H$ and for all $n \in \mathbb{N}$ we have*

$$n^{1/2} c_n(\mathrm{cov}(K)) \le c \inf_{\varepsilon > 0} \Big\{ \int\limits_{\varepsilon/4}^{1} \log^{1/2}(N(K;s))\, ds + n^{1/2}\varepsilon \Big\}.$$

(ii) *There is a universal constant $c > 0$ such that for each precompact subset $K$ of a Hilbert space $H$ and for all $k, n \in \mathbb{N}$ we have*

$$k^{1/2} c_{k+n}(\mathrm{cov}(K)) \le c \int\limits_{0}^{\varepsilon_n(K)} \log^{1/2}(N(K;s))\, ds.$$

The next theorem is a reformulation of Theorem A.

Theorem B. *Let $K$ be a precompact subset of a Hilbert space $H$. Then the following inequalities hold:*

(i) $$n^{1/2} c_n(\mathrm{cov}(K)) \le c_K \Big( 1 + \sum_{k=1}^{n} k^{-1/2} e_k(K) \Big) \quad \textit{for } n \in \mathbb{N},$$

*where* $c_K \le c(1+\sup_{t\in K}\|t\|)$ *and* $c>0$ *is an absolute constant*;

$$\text{(ii)}\quad k^{1/2}c_{k+n}(\operatorname{cov}(K)) \le c\Big[\log^{1/2}(n+1)\varepsilon_n(K) + \sum_{j=n+1}^{\infty}\frac{\varepsilon_j(K)}{j\log^{1/2}(j+1)}\Big]$$

*for* $k,n\in\mathbb{N}$, *where* $c>0$ *is an absolute constant.*

*Proof.* (i) First we assume that $K$ is contained in $B_H$. By Theorem A we get, with an absolute constant $c>0$,

$$n^{1/2}c_n(\operatorname{cov}(K)) \le c\Big(\int_{\varepsilon/4}^{1}\log_2^{1/2}(N(K;s))\,ds + n^{1/2}\varepsilon\Big)$$

for $n\in\mathbb{N}$ and $\varepsilon>0$ .

We suppose $e_n(K)>0$; the case $e_n(K)=0$ can be treated more easily. Put $\varepsilon := 4e_n(K)$. Then we have, with $e_0(K):=1$,

$$\begin{aligned}\int_{e_n(K)}^{1}\log_2^{1/2}(N(K;s))\,ds &= \sum_{k=1}^{n}\int_{e_k(K)}^{e_{k-1}(K)}\log_2^{1/2}(N(K;s))\,ds\\ &\le \sum_{k=1}^{n}\int_{e_k(K)}^{e_{k-1}(K)}k^{1/2}\,ds = \sum_{k=1}^{n}k^{1/2}(e_{k-1}(K)-e_k(K))\\ &= 1+\sum_{k=1}^{n-1}((k+1)^{1/2}-k^{1/2})e_k(K) - n^{1/2}e_n(K)\\ &\le 1+\frac12\sum_{k=1}^{n}k^{-1/2}e_k(K) - n^{1/2}e_n(K).\end{aligned}$$

Thus

$$\begin{aligned}&\int_{e_k(K)}^{1}\log_2^{1/2}(N(K;s))\,ds + 4n^{1/2}e_n(K)\\ &\qquad\le 1+\frac12\sum_{k=1}^{n-1}k^{-1/2}e_k(K) + 3n^{1/2}e_n(K)\\ &\qquad\le 1+\frac72\sum_{k=1}^{n}k^{-1/2}e_k(K).\end{aligned}$$

This implies, with an absolute constant $c>0$, the desired inequality for $K\subset B_H$:

$$n^{1/2}c_n(\operatorname{cov}(K)) \le c\Big(1+\sum_{k=1}^{n}k^{-1/2}e_k(K)\Big). \tag{1}$$

If $K\subset H$ is a precompact set with $d:=\sup_{t\in K}\|t\|\ge 1$, then by the previous inequality and the equalities

$$c_n(\operatorname{cov}(K))=dc_n(\operatorname{cov}(B)),\qquad e_n(K)=de_n(B),$$

where $B:=\{t/d\mid t\in K\}\subset B_H$, we get the estimate

$$n^{1/2}c_n(\operatorname{cov}(K))\le cd\Big(1+\sum_{k=1}^{n}k^{-1/2}e_k(K)\Big). \tag{2}$$

Combining (1) and (2) we obtain inequality (i) of the theorem with $c_K\le c(1+\sup_{t\in K}\|t\|)$.

(ii) This time the starting point is inequality (ii) of Theorem A for a precompact set $K\subset H$:

$$k^{1/2}c_{k+n}(\operatorname{cov}(K))\le c\int_0^{\varepsilon_n(K)}\log^{1/2}(N(K;s))\,ds.$$

Indeed, the right-hand side of the inequality can be estimated as follows:

$$\begin{aligned}\int_0^{\varepsilon_n(K)}\log^{1/2}(N(K;s))\,ds&=\sum_{j=n}^{\infty}\int_{\varepsilon_{j+1}(K)}^{\varepsilon_j(K)}\log^{1/2}(N(K;s))\,ds\\&\le\sum_{j=n}^{\infty}\log^{1/2}(N(K,\varepsilon_{j+1}(K)))(\varepsilon_j(K)-\varepsilon_{j+1}(K))\\&\le\sum_{j=n}^{\infty}\log^{1/2}(j+2)(\varepsilon_j(K)-\varepsilon_{j+1}(K))\\&=\log^{1/2}(n+2)\varepsilon_n(K)+\sum_{j=n+1}^{\infty}(\log^{1/2}(j+2)-\log^{1/2}(j+1))\varepsilon_j(K)\\&\le\log^{1/2}(n+2)\varepsilon_n(K)+\frac12\sum_{j=n+1}^{\infty}\frac{\varepsilon_j(K)}{(j+1)\log^{1/2}(j+1)}.\end{aligned}$$

This estimate yields the desired inequality (ii). ∎

The following inequality of Theorem 1.3 in [CKP] is a version of the corresponding inequality in [C1].

THEOREM C. *Let $(s_n)$ be a positive and increasing sequence with the property that there exists a constant $\gamma\ge 1$ such that $s_{2n}\le\gamma s_n$ for all $n\in\mathbb{N}$. Then there exists a constant $c_\gamma\ge 1$ such that for all operators $T:E\to F$ between Banach spaces $E,F$ and all $n\in\mathbb{N}$,*

$$\sup_{1\le k\le n}s_ke_k(T)\le c_\gamma\sup_{1\le k\le n}s_kc_k(T).$$

Finally, we need a refined version of the Sudakov-type inequality due to A. Pajor and N. Tomczak-Jaegermann.

THEOREM D ([PT]). *There is a constant $c \geq 1$ such that for all operators $T : E \to H$ from a Banach space $E$ into a Hilbert space $H$ and all $n \in \mathbb{N}$,*

$$n^{1/2} c_n(T) \leq c l(T').$$

*By Gordon* [Go] *we know that $c \leq \sqrt{2}$.*

**2. Results.** In this section we give several propositions, which will be proved in Section 3. The first result refines an inequality by Creutzig and Steinwart [CrSt] in the critical case which was originally given in terms of entropy numbers of absolutely convex hulls.

PROPOSITION 1. *Let $-\infty < \beta < 1$ and let $K$ be a precompact subset of a Hilbert space $H$. Then for all $n \in \mathbb{N}$,*

$$\sup_{1 \leq k \leq n} k^{1/2} \log^{\beta-1}(k+1) c_k(\mathrm{cov}(K)) \leq c_{K,\beta}(1 + \sup_{1 \leq k \leq n} k^{1/2} \log^{\beta}(k+1) e_k(K)),$$

*where*

$$c_{K,\beta} \leq c\left(1 + \frac{1}{1-\beta}\right)(1 + \sup_{t \in K} \|t\|)$$

*and $c > 0$ is an absolute constant.*

Moreover, by using the basic tools we also get the following result of Steinwart [St].

PROPOSITION 2. *Let $0 < \alpha < 1/2$ and $K$ be a precompact subset of a Hilbert space $H$. Then for all $n \in \mathbb{N}$,*

$$\sup_{1 \leq k \leq n} k^{\alpha} c_k(\mathrm{cov}(K)) \leq c_{K,\alpha}(1 + \sup_{1 \leq k \leq n} k^{\alpha} e_k(K)),$$

*where*

$$c_{K,\alpha} \leq \frac{c}{1-2\alpha}\,(1 + \sup_{t \in K} \|t\|)$$

*and $c > 0$ is an absolute constant.*

The next result gives the asymptotic behaviour of Gelfand numbers of absolutely convex hulls, which in the non-critical case $0 < \alpha < \infty$, $\alpha \neq 1/2$ can already be found in [CKP].

PROPOSITION 3. *Let $0 < \alpha < \infty$, $-\infty < \beta < \infty$ and let $K$ be a precompact subset of a Hilbert space $H$. If $e_n(K) \preceq n^{-\alpha} \log^{-\beta}(n+1)$, then we have, in the non-critical case $\alpha \neq 1/2$,*

$$c_n(\mathrm{cov}(K)) \preceq \begin{cases} n^{-\alpha} \log^{-\beta}(n+1) & \text{for } 0 < \alpha < 1/2,\ -\infty < \beta < \infty, \\ n^{-1/2} \log^{1/2-\alpha}(n+1) \log^{-\beta}(\log(n+1)+1) & \\ \qquad\qquad\qquad \text{for } 1/2 < \alpha < \infty,\ -\infty < \beta < \infty, \end{cases}$$

*and in the critical case* $\alpha = 1/2$, $-\infty < \beta < 1$,

$$c_n(\mathrm{cov}(K)) \preceq n^{-1/2} \log^{1-\beta}(n+1).$$

*The estimates are asymptotically optimal.*

REMARK. From the estimates of the previous proposition it is interesting to see that the asymptotic behaviour of Gelfand numbers of absolutely convex hulls has a sudden jump if $\alpha$ crosses the point $1/2$. This is why we call $\alpha = 1/2$ the critical case.

Finally we turn to Gelfand numbers of absolutely convex hulls generated by "few" extremal points. In [LL], Li and Linde studied the Gelfand numbers and metric entropy of $\mathrm{cov}(K)$ via certain quantities originating in the theory of majorizing measures. Among other results, they obtained some finer estimates of $c_n(\mathrm{cov}(K))$ for absolutely convex hulls generated by few extremal points which lead to sharper results in the critical case $\alpha = 1/2$. The result is stated in the next proposition. Moreover, we are going to give an alternative proof of it.

PROPOSITION 4. *Let* $K = \{t_1, t_2, \ldots\} \subset H$ *be a precompact set such that* $\|t_n\| \le \sigma_n$, *where* $\sigma_1 \ge \sigma_2 \ge \ldots \ge 0$ *and* $\lim_{n\to\infty} \sigma_n = 0$. *Then the following estimates hold*:

(i) *If* $\log^{1/2}(n+1)\sigma_n$ *is decreasing*, *then for all* $n \in \mathbb{N}$,

$$n^{1/2} c_{2n-1}(\mathrm{cov}(K)) \le c \log^{1/2}(n+1)\sigma_n,$$

*where* $c > 0$ *is an absolute constant.*

(ii) *If* $\log^{1/2}(n+1)\sigma_n$ *is increasing*, *then for all* $n \in \mathbb{N}$,

$$c_n(\mathrm{cov}(K)) \le c\, \sigma_{2^n},$$

*where* $c > 0$ *is an absolute constant.*

*The estimates are asymptotically optimal for slowly decreasing sequences* $(\sigma_n)$. *In particular*, *for* $\sigma_n \preceq \log^{-\alpha}(n+1)$, $0 < \alpha < \infty$, *we have*

$$c_n(\mathrm{cov}(K)) \preceq \begin{cases} n^{-\alpha} & \text{for } 0 < \alpha < 1/2, \\ n^{-1/2} \log^{1/2-\alpha}(n+1) & \text{for } 1/2 \le \alpha < \infty. \end{cases}$$

*From the previous estimate we see that in the case* $\alpha = 1/2$ *we have*

$$e_n(K) \preceq n^{-1/2} \quad \text{and} \quad c_n(\mathrm{cov}(K)) \preceq n^{-1/2}.$$

If we compare this result with the general result of Proposition 3 for $\alpha = 1/2$ and $\beta = 0$, then we observe that the additional logarithmic term does not appear. So we have in the critical case a difference between the asymptotic behaviour of Gelfand numbers of absolutely convex hulls although the (dyadic) entropy numbers have the same asymptotic behaviour: $e_n(K) \preceq n^{-1/2}$ . This phenomenon depends on the fact that the precompact set $K$ either contains "many" extremal or "few" extremal points.

Concluding remarks. (a) From the inequality of Theorem C we see that the estimates of Propositions 1–3 remain valid for the dyadic entropy numbers $e_n(\mathrm{cov}(K))$ instead of the Gelfand numbers $c_n(\mathrm{cov}(K))$ of the absolutely convex hull of $K$. In particular, for a precompact subset $K$ of a Hilbert space $H$ we recover the estimate of Gao [Ga],

$$e_n(\mathrm{cov}(K)) \preceq n^{-1/2}\log(n+1) \quad \text{for } e_n(K) \preceq n^{-1/2}.$$

(b) Moreover, from the proof of Proposition 6.4 in [CKP] we can also conclude that for a precompact subset $K$ of a Banach space $E$ of type $p$ the asymptotic estimate

$$e_n(\mathrm{cov}(K)) \preceq n^{-(1-1/p)}\log(n+1) \quad \text{for } e_n(K) \preceq n^{-(1-1/p)}$$

is valid. This estimate is the critical case for a Banach space of type $p$. It has been recently obtained by Creutzig and Steinwart in a more general setting [CrSt]. They also showed that this estimate is asymptotically optimal, thus extending the Hilbert space result of Gao [Ga].

## 3. Proofs of the results

*Proof of Proposition 1.* The inequality of Proposition 1 is an easy consequence of inequality (i) of Theorem B. Indeed, if $\beta < 1$ the right-hand side of (i) can be estimated by

$$\begin{aligned}\sum_{k=1}^{n} k^{-1/2}e_k(K) &\le \Big(\sum_{k=1}^{n} k^{-1}\log^{-\beta}(k+1)\Big)\sup_{1\le k\le n} k^{1/2}\log^{\beta}(k+1)e_k(K)\\ &\le 2\int_0^n \frac{ds}{(s+1)\log^{\beta}(s+1)}\sup_{1\le k\le n} k^{1/2}\log^{\beta}(k+1)e_k(K)\\ &\le \frac{2}{1-\beta}\log^{1-\beta}(n+1)\sup_{1\le k\le n} k^{1/2}\log^{\beta}(k+1)e_k(K).\end{aligned}$$

Thus, by (i) of Theorem B we get the estimate

$$n^{1/2}c_n(\mathrm{cov}(K)) \le c_K\bigg(1+\frac{2}{1-\beta}\log^{1-\beta}(n+1)\sup_{1\le k\le n} k^{1/2}\log(k+1)e_n(K)\bigg),$$

yielding the inequality

$$n^{1/2}\log^{\beta-1}(n+1)c_n(\mathrm{cov}(K)) \le c_{K,\beta}(1+\sup_{1\le k\le n} k^{1/2}\log^{\beta}(k+1)e_k(K))$$

for $n \in \mathbb{N}$, where

$$c_{K,\beta} \le c\bigg(1+\frac{1}{1-\beta}\bigg)(1+\sup_{t\in K}\|t\|)$$

and $c > 0$ is an absolute constant. This implies the desired inequality of Proposition 1. ■

*Proof of Proposition 2.* The inequality of Proposition 2 again easily follows from (i) of Theorem B. Indeed, for $0 < \alpha < 1/2$ we have

$$\begin{aligned} n^{1/2} c_n(\mathrm{cov}(K)) &\le c_K\Big(1 + \sum_{k=1}^{n} k^{-1/2} e_k(K)\Big) \\ &\le c_K\Big(1 + \Big(\sum_{k=1}^{n} k^{-1/2-\alpha}\Big) \sup_{1\le k\le n} k^\alpha e_k(K)\Big) \\ &\le c_K\bigg(1 + \frac{1}{1/2-\alpha}\, n^{1/2-\alpha} \sup_{1\le k\le n} k^\alpha e_k(K)\bigg), \end{aligned}$$

yielding, for all $n \in \mathbb{N}$,

$$n^\alpha c_n(\mathrm{cov}(K)) \le c_K \frac{2}{1-2\alpha}\,(1 + \sup_{1\le k\le n} k^\alpha e_n(K)).$$

This estimate implies the inequality of Proposition 2. ∎

*Proof of Proposition 3.* The estimates from above in the cases $0 < \alpha < 1/2$ and $\alpha = 1/2$ easily follow from (i) of Theorem B or from Propositions 1 and 2, whereas the estimate from above in the case $\alpha > 1/2$ follows from (ii) of Theorem B. Now we show that the results are asymptotically optimal. The optimality in the case $0 < \alpha < \infty$ and $\alpha \ne 1/2$ has already been proved in [CKP]. It remains to show the optimality in the critical case $\alpha = 1/2$.

For this purpose we use the ingenious example of Gao [Ga] in the version of [CrSt]. Namely, there is a precompact subset $K$ of a Hilbert space $H$ such that for $n \in \mathbb{N}$, $e_n(K) \le n^{-1/2}\log^{-\beta}(n+1)$ and $e_n(\mathrm{cov}(K)) \ge \gamma n^{-1/2}\log^{1-\beta}(n+1)$, where $\gamma > 0$ is an absolute constant. Fix $m \in \mathbb{N}$. Then from the inequality of Theorem C we get, with the sequence $s_n := n\log^{\beta-1}(n+1)$,

$$mn\log^{\beta-1}(mn+1) e_{mn}(\mathrm{cov}(K)) \le c_\beta \sup_{1\le k\le mn} k\log^{\beta-1}(k+1) c_k(\mathrm{cov}(K))$$

$$\le c_\beta[\sup_{1\le k<n} k\log^{\beta-1}(k+1)c_k(\mathrm{cov}(K)) + \sup_{n\le k\le mn} k\log^{\beta-1}(k+1)c_k(\mathrm{cov}(K))].$$

Inserting the entropy estimate $e_k(\mathrm{cov}(K))$ from below and the estimate of the Gelfand numbers $c_k(\mathrm{cov}(K)) \le \alpha k^{-1/2}\log^{1-\beta}(k+1)$ of Proposition 3 we arrive at

$$\begin{aligned} \gamma(mn)^{1/2} &\le c_\beta(\alpha n^{1/2} + \sup_{1\le k\le mn} k\log^{\beta-1}(k+1)c_k(\mathrm{cov}(K))) \\ &\le c_\beta(\alpha n^{1/2} + mn\log^{\beta-1}(mn+1)c_n(\mathrm{cov}(K))) \\ &\le c_\beta(\alpha n^{1/2} + mn\log^{\beta-1}(n+1)c_n(\mathrm{cov}(K))), \end{aligned}$$

yielding

$$(\gamma m^{1/2} - \alpha c_\beta)n^{1/2} \le c_\beta mn\log^{\beta-1}(n+1)c_n(\mathrm{cov}(K)).$$

Choose $m_0 = m(\gamma, \alpha, \beta)$ such that $\gamma m_0^{1/2} - \alpha c_\beta \geq 1$. Then we get, with a positive constant $c_{\alpha,\beta,\gamma}$, the estimate

$$c_n(\operatorname{cov}(K)) \geq c_{\alpha,\beta,\gamma} n^{-1/2} \log^{1-\beta}(n+1) \quad \text{for } n \in \mathbb{N},$$

which shows that the asymptotic behaviour of $c_n(\operatorname{cov}(K))$ is optimal. ∎

Finally, we turn to the proof of Proposition 4. For this purpose we need an additional version of Pajor and Tomczak-Jaegermann's inequality. In order to formulate it we introduce the approximation numbers with respect to the $l$-norm. For an operator $T : E \to F$ between Banach spaces $E$ and $F$ the approximation numbers with respect to $l$ are defined by

$$a_n(T; l) := \inf\{l(T - A) \mid \operatorname{rank}(A) < n\}, \quad n \in \mathbb{N}.$$

Lemma A. *For an operator $T : E \to H$ from a Banach space $E$ into a Hilbert space $H$ we have the inequality*

$$k^{1/2} c_{k+n-1}(T) \leq \sqrt{2}\, a_n(T'; l) \quad \textit{for } k, n \in \mathbb{N}.$$

*Proof.* For the proof of this inequality we assume $l(T') < \infty$. Let $A : H \to E'$ be an operator with $\operatorname{rank}(A) < n$. Then

$$c_{k+n-1}(T) = d_{k+n-1}(T') \leq d_k(T' - A) + d_n(A) = d_k(T' - A).$$

Thus by Theorem D we get

$$k^{1/2} c_{k+n-1}(T) \leq k^{1/2} d_k(T' - A) \leq \sqrt{2}\, l(T' - A),$$

and therefore,

$$k^{1/2} c_{k+n-1}(T) \leq \sqrt{2}\, a_n l(T', l) \quad \text{for } k, n \in \mathbb{N}. \ ∎$$

Finally, we need the following result of Linde and Pietsch [LP]. In the following, $l_\infty$ as usual denotes the Banach space of all bounded sequences.

Lemma B. *Let $D : l_\infty \to l_\infty$, $D(\xi_i) = (\sigma_i \xi_i)$, be a diagonal operator generated by a non-negative decreasing sequence $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ such that* $\sup \log^{1/2}(n+1)\sigma_n < \infty$. *Then there is a constant $c > 0$ such that*

$$l(D) \leq c \sup_n \log^{1/2}(n+1)\sigma_n.$$

*Proof of Proposition 4.* It will be convenient to couch the arguments in the language of Gelfand numbers of operators. Indeed, if

$$K = \{t_1, t_2, \ldots\} \subset H \quad \text{with } \|t_n\| \leq \sigma_n,\ \sigma_1 \geq \sigma_2 \geq \ldots \geq 0,$$

and $\lim \sigma_n = 0$, then we have

$$c_n(\operatorname{cov}(K)) = c_n(SD : l_1(\mathbb{N}) \to H),$$

where the diagonal operator $D : l_1 \to l_1$ is defined by $D(\xi_i) = (\sigma_i \xi_i)$, and $S : l_1 \to H$ by $Se_i := t_i/\sigma_i$ for $\sigma_i > 0$ and $Se_i := 0$ for $\sigma_i = 0$.

*Proof of (i).* We factorize the operator $D$ as $D = D_2 D_1$, where

$$D_1 : l_1 \to l_1, \quad (\xi_i) \mapsto (\log^{1/2}(i+1)\sigma_i \xi_i),$$
$$D_2 : l_1 \to l_1, \quad (\xi_i) \mapsto (\log^{-1/2}(i+1)\xi_i).$$

By Lemma 4 we get

$$\begin{aligned} n^{1/2}\, c_{2n-1}(SD_2D_1) &\le \sqrt{2}\, a_n(D_1' D_2' S'; l) \le \sqrt{2}\, a_n(D_1') l(D_2' S') \\ &\le \sqrt{2}\, a_n(D_1) l(D_2') \|S'\| \le \sqrt{2}\, a_n(D_1) l(D_2') \end{aligned}$$

because of $\|S'\| = \|S\| \le 2$.

Since $\log^{1/2}(n+1)\sigma_n$ is decreasing it follows that for all $n \in \mathbb{N}$,

$$a_n(D_1) \le \log^{1/2}(n+1)\sigma_n.$$

Moreover, by Lemma 6 we have

$$l(D_2') \le c.$$

Combining the previous estimates we obtain the desired estimate (i) of Proposition 4:

$$n^{1/2} c_{2n-1}(\mathrm{cov}(K)) = n^{1/2} c_{2n-1}(SD) \le \sqrt{2}\, c \log^{1/2}(n+1)\sigma_n.$$

*Proof of (ii).* This time we decompose the operator $D : l_1 \to l_1$ as

$$D = D - D_{2^n} + D_{2^n},$$

where

$$D_{2^n} : l_1 \to l_1, \quad (\xi_i) \mapsto \begin{cases} \sigma_i \xi_i, & i \le 2^n, \\ 0, & i > 2^n. \end{cases}$$

Furthermore, we factorize $D_{2^n}$ as

$$D_{2^n} = D_{2^n}^{(2)} D_{2^n}^{(1)},$$

where

$$D_{2^n}^{(1)} : l_1 \to l_1, \quad (\xi_i) \mapsto \begin{cases} \log^{1/2}(i+1)\sigma_i \xi_i, & i \le 2^n, \\ 0, & i > 2^n, \end{cases}$$
$$D_{2^n}^{(2)} : l_1 \to l_1, \quad (\xi_i) \mapsto \begin{cases} \log^{-1/2}(i+1)\xi_i, & i \le 2^n, \\ 0, & i > 2^n. \end{cases}$$

Hence,

$$\begin{aligned} c_n(SD) = c_n(S(D - D_{2^n}) + SD_{2^n}) &\le \|S\| \cdot \|D - D_{2^n}\| + c_n(SD_{2^n}) \\ &\le \sigma_{2^n+1} + c_n(SD_{2^n}) \le \sigma_{2^n} + c_n(SD_{2^n}) \end{aligned}$$

and

$$\begin{aligned} c_n(SD_{2^n}) &\le c_n(SD_{2^n}^{(2)}) \|D_{2^n}^{(1)}\| \le c_n(SD_{2^n}^{(2)}) \log^{1/2}(2^n+1)\sigma_{2^n} \\ &\le c_n(SD_{2^n}^{(2)})(n+1)^{1/2}\sigma_{2^n}. \end{aligned}$$

Moreover, by Theorem D and Lemma 6 we get

$$n^{1/2}c_n(SD^{(2)}_{2^n}) \le \sqrt{2}\, l((D^{(2)}_{2^n})'S') \le \sqrt{2}\, l((D^{(2)}_{2^n})')\|S'\| \le \sqrt{2}\, c\|S\| \le \sqrt{2}\, c.$$

Combining the previous estimates we obtain the desired estimate:

$$c_n(\mathrm{cov}(K)) = c_n(SD) \le \sigma_{2^n} + c_n(SD_{2^n}) \le \sigma_{2^n} + \sqrt{2}\, c\left(\frac{n+1}{n}\right)^{1/2}\sigma_{2^n} \le (1+2c)\sigma_{2^n}.$$

The results are asymptotically optimal. Indeed, by Proposition 5.5 in [CKP] we deduce that for $K = \{\sigma_n e_n : n \in \mathbb{N}\} \subset l_2$, where $e_n$ is the unit vector basis of $l_2$ and $\sigma_n = \log^{-\alpha}(n+1)$, $0 < \alpha < \infty$,

$$c_n(\mathrm{cov}(K)) \ge \begin{cases} cn^{-\alpha} & \text{for } 0 < \alpha \le 1/2, \\ cn^{-1/2}\log^{1/2-\alpha}(n+1) & \text{for } \alpha > 1/2. \end{cases} \blacksquare$$

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