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Three-space problems and bounded approximation properties

by

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Dedicated to Prof. Olek Pełczyński on the occasion of his 70th birthday

Abstract. Let $\{R_n\}_{n=1}^{\infty}$ be a commuting approximating sequence of the Banach space X leaving the closed subspace $A \subset X$ invariant. Then we prove three-space results of the following kind: If the operators R_n induce basis projections on X/A, and X or A is an $\mathcal{L}_p$-space, then both X and A have bases. We apply these results to show that the spaces $C_\Lambda = \overline{\text{span}}\{z^k : k \in \Lambda\} \subset C(\mathbb{T})$ and $L_\Lambda = \overline{\text{span}}\{z^k : k \in \Lambda\} \subset L_1(\mathbb{T})$ have bases whenever $\Lambda \subset \mathbb{Z}$ and $\mathbb{Z} \setminus \Lambda$ is a Sidon set.

1. Introduction. Let X be a separable Banach space (over $\mathbb{R}$ or $\mathbb{C}$), $A \subset X$ a closed subspace and (P) a Banach space property. Then the paper deals with three-space problems of the following kind:

If X/A satisfies (P), do X and A also have (P)?

Let $B \subset X$ be another closed subspace such that $X = \overline{A + B}$. If B satisfies (P), do A and X also have (P)?

It turns out that these questions are meaningful if (P) is a bounded approximation property, X, A or X/B is an $\mathcal{L}_p$-space, and A, B are invariant under a sequence of finite rank operators which approximate the identity on X. We obtain basis and FDD existence theorems supplementing the results of [9]. In Section 3 we apply these methods to C_Λ- and L_Λ-spaces (over $\mathbb{T}$) and show that C_Λ and L_Λ have bases whenever $\Lambda \subset \mathbb{Z}$ is a co-Sidon set.

First we recall some basic definitions. X is called an *$\mathcal{L}_p$-space* (or *$\mathcal{L}_{p,\lambda}$-space*) if there exists a $\lambda \ge 1$ such that, for every finite-dimensional $E \subset X$, there is a finite-dimensional subspace $F \subset X$ with $E \subset F$ and $d(F, l_p^{\dim F}) \le \lambda$. ($d(\cdot,\cdot)$ is the Banach–Mazur distance.) It is known ([6]) that in this situation we can even find such F which are uniformly complemented in X.

X has the *bounded approximation property* (BAP) if there is a sequence of bounded linear finite rank operators $R_n : X \to X$ with $\lim_n R_n x = x$ for all $x \in X$; $\{R_n\}_{n=1}^{\infty}$ is then called an *approximating sequence* (a.s.).

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If in addition $R_nR_m = R_{\min(n,m)}$ for $n \neq m$ then $\{R_n\}_{n=1}^{\infty}$ is called a *commuting approximating sequence* (c.a.s.) and X is said to have the *commuting bounded approximation property* (CBAP).

X has a *finite-dimensional Schauder decomposition* (FDD) if there is a c.a.s. $\{R_n\}_{n=1}^{\infty}$ of X where all R_n are projections. (In this case we have $X = \sum_n \oplus (R_{n+1} - R_n)X$.)

Finally, X has a *basis* provided that X has a c.a.s. $\{R_n\}_{n=1}^{\infty}$ consisting of projections such that $\dim (R_{n+1} - R_n)X = 1$ for all n.

It is clear that basis $\Rightarrow$ FDD $\Rightarrow$ CBAP $\Rightarrow$ BAP. On the other hand it is well known that CBAP $\not\Rightarrow$ FDD $\not\Rightarrow$ basis ([1], [11], [12]; see also [10]).

In the following, "$\sim$" means "is isomorphic to". If $U_n : X \to X$, $n = 1, 2, \ldots$, are linear operators we always put $U_0 = U_{-1} = \ldots = 0$.

We say that the U_n *factor uniformly* through an $\mathcal{L}_p$-space Y if there are linear operators $T_n : X \to Y$ and $S_n : Y \to X$ with $S_nT_n = U_n$ and $\sup_n \|S_n\| \cdot \|T_n\| < \infty$.

2. The main results. Again, assume that X is a separable Banach space. Let $A \subset X$ and $B \subset X$ be closed subspaces. Recall that a linear operator $R : X \to X$ with $RA \subset A$ induces a linear operator $\widehat{R}$ on X/A with $\|\widehat{R}\| \le \|R\|$, namely $\widehat{R}(x + A) = Rx + A$, $x \in X$.

2.1. Theorem. *Let $\{R_n\}_{n=1}^{\infty}$ be a c.a.s. of X with $R_nA \subset A$, $n = 1, 2, \ldots$*

(a) *Assume that the operators R_n induce the projections of a basis* (*or FDD*, *resp.*) *on X/A. If X or A is an $\mathcal{L}_p$-space for some $p \in [1, \infty[$ then $X \oplus l_p$ has a basis* (*or an FDD*, *resp.*) *with projections P_n which leave $A \oplus l_p$ invariant. In particular*, *$A \oplus l_p$ also has a basis* (*or an FDD*, *resp.*) *with projections $P_n|_{A\oplus l_p}$.*

(b) *Assume that $X = \overline{A + B}$ and that $R_n|_B$, $n = 1, 2, \ldots$, are the projections of a basis* (*or an FDD*, *resp.*) *of B. If X, X/B or A is an $\mathcal{L}_p$-space for some $p \in [1, \infty[$ then $X \oplus l_p$ has a basis* (*or an FDD*, *resp.*) *with projections P_n satisfying $P_n(A \oplus l_p) \subset A \oplus l_p$ and $P_n|_B = R_n|_B$, $n = 1, 2, \ldots$ In particular*, *$A \oplus l_p$ has a basis* (*or an FDD*, *resp.*) *with projections $P_n|_{A\oplus l_p}$.*

We postpone the proof of 2.1 to Section 4. Here we make a few remarks.

Remarks. The proof of 2.1 shows that the theorem remains true for $p = \infty$. Here we have to replace l_p by c_0.

In 2.1(b) we do not require $A \cap B = \{0\}$. Moreover, we can admit the case that $R_n|_B = R_{n+1}|_B$ for some n. On the other hand, we do not claim that the $R_n|_A$ themselves are the projections of a basis or FDD of A. The theorem is certainly false if we drop the assumption that X, A or X/B is an $\mathcal{L}_p$-space (e.g. take $B = \{0\}$ and $A = X$).

In some cases one obtains slightly better results. Then we do not need to add l_p or c_0:

2.2. THEOREM. *Let $\{R_n\}_{n=1}^{\infty}$ be a c.a.s. of X which leaves A invariant and defines a sequence of projections for a basis of X/A. If X or A is an $\mathcal{L}_p$-space for some $p \in [1,\infty]$ then both X and A have bases.*

Proof. $\{R_n|_A\}_{n=1}^{\infty}$ is a c.a.s. of A. We claim that $R_n - R_{n-1}$ and $(R_n - R_{n-1})|_A$ factor uniformly through an $\mathcal{L}_p$-space. Indeed, by our assumption, $A \cap (R_n - R_{n-1})X$ is at most 1-codimensional in $(R_n - R_{n-1})X$. Hence we find uniformly bounded projections $P_n : (R_n - R_{n-1})X \to A \cap (R_n - R_{n-1})X$.

If X is an $\mathcal{L}_p$-space then define

$$T_n : A \to X \quad \text{by} \quad T_n a = (R_{n+1} - R_{n-2})a, \quad a \in A,$$
$$S_n : X \to A \quad \text{by} \quad S_n x = P_n(R_n - R_{n-1})x, \; x \in X.$$

We obtain $S_n T_n = (R_n - R_{n-1})|_A$. Hence the operators $(R_n - R_{n-1})|_A$ factor uniformly through X. By [8], A has a basis.

If A is an $\mathcal{L}_p$-space then set $W = (\mathrm{id} - P_n)(R_n - R_{n-1})X$ and define $T_n : X \to A \oplus W$ by

$$T_n x = (P_n(R_n - R_{n-1})x, (\mathrm{id} - P_n)(R_n - R_{n-1})x),$$

and $S_n : A \oplus W \to X$ by

$$S_n(a, w) = (R_{n+1} - R_{n-2})a + w.$$

Here $S_n T_n = R_n - R_{n-1}$ and $R_n - R_{n-1}$ factors uniformly through $A \oplus W$. The latter space is an $\mathcal{L}_{p,\lambda}$-space (where λ does not depend on n) because $\dim W \le 1$. Hence X has a basis (in view of [8]).

This proves 2.2, since separable $\mathcal{L}_p$-spaces always have bases ([4]). ■

In the case $p = 1$ and X an $\mathcal{L}_1$-space Theorem 2.1(a) can be proved under the considerably weaker assumption that $\{R_n\}_{n=1}^{\infty}$ be an approximating sequence. Similarly the basis version of 2.1 for $p = \infty$ can also be inferred under this assumption.

2.3. THEOREM. *Let $\{R_n\}_{n=1}^{\infty}$ be an a.s. of X with $R_n A \subset A$, $n = 1, 2, \ldots$ Assume that the operators R_n induce the projections of a basis* (*or an FDD*, *resp.*) *on X/A. If X is an $\mathcal{L}_1$-space then $X \oplus l_1$ has a basis* (*or an FDD*, *resp.*) *with projections P_n which leave $A \oplus l_1$ invariant. In particular*, *$A \oplus l_1$ also has a basis* (*or an FDD*, *resp.*) *with projections $P_n|_{A \oplus l_1}$.*

2.4. THEOREM. *Let X be an $\mathcal{L}_\infty$-space and let $\{R_n\}_{n=1}^{\infty}$ be an a.s. of X with $R_n A \subset A$ for all n. Assume that the R_n*, *$n = 1, 2, \ldots$*, *induce the projections of a basis of X/A. Then $X \oplus c_0$ has a basis with projections P_n satisfying $P_n(A \oplus c_0) \subset A \oplus c_0$, $n = 1, 2, \ldots$ In particular*, *$A \oplus c_0$ has a basis with projections $P_n|_{A \oplus c_0}$. Finally*, *there is a subspace $B \sim c_0$ of $X \oplus c_0$ such* *that* $\overline{(A \oplus c_0) + B} = X \oplus c_0$ *and the operators* $P_n|_B$ *are the basis projections of the unit vector basis of* c_0.

We also postpone the proofs of Theorems 2.3 and 2.4 to Section 4.

Recall that $A \oplus l_p \sim A$ provided that A contains a complemented isomorphic copy of l_p, and $A \oplus c_0 \sim A$ provided that A contains an isomorphic copy of c_0 (see [7]). Together with 2.1 and the remark following it we obtain

2.5. Corollary. *Let* $\{R_n\}_{n=1}^{\infty}$ *be a c.a.s. of* X *and let* $A \subset X$ *be an* $\mathcal{L}_p$-*space for some* $p \in [1, \infty]$ *such that* $R_n(\mathrm{id} - R_n)X \subset A$, $n = 1, 2, \ldots$ *Then* $X \oplus l_p$, *if* $p < \infty$, *and* $X \oplus c_0$, *if* $p = \infty$, *has an FDD.*

Reformulating the basis version of 2.1(b) (with $A = X$) we obtain the following basis extension result.

2.6. Corollary. *Let* $B \subset X$ *be a closed subspace with a basis* Ω *and assume that* X *or* X/B *is an* $\mathcal{L}_p$-*space. If the basis projections of* Ω *can be extended to a c.a.s. of* X *then* $X \oplus l_p$, *for* $1 \le p < \infty$, *and* $X \oplus c_0$, *for* $p = \infty$, *has a basis which contains* Ω *as a subsequence.*

Remarks. Here we identify $x \in X$ with $(x, 0) \in X \oplus l_p$. Note that Ω is not just equivalent to a subsequence but the elements of Ω coincide with some elements of the extended basis.

Theorem 2.3 also includes a result of [9]. Recall that every separable Banach space Y is isomorphic to a quotient space of l_1.

2.7. Corollary. *Let* Y *be a Banach space with basis and let* $q : l_1 \to Y$ *be a quotient map. Then* $\ker q$ *has a basis.*

Proof. Let $\widehat{R}_n : Y \to Y$ be the basis projections of a given basis of Y. Moreover, let $\{e_k\}_{k=1}^{\infty}$ be the unit vector basis of l_1. Find $y_j \in Y$ with $\|y_j\| = 1$ and integers $0 < m_1 < m_2 < \ldots$ such that $y_j \in \widehat{R}_n Y$, $j \le m_n$, satisfying the following:

For each $y \in \widehat{R}_n Y$ with $\|y\| = 1$ there are $\lambda_1, \ldots, \lambda_{m_n}$ such that

$$y = \sum_{j=1}^{m_n} \lambda_j y_j, \qquad \sum_{j=1}^{m_n} |\lambda_j| \le 2.$$

Then define the quotient map $q_0 : l_1 \to Y$ by $q_0 e_j = y_j$ for all j. It is well known ([7]) that $\ker q_0 \sim \ker q$. Put $A = \ker q_0$. We can assume that $\dim A = \infty$, hence $A \sim A \oplus l_1$ ([7]).

Define the linear operators $R_n : l_1 \to l_1$ by $R_n e_j = e_j$ for $j \le m_n$. If $k > m_n$ find $\lambda_1, \ldots, \lambda_{m_n}$ with

$$\widehat{R}_n y_k = \|\widehat{R}_n\| \sum_{j=1}^{m_n} \lambda_j y_j, \qquad \sum_{j=1}^{m_n} |\lambda_j| \le 2.$$

Put $R_n e_k = \|\widehat{R}_n\| \sum_{j=1}^{m_n} \lambda_j e_j$. Then we obtain an a.s. $\{R_n\}_{n=1}^{\infty}$ with $q_0 R_n = \widehat{R}_n q_0$ for all n. Now Theorem 2.3 completes the proof. ∎

3. Co-Sidon sets. Now we turn to complex Banach spaces. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Fix $\Lambda \subset \mathbb{Z}$ and let C_Λ be the closed linear span of the functions z^k, $k \in \Lambda$, on $\mathbb{T}$ with respect to the sup-norm (denoted by $\|\cdot\|_\infty$). Moreover, let L_Λ be the closed linear span of z^k, $k \in \Lambda$, with respect to the L_1-norm on $\mathbb{T}$ (denoted by $\|\cdot\|_1$).

We make use of some classical finite rank operators. Fix n and put

$$\sigma_n\Big(\sum_k \alpha_k z^k\Big) = \sum_{k=-n}^{n} \frac{n-|k|}{n}\, \alpha_k z^k.$$

It is well known ([3]) that $\|\sigma_n\| = 1$ on $C(\mathbb{T}) = C_{\mathbb{Z}}$ as well as on $L_1(\mathbb{T}) = L_{\mathbb{Z}}$. Clearly, $\sigma_n(C_\Lambda) \subset C_\Lambda$ and $\sigma_n(L_\Lambda) \subset L_\Lambda$ for each $\Lambda \subset \mathbb{Z}$. For $0 < m < n$ put

$$V_{n,m} = \frac{n\sigma_n - m\sigma_m}{n-m}.$$

Then

$$V_{n,m}\Big(\sum_k \alpha_k z^k\Big) = \sum_{|k|\le m} \alpha_k z^k + \sum_{m<|k|\le n} \frac{n-|k|}{n-m}\, \alpha_k z^k$$

and $\|V_{n,m}\| \le (n+m)/(n-m)$ (as an operator on $C(\mathbb{T})$ as well as on $L_1(\mathbb{T})$).

$\Lambda \subset \mathbb{Z}$ is called a *Sidon set* if $\{z^k\}_{k\in\Lambda}$ (regarded as a sequence in $C(\mathbb{T})$) is equivalent to the unit vector basis of l_1. It is well known ([2]) that lacunary sets are Sidon sets and finite unions of Sidon sets are Sidon sets.

3.2. Theorem. *Let $\Lambda \subset \mathbb{Z}$ be such that $\mathbb{Z} \setminus \Lambda$ is a Sidon set. Then C_Λ and L_Λ have a basis.*

Further results on C_Λ and L_Λ, where $\mathbb{Z} \setminus \Lambda$ is a Sidon set, can be found in [5].

For the proof of 3.2 we need the following

3.3. Lemma. *Let $\Lambda \subset \mathbb{Z}$ be such that $\mathbb{Z} \setminus \Lambda$ is a Sidon set. Then C_Λ contains an isomorphic copy of c_0 and hence $C_\Lambda \sim C_\Lambda \oplus c_0$. Moreover, L_Λ contains a complemented copy of l_1 and $L_\Lambda \sim L_\Lambda \oplus l_1$.*

Proof. It is well known ([2]) that $C(\mathbb{T})/C_\Lambda$ is isomorphic to l_2 since $\mathbb{Z} \setminus \Lambda$ is a Sidon set. Now find $e_n \in C(\mathbb{T})$ of norm one with mutually disjoint supports, which implies that $\{e_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of c_0. Let $q : C(\mathbb{T}) \to C(\mathbb{T})/C_\Lambda$ be the quotient map. Then we must have $\lim_n \|qe_n\|_\infty = 0$ because otherwise we could find a subsequence $\{e_{n_k}\}_{k=1}^\infty$ such that $q|_{\mathrm{span}\{e_{n_k}\}_{k=1}^\infty}$ is an isomorphism, which is impossible since $qC(\mathbb{T}) \sim l_2$. So we find $\widetilde{e}_n \in C_\Lambda$ with $\lim_n \|e_n - \widetilde{e}_n\|_\infty = 0$ and hence a subsequence $\{\widetilde{e}_{n_m}\}_{m=1}^\infty$ which is equivalent to $\{e_{n_m}\}_{m=1}^\infty$. Thus $\overline{\mathrm{span}}\{\widetilde{e}_{n_m}\}_{m=1}^\infty \subset C_\Lambda$ is isomorphic to c_0.

It is well known ([2]) that $L_1(\mathbb{T})/L_\Lambda$ is isomorphic to c_0 since $\mathbb{Z} \setminus \Lambda$ is a Sidon set. Take $f_n \in L_1(\mathbb{T})$ of norm one with disjoint supports. Then $\{f_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of l_1 and $\overline{\mathrm{span}}\{f_n\}_{n=1}^\infty$ is complemented in $L_1(\mathbb{T})$. Let $q_1 : L_1(\mathbb{T}) \to L_1(\mathbb{T})/L_\Lambda$ be the corresponding quotient map. Since we are in c_0 we find a subsequence $\{q_1 f_{n_k}\}_{k=1}^\infty$ of $\{q_1 f_n\}_{n=1}^\infty$ such that

$$\text{weak-}\lim_{k\to\infty} q_1(f_{n_{2k+1}} - f_{n_{2k}}) = 0.$$

By Mazur's theorem there are $m_1 < m_2 < \ldots$ and suitable convex combinations $g_k = \sum_{j=m_k+1}^{m_{k+1}} \lambda_{k,j}(f_{n_{2j+1}} - f_{n_{2j}})$ such that $\lim_{k\to\infty} \|q_1 g_k\| = 0$. So we find $\widetilde{g}_k \in L_\Lambda$ with $\lim_{k\to\infty} \|g_k - \widetilde{g}_k\|_1 = 0$. The elements g_k, $k = 1, 2, \ldots$, have disjoint supports and are equivalent to the unit vector basis of l_1. By going over to a suitable subsequence $\{g_{k_j}\}_{j=1}^\infty$ we see that $\overline{\mathrm{span}}\{g_{k_j}\}_{j=1}^\infty$, and then also $\overline{\mathrm{span}}\{\widetilde{g}_{k_j}\}_{j=1}^\infty \subset L_\Lambda$, is isomorphic to l_1 and complemented in $L_1(\mathbb{T})$. ∎

Proof of Theorem 3.2. Put $\Omega = \mathbb{Z} \setminus \Lambda$. Fix an integer $n > 0$ and take m with $0 < m < n$ such that $(n+m)/(n-m) \le 2$. Consider

$$V_{n,m}\Big(\sum_k \alpha_k z^k\Big) = \sum_{|k|\le m} \alpha_k z^k + \sum_{m<|k|\le n} \frac{n-|k|}{n-m}\,\alpha_k z^k.$$

For $h \in L_1(\mathbb{T})$ we define $\mu_h : C(\mathbb{T}) \to \mathbb{C}$ by

$$\mu_h(f) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{-it}) f(e^{it})\,dt, \qquad f \in C(\mathbb{T}).$$

Then $\|\mu_h\| = \|h\|_1$.

Regard $g_n := \sum_{m<|k|\le n} z^k$ as an element of $L_1(\mathbb{T})$. Since Ω is a Sidon set we obtain a constant $c > 0$ independent of n with $|\mu_{g_n}(f)| \le c\|f\|_\infty$ for every $f \in C_\Omega$. Put

$$M_\Lambda = \{\mu \in C(\mathbb{T})^* : \widehat{\mu}(k) = 0 \text{ if } k \in \mathbb{Z} \setminus \Lambda\},$$

where $\widehat{\mu}(k) = \mu(z^{-k})$. Then $C_\Omega^* = C(\mathbb{T})^*/M_\Lambda$. Hence we find $\mu \in M_\Lambda$ with $\|\mu + \mu_{g_n}\| \le 2c$. Consider

$$\nu := (\mu + \mu_{g_n}) \circ (V_{2n,n} - V_{n,m}).$$

By definition of the $V_{j,k}$ we can find a trigonometric polynomial $h_n \in \mathrm{span}\{z^k : m \le |k| \le 2n\}$ such that $\nu = \mu_{h_n}$. Moreover, $\|h_n\|_1 \le 12c$.

Put

$$P_\Omega\Big(\sum_k \alpha_k z^k\Big) = \sum_{k\in\Omega} \alpha_k z^k.$$

Since

$$\mu_{h_n}(f) = \mu_{g_n}((V_{2n,n} - V_{n,m})f)$$

for all $f \in C_\Omega$ we obtain

$$P_\Omega h_n = P_\Omega(V_{2n,n} - V_{n,m})g_n = \sum_{\substack{m<|k|\le n\\ k\in\Omega}} \frac{|k|-m}{n-m}\, z^k.$$

Now, define $R_n f = V_{n,m}f + h_n * f$, $f \in C(\mathbb{T})$. Then the R_n are uniformly bounded finite rank operators. Moreover, $\lim_n R_n f = f$ for all $f \in C(\mathbb{T})$. On C_Ω we have

$$R_n\Big(\sum_{k\in\Omega} \alpha_k z^k\Big) = \sum_{\substack{|k|\le n\\ k\in\Omega}} \alpha_k z^k.$$

Hence the $R_n|_{C_\Omega}$ are basis projections. We have $C(\mathbb{T}) = \overline{C_\Omega + C_\Lambda}$. Since $R_n C_\Lambda \subset C_\Lambda$ for all n, the operators R_n define basis projections on $C(\mathbb{T})/C_\Lambda$. By 2.4 with $A = C_\Lambda$ the space $C_\Lambda \oplus c_0$ has a basis. Now, 3.3 shows that C_Λ has a basis.

The operators $R_n^*|_{L_1(\mathbb{T})}$ define an a.s. on $L_1(\mathbb{T})$. (Indeed, $R_n^* f = R_n f$ for any trigonometric polynomial f.) The operators $R_n^*|_{L_\Omega}$ are basis projections. Since $L_1(\mathbb{T}) = \overline{L_\Lambda \oplus L_\Omega}$ the operators R_n^* define basis projections on $L_1(\mathbb{T})/L_\Lambda$. By 2.3, $L_\Lambda \oplus l_1$ has a basis. Now 3.3 concludes the proof. ■

4. Proofs of the main results. In the following let X be a separable Banach space, and $A \subset X$ and $B \subset X$ closed subspaces such that $X = \overline{A+B}$. Put

$$W_p = \begin{cases} l_p, & 1 \le p < \infty,\\ c_0, & p = \infty.\end{cases}$$

4.1. Proposition. *Let $\{R_n\}_{n=1}^\infty$ be an a.s. of X with $R_n(\mathrm{id}-R_n)X \subset A$ and $R_n A \subset A$ for all n. Assume that either*

(i) *X or A is an $\mathcal{L}_p$-space, or*

(ii) *$R_n B \subset B$ for all n, the operators $R_n|_B$ are projections, and X, A or X/B is an $\mathcal{L}_p$-space.*

Then there is an a.s. $\{P_n\}_{n=1}^\infty$ of $X \oplus W_p$ consisting of projections with $P_n(A \oplus W_p) \subset A \oplus W_p$.

If (ii) *holds then in addition $P_n|_B = R_n|_B$ for all n.*

Moreover, if $R_n R_m = R_{\min(m,n)}$ for some m and n then also

$$P_n P_m = P_{\min(m,n)}.$$

If $(R_nR_m - R_{\min(m,n)})X \subset A$ *for some* m *and* n *then also*

$$(P_nP_m - P_{\min(m,n)})(X \oplus W_p) \subset A \oplus W_p.$$

If either (i) *holds and the* $R_n - R_{n-1}$ *define rank one operators on* X/A, *or* (ii) *holds and* $\dim(R_n - R_{n-1})B \le 1$ *for all* n, *then the operators* $P_n - P_{n-1}$ *induce rank one operators on* $(X \oplus W_p)/(A \oplus W_p)$.

Proof. If $R_nB \subset B$, $n = 1, 2, \ldots$, then the R_n define operators (called R_n again) on X/B. Moreover, if the operators $R_n|_B$ are projections then the map

$$x + B \mapsto R_n(\mathrm{id} - R_n)x, \quad x \in X,$$

makes sense, has norm $\le \|R_n(\mathrm{id} - R_n)\|$ and will be called $R_n(\mathrm{id} - R_n)$ again.

Let V be that space among X, A or X/B which is an $\mathcal{L}_p$-space. Fix m_n-dimensional subspaces $F_n \subset V$ with $\sup_n d(F_n, l_p^{m_n}) < \infty$ and $R_nV \subset F_n$. Put

$$W = \begin{cases} (\sum_n \oplus F_n)_{(p)} & \text{if } 1 \le p < \infty, \\ (\sum_n \oplus F_n)_{(0)} & \text{if } p = \infty. \end{cases}$$

Then $W \sim W_p$. Now define $P_n : X \oplus W \to X \oplus W$ by

$$P_n(x, (f_k)) = ((2\mathrm{id} - R_n^2)R_n(R_nx + (\mathrm{id} - R_n^2)f_n),$$
$$(f_1, \ldots, f_{n-1}, (\mathrm{id} - R_n^2)(R_nx + (\mathrm{id} - R_n^2)f_n), 0, 0, \ldots))$$

for $x \in X$, $f_k \in F_k$, $k = 1, 2, \ldots$

(In the case $F_n \subset X/B$ take $(\mathrm{id} - R_n^2)(R_nx + B + (\mathrm{id} - R_n^2)f_n)$ in the nth component instead.)

The operators P_n are of finite rank and we have $P_n \to \mathrm{id}$ pointwise on $X \oplus W$. It is easily checked that the P_n are projections, that $R_n|_B = P_n|_B$ if we are in case (ii), and that $P_nP_m = P_{\min(m,n)}$ provided that $R_nR_m = R_{\min(m,n)}$.

Similarly, if $(R_nR_m - R_{\min(m,n)})X \subset A$ for some m and n then $R_nR_m - R_{\min(m,n)}$ induces the zero operator on X/A. We can easily check that then $P_nP_m - P_{\min(m,n)}$ induces the zero operator on $(X \oplus W_p)/(A \oplus W_p)$. Moreover, clearly $P_n(A \oplus W) \subset A \oplus W$ (since, by assumption, $R_n(\mathrm{id} - R_n)X \subset A$).

Finally, assume that the operators $R_n - R_{n-1}$ define rank one operators on X/A. By assumption, $R_n(\mathrm{id} - R_n)X \subset A$ and the operators on X/A induced by R_n and $(2\mathrm{id} - R_n^2)R_n^2$ coincide. Hence, by definition of the P_n, the operators $P_n - P_{n-1}$ define rank one operators on $(X \oplus W_p)/(A \oplus W_p)$. ∎

Remark. The proof of 4.1 shows that actually

$$\sup_n \|P_n\| \le 18 \sup_n \|R_n\|^4.$$

Proof of Theorem 2.1. If the operators R_n define projections on X/A or the operators $R_n|_B$ are projections then $R_n(\mathrm{id} - R_n)X \subset A$ for all n. Hence Proposition 4.1 proves the FDD version of Theorem 2.1.

If the operators R_n define basis projections on X/A or on B then $R_n - R_{n-1}$ define rank one operators on X/A. Using Proposition 4.1 we find a sequence $\{P_n\}_{n=1}^{\infty}$ of FDD-projections of $X \oplus W_p$ with all the properties of 4.1 such that the operators $P_n - P_{n-1}$ induce rank one operators on $(X \oplus W_p)/(A \oplus W_p)$. This implies that there are subspaces $U_n \subset X \oplus W_p$ such that

$$(P_n - P_{n-1})(X \oplus W_p) = (P_n - P_{n-1})(A \oplus W_p) \oplus U_n$$

with $\dim U_n \leq 1$. Since the $P_n - P_{n-1}$ are uniformly bounded projections we find uniformly complemented subspaces $G_n \subset l_p^{m_n}$ for suitable m_n such that

$$\sup_n d((P_n - P_{n-1})(A \oplus W_p) \oplus G_n, l_p^{m_n}) < \infty.$$

This is possible if any of X, A or X/B is an $\mathcal{L}_p$-space. (In the latter case we have $U_n = (P_n - P_{n-1})B$.)

Since $\dim (P_n - P_{n-1})(X \oplus W_p)/(P_n - P_{n-1})(A \oplus W_p) \leq 1$ we also have

$$\sup_n d((P_n - P_{n-1})(X \oplus W_p) \oplus G_n, l_p^{k_n}) < \infty \quad \text{for suitable } k_n.$$

Put

$$V_p = \begin{cases} (\sum_n \oplus G_n)_{(p)} & \text{if } 1 \leq p < \infty, \\ (\sum_n \oplus G_n)_{(0)} & \text{if } p = \infty. \end{cases}$$

Then V_p is complemented in l_p or c_0, resp. Hence $V_p \sim l_p$ if $1 \leq p < \infty$, and $V_\infty \sim c_0$ ([7]). Put $Y_p = X \oplus W_p \oplus V_p$. Then $Y_p \sim X \oplus l_p$ if $1 \leq p < \infty$, and $Y_\infty \sim X \oplus c_0$. Define $\widehat{P}_n : Y_p \to Y_p$ by

$$\widehat{P}_n(y, (g_1, g_2, \ldots)) = (P_n y, (g_1, \ldots, g_n, 0, 0, \ldots))$$

for $y \in X \oplus W_p$ and $g_k \in G_k$, $k = 1, 2, \ldots$ Then the $\widehat{P}_n$ are FDD-projections on Y_p with $\widehat{P}_n(A \oplus W_p \oplus V_p) \subset A \oplus W_p \oplus V_p$, and in the case of 2.1(b), $\widehat{P}_n|_B = R_n|_B$.

Finally, we have

$$(\widehat{P}_n - \widehat{P}_{n-1})Y_p = (P_n - P_{n-1})(X \oplus W_p) \oplus G_n$$

and $\dim(\widehat{P}_n - \widehat{P}_{n-1})Y_p/(\widehat{P}_n - \widehat{P}_{n-1})(A \oplus W_p \oplus V_p) \leq 1$. Since the summands $(\widehat{P}_n - \widehat{P}_{n-1})Y_p$ are $l_p^{k_n}$-spaces they have bases with uniformly bounded basis constants, and suitable subsets are bases of $(\widehat{P}_n - \widehat{P}_{n-1})(A \oplus W_p \oplus V_p)$. This shows that Y_p has a basis with a suitable subsequence being a basis of $A \oplus W_p \oplus V_p$. Let Q_j be the corresponding basis projections. In the case of 2.1(b) we have $\dim (\widehat{P}_n - \widehat{P}_{n-1})B \leq 1$ for all n. Hence for every

j there is n such that $Q_j|_B = \widehat{P}_n|_B = R_n|_B$. This completes the proof of 2.1(b). ∎

To prove 2.3 we need

4.2. Lemma. *Let X be an $\mathcal{L}_p$-space for some $p \in [1,\infty]$ and let $R_k : X \to X$, $k = 1,\ldots,n$, be linear, bounded and of finite rank. Then there is a finite rank projection $Q : X \to X$ with $R_k Q = Q R_k = R_k$, $k = 1,\ldots,n$, where $\|Q\|$ does not depend on the operators R_k.*

Proof. It is well known that X^* is an $\mathcal{L}_q$-space where $p^{-1}+q^{-1} = 1$ ([6]). Since the $R_k^* : X^* \to X^*$ are of finite rank we find a finite rank projection $P : X^* \to X^*$ with $R_k^* X^* \subset PX^*$, where $\|P\|$ does not depend on the R_k. By [6, Corollary 3.2] we can choose P to be w*-continuous. Regard X as a natural subspace of X^{**}. Then $Q_1 = P^*|_X$ is a projection with $Q_1 X \subset X$. Since $PR_k^* = R_k^*$ we obtain $R_k Q_1 = R_k$ for all k. Using the fact that X is an $\mathcal{L}_p$-space we find a finite rank projection $Q_2 : X \to X$ with $Q_1 X \subset Q_2 X$ and $R_k X \subset Q_2 X$, $k = 1,\ldots,n$, where $\|Q_2\|$ does not depend on the R_k. Hence we obtain $Q_2 R_k = R_k$, $k = 1,\ldots,n$, and $Q_2 Q_1 = Q_1$. Put $Q = Q_1 + (\mathrm{id} - Q_1)Q_2$. Then Q is a finite rank projection satisfying $R_k Q = Q R_k = R_k$, $k = 1,\ldots,n$, and $\|Q\|$ does not depend on the R_k. ∎

Proof of Theorem 2.3. Let $\{R_n\}_{n=1}^\infty$ be an a.s. of X such that $R_n A \subset A$ for all n and the operators R_n define FDD-projections $\widehat{R}_n$ on X/A. Moreover, let X be an $\mathcal{L}_1$-space. We prove the theorem in two steps.

(a) First we construct an a.s. $\{P_n\}_{n=1}^\infty$ of $X \oplus l_1$ such that $P_n(A \oplus l_1) \subset A \oplus l_1$ for all n, the P_n define FDD-projections on $(X \oplus l_1)/(A \oplus l_1)$, and $P_m P_n = P_m$ whenever $m \le n$. Moreover, $P_n - P_{n-1}$ shall induce rank one operators on $(X \oplus l_1)/(A \oplus l_1)$ if $\dim(\widehat{R}_n - \widehat{R}_{n-1})(X/A) \le 1$ for all n.

Indeed, by Proposition 4.1 we may assume that the R_n are already projections. Fix uniformly bounded finite rank projections $Q_n : X \to X$ with $R_j Q_n = R_j = Q_n R_j$ and $Q_j Q_n = Q_n = Q_n Q_j$, $j = 1,\ldots,n$, which exist according to 4.2. Consider subspaces $F_n \subset X$ with $Q_n X \subset F_n$ and $\sup_n d(F_n, l_1^{m_n}) < \infty$ for some m_n. Then $F := (\sum_n \oplus F_n)_{(1)} \sim l_1$. Now define $P_n : X \oplus F \to X \oplus F$ by

$$P_n(x,(f_1,f_2,\ldots)) = \Big(R_n\Big(x + \sum_{k=n}^{\infty}(\mathrm{id} - R_k)f_k\Big),$$
$$\Big(f_1,\ldots,f_{n-1},(Q_n - R_n)\Big(x + \sum_{k=n}^{\infty}(\mathrm{id} - R_k)f_k\Big),0,\ldots 0\Big)\Big).$$

If $x \in A$ then clearly $P_n(x,(f_k)) \subset A \oplus F$ since $R_n(\mathrm{id} - R_k)X \subset A$ for all $k \ge n$. Moreover, $R_n^2 = R_n$ and $(Q_n - R_n)^2 = Q_n - R_n$. This implies that $P_m P_n = P_m$ whenever $m \le n$. If $j \ge n$ then we easily see that

$$(P_jP_n - P_n)(x,(f_k)) = \Big((R_j - \mathrm{id})R_n\Big(x + \sum_{k=n}^{\infty}(\mathrm{id} - R_k)f_k\Big),$$

$$\Big(\underbrace{0,\dots,0}_{j-1}, (Q_j - R_j)R_n\Big(x + \sum_{k=n}^{\infty}(\mathrm{id} - R_k)f_k\Big), 0, \dots\Big)\Big) \in A \oplus F.$$

This implies $(P_mP_n - P_{\min(m,n)})(X \oplus l_1) \subset A \oplus F$ and hence the P_n define FDD-projections on $(X \oplus l_1)/(A \oplus l_1)$.

Finally, since the R_n are projections, we have

$$(P_n - P_{n-1})(x,(f_k)) = \Big((R_n - R_{n-1})\Big(x + \sum_{k=n}^{\infty}(\mathrm{id} - R_k)f_k\Big),$$

$$\Big(\underbrace{0,\dots,0}_{n-2}, f_{n-1} - (Q_{n-1} - R_{n-1})\Big(x + \sum_{k=n-1}^{\infty}(\mathrm{id} - R_k)f_k\Big),$$

$$(Q_n - R_n)\Big(x + \sum_{k=n}^{\infty}(\mathrm{id} - R_k)f_k\Big), 0, \dots\Big)\Big).$$

Since $(R_n - R_{n-1})\sum_{k=n}^{\infty}(\mathrm{id} - R_k)f_k \in A$, the $P_n - P_{n-1}$ define rank one operators on $(X \oplus l_1)/(A \oplus l_1)$ provided that $\dim(\widehat{R}_n - \widehat{R}_{n-1})(X/A) \le 1$ for all n.

(b) Now we prove 2.3. According to (a), taking $X \oplus l_1$ instead of X and $A \oplus l_1$ instead of A, we can assume that $\{R_n\}_{n=1}^{\infty}$ is an a.s. of X with $R_nA \subset A$, $R_mR_n = R_m$ for $m \le n$ such that the R_n define FDD-projections $\widehat{R}_n$ on X/A. Since $R_n \to \mathrm{id}$ and the R_n are of finite rank, using a perturbation argument, we may as well assume that there is a subsequence $\{R_{n_m}\}_{m=1}^{\infty}$ which is a c.a.s. of X. Then the operators R_{n_m} are FDD-projections. This proves the FDD-part of 2.2(a).

Now assume that in addition $\dim(\widehat{R}_n - \widehat{R}_{n-1})(X/A) \le 1$ for all n. Put $X_m = (R_{n_m} - R_{n_{m-1}})X$ and $A_m = (R_{n_m} - R_{n_{m-1}})A$. We may assume that $R_jX_m \subset X_m$ for all $j = n_{m-1}+1, \dots, n_m$. (Otherwise replace R_j by $R_{n_{m-1}} + (R_{n_m} - R_{n_{m-1}})R_j(R_{n_m} - R_{n_{m-1}})$.) Find subspaces $G_m \subset X$ with $X_m \subset G_m$ and $\sup_m d(G_m, l_1^{k_m}) < \infty$ for some k_m. Put

$$Y_m = (\underbrace{G_m \oplus \dots \oplus G_m}_{n_m - n_{m-1}\ \text{times}})_{(1)}.$$

Then $Z := X \oplus (\sum_m \oplus Y_m)_{(1)} \sim X \oplus l_1$. Moreover, the $X_m \oplus Y_m$ are the summands of an FDD of Z. Let P_{n_m} be the corresponding projections and put $W = A \oplus (\sum_m \oplus Y_m)_{(1)}$. Then $P_{n_m}W \subset W$ for all m. We complete the P_{n_m} to a c.a.s. $\{P_j\}_{j=1}^{\infty}$ of Z with $P_jW \subset W$ such that the P_j define basis projections on Z/W. Then an application of 2.1(a) finishes the proof.

To this end put $M = n_m - n_{m-1}$ and define $\overline{P}_l : X_m \oplus Y_m \to X_m \oplus Y_m$ by

$$\overline{P}_l(x,(g_1,\dots,g_M)) = \Big(R_l x - (\mathrm{id} - R_l)\sum_{k=1}^{l-1-n_{m-1}} R_{k+n_{m-1}} g_k,$$
$$\Big(g_1,\dots,g_{l-1}, -R_l\Big(x + \sum_{k=1}^{l-1-n_{m-1}} R_{k+n_{m-1}} g_k\Big), 0,\dots,0\Big)\Big)$$

for $l = n_{m-1}+1,\dots,n_m$. Then the $\overline{P}_l$ are uniformly bounded and $\overline{P}_l\overline{P}_k = \overline{P}_{\min(k,l)}$ for all k and l. Moreover, $\overline{P}_l(A_m \oplus Y_m) \subset A_m \oplus Y_m$ since $(\mathrm{id} - R_l)R_h X \subset A$ if $l \ge h$. From the latter fact we also infer that the $\overline{P}_l - \overline{P}_{l-1}$ define rank one operators on $(X_m \oplus Y_m)/(A_m \oplus Y_m)$. Now, finally, put

$$P_l = P_{n_m} + (P_{n_m} - P_{n_{m-1}})\overline{P}_l(P_{n_m} - P_{n_{m-1}}), \quad n_{m-1}+1 \le l \le n_m,\ l = 1,2,\dots$$

Then the P_l satisfy all the requirements to apply 2.1(a). ∎

To prove Theorem 2.4, let X be an $\mathcal{L}_\infty$-space, $A \subset X$ a closed subspace, and $\{R_k\}_{k=1}^\infty$ an a.s. of X such that $R_n A \subset A$ for all n and the operators R_n induce basis projections on X/A. By Proposition 4.1 we may assume that the operators R_k are projections.

4.3. Lemma. *Put $X_1 = X \oplus c_0$ and $A_1 = A \oplus c_0$. Then there is a closed subspace $B \subset X_1$ such that*

$$B \sim c_0, \quad A_1 \cap B = \{0\}, \quad \overline{A_1 + B} = X_1. \tag{4.1}$$

Furthermore, there exists an a.s. $\{S_k\}_{k=1}^\infty$ of X_1 consisting of projections such that

$$S_n|_X = R_n \quad \text{for all } n, \tag{4.2}$$

A_1 and B are invariant under S_n and the operators $S_n|_B$ are the projections of the unit vector basis of c_0. Moreover,

$$S_m(\mathrm{id} - S_n)X_1 \subset A, \quad (\mathrm{id} - S_n)S_m X_1 \subset A \quad \text{whenever } m \le n. \tag{4.3}$$

Finally, there is a bounded projection $Q : X_1 \to B$ with

$$X = \ker Q \tag{4.4}$$

(where we identify $x \in X$ with $(x,0) \in X_1$).

Proof. Find $x_n \in X$ such that $\|x_n\| \le 2$ and the elements $x_n + A$ are the elements of a normalized basis of X/A whose basis projections are induced by the operators R_n. This implies

$$R_k x_n \in A \quad \text{whenever } k < n, \quad (\mathrm{id} - R_k)x_n \in A \quad \text{whenever } k > n. \tag{4.5}$$

Let $\{e_k\}_{k=1}^\infty$ be the unit vector basis of c_0. Put $b_n = (2^{-n}x_n, e_n) \in X_1$ and let $B = \overline{\text{span}}\{b_n\}_{n=1}^\infty \subset X_1$. Define $Q : X_1 \to B$ by

$$Q\Big(x, \sum_k \alpha_k e_k\Big) = \Big(\sum_k \frac{1}{2^k}\,\alpha_k x_k, \sum_k \alpha_k e_k\Big).$$

Then clearly (4.1) is satisfied since $\{b_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of c_0. We also obtain (4.4).

Now define $S_k : X_1 \to X_1$ by $S_k(x,0) = (R_k x, 0)$ for $x \in X$, and

$$(4.6)\qquad S_k\Big(0, \sum_{n=1}^\infty \alpha_n e_n\Big) = \Big(-\sum_{n=k+1}^\infty \frac{\alpha_n}{2^n}\,R_k x_n + \sum_{n=1}^k \frac{\alpha_n}{2^n}\,(\mathrm{id} - R_k)x_n, \sum_{n=1}^k \alpha_n e_n\Big).$$

This implies in particular that $\{S_k\}_{k=1}^\infty$ is an a.s. of X_1 and that $S_k A_1 \subset A_1$ for all k (see (4.5)). Furthermore, we have

$$S_k b_n = \begin{cases} b_n & \text{if } k \ge n,\\ 0 & \text{if } k < n.\end{cases}$$

This shows that the S_k are projections (since the R_k are assumed to be projections). We obtain (4.2).

We have $S_k(\mathrm{id} - S_m)X = R_k(\mathrm{id} - R_m)X \subset A$ whenever $k \le m$ by assumption on X/A. Similarly $(\mathrm{id} - S_m)S_k X \subset A$ if $k \le m$. Using (4.6) we see that, for $k \le m$ and any n,

$$\begin{aligned} S_k(\mathrm{id} - S_m)(0, e_n) &= S_k(\mathrm{id} - S_m)\Big(b_n - \Big(\frac{x_n}{2^n}, 0\Big)\Big)\\ &= \Big(-\frac{1}{2^n}\,R_k(\mathrm{id} - R_m)x_n, 0\Big) \in A, \end{aligned}$$

and similarly,

$$(\mathrm{id} - S_m)S_k(0, e_n) = \Big(-\frac{1}{2^n}(\mathrm{id} - R_m)R_k x_n, 0\Big) \in A.$$

Hence we obtain (4.3), which completes the proof of Lemma 4.3. ∎

REMARK. (4.4) implies that $X \oplus B = X_1$. This means in particular that $S_m S_n = S_{\min(m,n)}$ whenever $R_m R_n = R_{\min(m,n)}$.

We retain the notation of Lemma 4.3. Consider the unit vector basis $\{b_n\}_{n=1}^\infty$ of $B \sim c_0$. The projections S_k induce the basis projections for the basis $\{b_n + A_1\}_{n=1}^\infty$ of X_1/A_1. Hence we find $b_n^* \in A_1^\perp \subset X_1^*$ with

$$(4.7)\qquad b_n^*(b_m) = \begin{cases} 1, & n \ne m,\\ 0, & n = m.\end{cases}$$

(The functionals b_n^* may not be uniformly bounded.)

Let $\widetilde{Q}_n : X_1 \to \mathrm{span}\{b_n\}$ be defined by $\widetilde{Q}_n x = b_n^*(x)b_n$. Then

$$(4.8)\qquad S_m\widetilde{Q}_n = \widetilde{Q}_nS_m \quad \text{for all } n,m, \quad \widetilde{Q}_nS_m = \begin{cases} \widetilde{Q}_n, & n \le m,\\ 0, & m < n.\end{cases}$$

Moreover,

$$(4.9)\qquad S_nx - \sum_{k=1}^{n}\widetilde{Q}_kx \in A_1 \quad \text{for any } n \text{ and any } x \in X_1.$$

($\sum_{k=1}^{n}\widetilde{Q}_kx$ is the projection of S_nx onto B along A_1.) Now (4.3) implies

$$(4.10)\quad \widetilde{Q}_nS_m(\mathrm{id}-S_n) = 0, \quad \widetilde{Q}_n(\mathrm{id}-S_n)S_m = 0 \quad \text{whenever } m \le n.$$

Also, $(\ker\widetilde{Q}_n)\cap X$ is an $\mathcal{L}_{\infty,\lambda}$-space where λ does not depend on n since this space is 1-codimensional in X.

4.4. Proposition. *There is an a.s.* $\{T_n\}_{n=1}^{\infty}$ *of* $X_1 \oplus c_0$ *consisting of projections and leaving* $A_1 \oplus c_0$ *and* B *invariant such that*

$$(4.11)\qquad T_mT_n = T_m \quad \textit{whenever } m \le n, \quad T_k|_B = S_k|_B \quad \textit{for all } k.$$

(*We identify* $x \in X_1$ *with* $(x,0) \in X_1 \oplus c_0$.)

Proof. Find finite-dimensional subspaces $F_n \subset \ker\widetilde{Q}_n \cap X$ with

$$(4.12)\qquad \bigcup_{k=n}^{\infty} S_n(\mathrm{id}-S_k)X_1 \subset F_n, \quad \bigcup_{m=1}^{n}(\mathrm{id}-\widetilde{Q}_n)S_mX \subset F_n$$

and $\sup_n d(F_n, l_\infty^{m_n}) < \infty$ where $m_n = \dim F_n$. Put $G_n = F_n + \mathrm{span}\{b_n\}$. Hence $\sup_n d(G_n, l_\infty^{m_n+1}) < \infty$. Note that with the projection $Q : X_1 \to B$ of Lemma 4.3, using (4.4) we have

$$(4.13)\qquad QG_n = \mathrm{span}\{b_n\}, \quad \ker Q \cap G_n = F_n.$$

Put $X_2 = X_1 \oplus (\sum_n \oplus G_n)_{(0)}$. Hence $X_2 \sim X_1 \oplus c_0$. Define $T_n : X_2 \to X_2$ by

$$(4.14)\quad T_n(x,(g_1,g_2,\ldots)) = (S_n(x+g_n),(g_1 + S_1(\mathrm{id}-S_n)x - S_1S_ng_n, \ldots, g_{n-1} + S_{n-1}(\mathrm{id}-S_n)x - S_{n-1}S_ng_n, 0,0,\ldots)).$$

The definition of T_n makes sense since we have $S_m(\mathrm{id}-S_n)X_1 \subset F_m$ and $S_mS_nG_n \subset G_m$ for $m \le n$. The latter inclusion follows from the fact that $S_mS_nb_n = 0$ if $m < n$ and $S_mS_nF_n \subset G_m$ (in view of (4.12)). The T_n are uniformly bounded projections and $T_n|_B = S_n|_B$ since $S_m(\mathrm{id}-S_n)|_B = 0$ for $m \le n$. We easily check that (4.11) is satisfied. Put

$$b(n) = (b_n, (\underbrace{0,\ldots,0}_{n-1}, -b_n, 0, 0, \ldots)) \in X_2$$

and $W = \overline{\mathrm{span}}\{b(n)\}_{n=1}^{\infty}$. Moreover, put $V = W + (\sum_n \oplus F_n)_{(0)}$. Then, in view of (4.13), we have $V \sim c_0$ and $X_2 \sim X_1 \oplus V$. Also, (4.14) implies

$$T_n b(m) = \begin{cases} b(m), & m < n, \\ 0, & n \le m. \end{cases}$$

(Recall that $S_n b_m = 0$ if $m > n$, since the operators $S_n|_B$ are the basis projections of $\{b_k\}_{k=1}^\infty$.) Moreover, if $f \in F_m$ and $f(m) = (0, (\underbrace{0, \ldots, 0}_{m-1}, f, 0, 0, \ldots))$, then

$$T_n f(m) = \begin{cases} f(m), & m < n, \\ 0, & m > n, \end{cases}$$

and

$$T_n f(n) + V = \Big(S_n f - \sum_{k=1}^n \widetilde{Q}_k f, \\ (-S_1 S_n f + \widetilde{Q}_1 f, \ldots, -S_{n-1} S_n f + \widetilde{Q}_{n-1} f, 0, 0, \ldots)\Big).$$

(Recall that $\widetilde{Q}_n f = 0$ if $f \in F_n \subset \ker \widetilde{Q}_n \cap X$.) In view of (4.8) and (4.9) this implies $T_n f(n) \in A_1 + V$. On the other hand, if $a \in A_1$, then

$$T_n(a, (0, 0, \ldots)) \in A_1 + \Big(\sum_k \oplus F_k\Big)_{(0)}$$

according to (4.12) and (4.14). Hence $T_n(A_1 + V) \subset A_1 + V$ for all n. We clearly have, in view of (4.13), $A_1 + V \sim A_1 \oplus c_0$. ∎

4.5. PROPOSITION. *There is an a.s.* $\{P_n\}_{n=1}^\infty$ *of* $X_1 \oplus c_0$ *consisting of projections and leaving* $A_1 \oplus c_0$ *and* B *invariant such that*

$$(4.15) \qquad P_n P_m = P_m \quad \textit{whenever } m \le n, \qquad P_k|_B = S_k|_B \quad \textit{for all } k.$$

Moreover,

$$(4.16) \qquad P_m P_n = P_m \quad \textit{whenever } R_m R_n = R_m \textit{ and } m \le n.$$

Proof. Since $\lim_{k\to\infty} S_k x = x$ for all $x \in X_1$ the space $\mathrm{span}(\bigcup_k S_k X_1)$ is dense in X_1. Find finite-dimensional subspaces

$$(4.17) \qquad F_n \subset \mathrm{span}\Big(\bigcup_k S_k X_1\Big) \cap \ker \widetilde{Q}_{n+1} \cap X$$

with

$$(4.18) \qquad \bigcup_{m=1}^n (\mathrm{id} - S_n) S_m X_1 \subset F_n, \qquad \bigcup_{m=1}^n S_m X \subset F_n.$$

This is possible, since by (4.3),

$$(\mathrm{id} - S_n) S_m X_1 \subset A \cap \mathrm{span}\Big(\bigcup_k S_k X_1\Big) \subset X \cap \mathrm{span}\Big(\bigcup_k S_k X\Big),$$

and by (4.8), $\widetilde{Q}_{n+1}(\bigcup_{m=1}^n S_m X) = 0$. (Recall that $S_k X \subset X$ for all k in view of (4.2).) Finally, F_n can be arranged such that in addition $\sup_n d(F_n, l_\infty^{\dim F_n}) < \infty$, since $X \cap \ker \widetilde{Q}_{n+1}$ is an $\mathcal{L}_\infty$-space. Put $G_n = F_n + \operatorname{span}\{b_{n+1}\}$. Hence $\sup_n d(G_n, l_\infty^{\dim G_n}) < \infty$. We have

$$QG_n = \operatorname{span}\{b_{n+1}\}, \qquad \ker Q \cap G_n = F_n, \tag{4.19}$$

in view of (4.4) since $F_n \subset X$.

Now put $X_2 = X_1 \oplus (\sum_n \oplus G_n)_{(0)} \sim X_1 \oplus c_0$. Define $P_n : X_2 \to X_2$ by

$$\begin{aligned} P_n(x, (g_1, g_2, \ldots)) &= (S_n x + (\mathrm{id} - S_n) g_n, (g_1, \ldots, g_n, \\ &(\mathrm{id} - S_{n+1})(S_n x + (\mathrm{id} - S_n) g_n), (\mathrm{id} - S_{n+2})(S_n x + (\mathrm{id} - S_n) g_n), \ldots)). \end{aligned} \tag{4.20}$$

The definition of P_n makes sense in view of (4.18). In particular we have $(\mathrm{id} - S_{n+k})(\mathrm{id} - S_n) G_n \subset G_{n+k}$ since $(\mathrm{id} - S_{n+k})(\mathrm{id} - S_n) b_{n+1} = 0$ for $k \geq 1$. The operators P_n are uniformly bounded projections. We obtain $P_n|_B = S_n|_B$ since $(\mathrm{id} - S_{n+k}) S_n B = \{0\}$. It is easily checked that $P_n P_m = P_m$ whenever $n \geq m$. If $R_m R_n = R_m$ and $m \leq n$ then, in view of $X \oplus B = X_1$, by (4.2) we see that $S_m S_n = S_m$. Also, (4.20) implies that then $P_m P_n = P_m$.

Put

$$b(n) = (b_{n+1}, (\underbrace{0, \ldots, 0}_{n-1}, b_{n+1}, 0, 0, \ldots)) \in X_2$$

and $W = \overline{\operatorname{span}}\{b(n)\}_{n=1}^\infty$. Moreover, put $V = W + (\sum_n \oplus F_n)_{(0)}$. Then, in view of (4.19) and the fact that $\{b_n\}_{n=1}^\infty$ is the unit vector basis of c_0, we have $X_2 = X_1 \oplus V \sim X_1 \oplus c_0$. Equation (4.20) implies

$$P_n b(m) = \begin{cases} b(m), & m \leq n, \\ 0, & m > n. \end{cases}$$

(This follows since $S_n b_m$ is b_m if $m \leq n$, and 0 otherwise.)

Finally, take $f \in F_m$ and put

$$f(m) = (0, (\underbrace{0, \ldots, 0}_{m-1}, f, 0, 0, \ldots)).$$

Then (4.20) implies

$$P_n f(m) = \begin{cases} f(m), & m < n, \\ 0, & m > n, \end{cases}$$

and

$$\begin{aligned} P_n f(n) + V = \Big(&(\mathrm{id} - S_n) f - \sum_{k=n+1}^\infty \widetilde{Q}_k f, (\underbrace{0, \ldots, 0}_{n-1}, f, \\ &(\mathrm{id} - S_{n+1})(\mathrm{id} - S_n) f - \widetilde{Q}_{n+2} f, (\mathrm{id} - S_{n+2})(\mathrm{id} - S_n) f - \widetilde{Q}_{n+3} f, \ldots) \Big). \end{aligned}$$

Note that, in view of (4.17), $\widetilde{Q}_{n+k} f = 0$ for some k_0 and all $k \geq k_0$, and also $\widetilde{Q}_{n+1} f = 0$. In particular, there is no problem with the convergence of the series. Now, $(\mathrm{id} - S_n) f - \sum_{k=n+1}^\infty \widetilde{Q}_k f$ is the projection of $(\mathrm{id} - S_n) f$ onto

A_1 along B. Moreover, by (4.8), we have

$$\widetilde{Q}_{n+k+1}(\mathrm{id} - S_{n+k})(\mathrm{id} - S_n)f = \widetilde{Q}_{n+k+1}f.$$

Hence $P_n V \subset A_1 + V$. In view of (4.3) and (4.20) we also have $P_n A_1 \subset A_1 + V$. Since $A_1 + V \sim A \oplus c_0$ the proof is complete. ∎

Proof of Theorem 2.4. First we apply Lemma 4.3 to obtain $X_1 = X \oplus c_0$, $A_1 = A \oplus c_0$ and S_k. Then we continue with Proposition 4.4 to obtain

$$X_2 = X_1 \oplus c_0 \sim X \oplus c_0, \quad A_2 \sim A_1 \oplus c_0 \sim A \oplus c_0$$

and T_k.

Then we apply Lemma 4.3 and the remark following it again to X_2 instead of X, A_2 instead of A, and T_k instead of R_k. Finally, we apply Proposition 4.5 to find P_n on $X \oplus c_0$. Since $T_m T_n = T_m$ for $m \le n$, by (4.16) we also have $P_m P_n = P_m$. Hence $\{P_n\}_{n=1}^{\infty}$ is a sequence of FDD-projections on $X \oplus c_0$ leaving $A \oplus c_0$ invariant. Moreover, there is a subspace $B \subset X \oplus c_0$ with

$$\overline{(A \oplus c_0) + B} = X \oplus c_0, \quad B \sim c_0$$

such that the operators $P_n|_B$ are the projections of the unit vector basis of c_0. Now an application of Theorem 2.1(b) (and the following remark) finishes the proof. ∎

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