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Selecting basic sequences in φ-stable Banach spaces

by

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Dedicated to Aleksander Pełczyński on his seventieth birthday

Abstract. In this paper we make use of a new concept of φ-stability for Banach spaces, where φ is a function. If a Banach space X and the function φ satisfy some natural conditions, then X is saturated with subspaces that are φ-stable (cf. Lemma 2.1 and Corollary 7.8). In a φ-stable Banach space one can easily construct basic sequences which have a property $P(\varphi)$ defined in terms of φ (cf. Theorem 4.5).

This leads us, for appropriate functions φ, to new results on the existence of unconditional basic sequences with some special properties as well as new proofs of some known results. In particular, we get a new proof of the Gowers dichotomy theorem which produces the best unconditionality constant (also in the complex case).

1. Introduction. Let us recall briefly some definitions and facts from Banach space theory; the reader is referred to [L-T.1] and [J-L] for a more complete introduction to the subject.

A sequence (x_n) of non-zero elements of a Banach space X is said to be an *unconditional basic sequence* if it is M-unconditional for some $M < \infty$, i.e., if (x_n) satisfies the inequality

$$\Big\| \sum_n \lambda_n a_n x_n \Big\| \le M \Big\| \sum_n a_n x_n \Big\|$$

for each choice of scalars (a_n) and (λ_n) such that all but finitely many a_n's are zero and $|\lambda_n| \le 1$ for each n. The *unconditionality constant* of (x_n) is the least M with the above property. A basic sequence is unconditional if and only if it is a basic sequence in any ordering. An *unconditional basis* in X is an unconditional basic sequence which is a Schauder basis in X.

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A Banach space Y is said to be *decomposable* if there exist closed infinite-dimensional subspaces U, V of Y such that $U \cap V = \{0\}$ and $U+V=Y$ (this is equivalent to the natural projections of $U+V$ onto U and V being bounded). It is clear that every space with an unconditional basis is decomposable (let U and V be closed linear subspaces spanned by two infinite complementary subsets of the basis).

The long-standing question whether every infinite-dimensional Banach space contains an unconditional basic sequence was answered in the negative by W. T. Gowers and B. Maurey [G-M]. In fact, they constructed Banach spaces which are *hereditarily indecomposable*, i.e., contain no decomposable subspaces. The Gowers dichotomy discovered in [G] says that every Banach space either contains an unconditional basic sequence or has a hereditarily indecomposable subspace.

In this paper we present a method whose early version was suggested by Maurey's proof in [M.1]. Our method can also be used in other situations, thus leading to constructions of unconditional basic sequences with some special properties. Those properties are expressed and studied in terms of some functions defined on products of Grassmann manifolds.

Given an infinite-dimensional Banach space X, we let $\mathcal{G}(X)$ denote the set of all closed linear subspaces of X. We let $\mathcal{G}_{\mathrm{fin}}(X)$ denote the subset $\{E \in \mathcal{G}(X) : \dim E < \infty\}$ and put $\mathcal{G}_\infty(X) = \mathcal{G}(X) \setminus \mathcal{G}_{\mathrm{fin}}(X)$.

Observe that the decomposability of a subspace of X into a direct sum $U+V$, where $U, V \in \mathcal{G}_\infty(X)$, is equivalent to the finiteness of the expression

$$\varphi(U,V) := \sup\{\|u-v\| : u \in U,\ v \in V,\ \|u+v\| = 1\}, \tag{1}$$

while the existence of a decomposable subspace in a given subspace $Y \in \mathcal{G}_\infty(X)$ is equivalent to the finiteness of

$$\phi(Y) := \inf\{\varphi(U,V) : U, V \in \mathcal{G}_\infty(Y)\}.$$

Clearly, a subspace $Y \in \mathcal{G}_\infty(X)$ is hereditarily indecomposable if and only if $\phi(Y) = \infty$. Thus the dichotomy theorem is equivalent to the statement that if $\phi(\mathcal{G}_\infty(X)) \subseteq [1,\infty)$ then X contains an unconditional basic sequence.

Our result is more precise. If X is a real Banach space and $\phi(\mathcal{G}_\infty(X)) \subseteq [1, M)$ for some $M \le \infty$ then X contains a sequence (x_i) whose unconditionality constant is $< M$ (Theorem 6.1). We also obtain an analogous result for complex Banach spaces. These results yield nearly optimal bounds for unconditionality constants, because no basic sequence in Y can have its unconditionality constant $< \phi(Y)$. On the other hand, it is an easy fact that since the function ϕ is monotone on $\mathcal{G}_\infty(X)$, it has a stabilization property. Namely, one can find $Y \in \mathcal{G}_\infty(X)$ such that ϕ is constant on $\mathcal{G}_\infty(Y)$, i.e., $\phi(Z) = \phi(Y)$ for each $Z \in \mathcal{G}_\infty(Y)$. If Y is ϕ-stable in this sense, then for any given $M > \phi(Y)$ we can produce a basic sequence in Y whose unconditionality constant is $< M$. Thus for ϕ-stable spaces our bounds for unconditionality constants are essentially the best possible.

The proof given in this paper is somewhat more complicated than the description in the preceding paragraph. We introduce and exploit a much stronger stabilization property (cf. Lemma 2.1). Namely, a function $\widetilde{\varphi}$ we use in the selection procedure is defined on the Cartesian product $T \times \mathcal{G}_\infty(X)$, where T is a set of parameters. Thus $\widetilde{\varphi}$ may represent an uncountable family of $[1,\infty]$-valued functions on $\mathcal{G}_\infty(X)$ (those functions are indexed by elements $t \in T$). Then $Y \in \mathcal{G}_\infty(X)$ is said to be $\widetilde{\varphi}$-*stable* if each of the functions in that family is constant on $\mathcal{G}_\infty(Y)$.

The existence of $\widetilde{\varphi}$-stable elements in $\mathcal{G}_\infty(X)$ is assured if there is a topology on T such that T is separable and $\widetilde{\varphi}(\cdot, Y)$ is continuous on T for each fixed $Y \in \mathcal{G}_\infty(X)$. The continuity proofs can be found in Section 7. They are presented in a way that can provide an easy proof of the continuity for natural examples of functions φ different from those listed in Definition 5.1.

In Section 4 our inductive construction of basic sequences in φ-stable spaces is presented. Theorem 4.5 is really a scheme from which various such results can be obtained. The proofs of those theorems are quite simple, because they rely on specially designed combinatorial concepts and make a crucial use of φ-stability. In Section 6 we include some specific results which can be obtained by applying Theorem 4.5 to the functions φ defined in Section 5. We also give necessary and sufficient conditions for the existence of basic sequences with the $T(p)$-property and introduce the concept of generalized Tsirelson bases.

The scope of our approach is not restricted to Banach spaces. E.g., with a suitable choice of φ, one can also obtain results for operator spaces ([E-R], [Pi]). We are going to present those results in another paper.

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2. The stabilization property. We present here a general result which in particular can be applied to various functions Φ of the form $\widetilde{\varphi}$, defined in formula (2) below.

A partially ordered set $(S, \leq)$ is said to satisfy (CCB) (i.e., countable chains in S are bounded in S) provided that, for each sequence (s_i) in S such that $s_i \leq s_{i+1}$ for each i, there is an $s \in S$ such that $s_i \leq s$ for each i. A subset $S' \subseteq S$ is said to be *cofinal* in S if for each $s \in S$ there is an $s' \in S'$ such that $s \leq s'$. If Φ is a function defined on the product $T \times S$ where T is a set, then we denote by $\mathrm{St}(\Phi)$ the set of Φ-*stable* elements, i.e.,

$$\mathrm{St}(\Phi) = \{s \in S : \forall t \in T\ \forall s' \in S\ (s \leq s' \Rightarrow \Phi(t,s) = \Phi(t,s'))\}.$$

LEMMA 2.1. *Assume that* $(S, \leq)$ *satisfies* (CCB) *and* T *is a separable topological space. Let* $\Phi : T \times S \to [0, \infty]$ *be a function such that* $\Phi(t, \cdot)$ *is nondecreasing for each* $t \in T$, *while* $\Phi(\cdot, s)$ *is continuous for each* $s \in S$. *Then* $\mathrm{St}(\Phi)$ *is cofinal in* S.

Proof. First we consider Case 1 which occurs if $\mathrm{Card}(T) = 1$. In this case the assertion follows easily from (CCB). Case 2 occurs when $T = \{t_0, t_1, \ldots\}$ is countable. In this case we again make use of the (CCB) condition after applying Case 1 consecutively to each $\Phi(t_i, \cdot)$. In the general case, where T is a topological space with a dense countable subset T', we first apply Case 2 to the function $\Phi|_{T' \times S}$ and then we make use of the continuity assumption.

The proof of the following lemma is an easy exercise.

LEMMA 2.2. *Assume that* $(S, \leq)$ *satisfies* (CCB) *and suppose that* (Φ_i) *is a countable family of functions such that* $\Phi_i : T_i \times S \to [0, \infty]$ *and the set* $\mathrm{St}(\Phi_i)$ *is cofinal in* S *for each* i. *Then* $\bigcap_i \mathrm{St}(\Phi_i)$ *is also cofinal in* S.

In what follows we consider only those (CCB) partial orders which are described in the following lemma.

LEMMA 2.3. *Let* X *be an infinite-dimensional Banach space and let* $\preceq$ *denote the partial order on* $\mathcal{G}_\infty(X)$ *defined by the formula*: $Y \preceq Z$ *if and only if* $Z \subseteq Y + E$ *for some* $E \in \mathcal{G}_{\mathrm{fin}}(X)$. *Then* $(\mathcal{G}_\infty(X), \preceq)$ *satisfies* (CCB).

Proof. Let $(X_n)_{n=1}^\infty$ be a sequence in $\mathcal{G}_\infty(X)$ such that $X_n \preceq X_{n+1}$ for each n. Put $Y_n = \bigcap_{1 \leq i \leq n} X_i$. Then $Y_n \in \mathcal{G}_\infty(X)$, because $\dim(X_n/Y_n) < \infty$ for each n. Thus it is easy to construct by induction a linearly independent sequence (y_n) such that $y_n \in Y_n$ for each n. Let Y denote the closed linear span of (y_n). Then $X_n \preceq Y$ for each n, because $Y \subseteq \mathrm{span}\{y_1, \ldots, y_{n-1}\} + Y_n$ and $Y_n \subseteq X_n$. This completes the proof.

REMARK 2.4. Note that some results in [P] are deduced from Lemma 2.1 therein, which is a special case of our Lemma 2.1 (with no topology involved). Those two lemmas have been discovered independently of each other.

3. Notation and preliminaries. Observe that if $E, F \in \mathcal{G}_{\mathrm{fin}}(X)$, then $E + F \in \mathcal{G}_{\mathrm{fin}}(X)$, i.e., $\mathcal{G}_{\mathrm{fin}}(X)$ has a natural structure of an Abelian semigroup. Also if $E \in \mathcal{G}_{\mathrm{fin}}(X)$ then $E + U \in \mathcal{G}_\infty(X)$ for $U \in \mathcal{G}_\infty(X)$, hence $\mathcal{G}_{\mathrm{fin}}(X)$ acts in this way on the set $\mathcal{G}_\infty(X)$. For each positive integer k, there are analogous operations on the semigroup $\mathcal{G}_{\mathrm{fin}}(X)^k$ and the set $\mathcal{G}_\infty(X)^k$. The neutral element $(\{0\})_{i=1}^k$ of the semigroup $\mathcal{G}_{\mathrm{fin}}(X)^k$ will be denoted by $\vec{0}_k$.

We let $\mathcal{G}(X)_0^k$ denote the set of *nonsingular* elements of $\mathcal{G}_{\mathrm{fin}}(X)^k$, i.e.,

$$\mathcal{G}(X)_0^k = \Big\{\vec{E} \in \mathcal{G}_{\mathrm{fin}}(X)^k : \dim\Big(\sum_{i=1}^k E_i\Big) = \sum_{i=1}^k \dim E_i\Big\}.$$

If $\vec{U} = (U_1, \ldots, U_k) \in \mathcal{G}(X)^k$, $w \in X$ and $1 \le i \le k$, we let $\vec{U} +_i w$ denote the element $(V_1, \ldots, V_k) \in \mathcal{G}(X)^k$ such that $V_j = U_j$ for $j \neq i$ and $V_i = \mathrm{span}(U_i \cup \{w\})$.

The set $\mathcal{G}_{\mathrm{fin}}(X)$ will be given the topology defined by the Hausdorff metric ϱ, i.e., the distance of two subspaces $E, F \in \mathcal{G}_{\mathrm{fin}}(X)$ is defined to be

$$\varrho(E,F) = \inf\{t \ge 0 : B_E \subseteq B_F + tB_X \text{ and } B_F \subseteq B_E + tB_X\},$$

where $B_Z = \{z \in Z : \|z\| \le 1\}$ denotes the closed unit ball of a subspace $Z \in \mathcal{G}(X)$. It is an easy and well known fact that if the Banach space X is separable, then so is the metric space $(\mathcal{G}_{\mathrm{fin}}(X), \varrho)$.

The identity operator on a Banach space Y will be denoted by I_Y, and $\mathbb{N}$ will denote the set of positive integers.

4. Selecting sequences in φ-stable spaces. Let $k \in \mathbb{N}$ and let $\varphi : \mathcal{G}_\infty(X)^k \to [0,\infty]$ be a nondecreasing function, i.e.,

$$\varphi(\vec{U}) \le \varphi(\vec{V})$$

whenever $\vec{U}, \vec{V} \in \mathcal{G}_\infty(X)^k$ and $U_i \subseteq V_i$ for each i. Let $\widetilde{\varphi}$ denote the function which maps $\mathcal{G}_{\mathrm{fin}}(X)^k \times \mathcal{G}_\infty(X)$ into $[0,\infty]$ and is defined by the formula

$$\widetilde{\varphi}(\vec{E}, Y) = \inf\{\varphi(\vec{E} + \vec{U}) : \vec{U} \in \mathcal{G}_\infty(Y)^k\}. \tag{2}$$

Definition 4.1. Let $T \subseteq \mathcal{G}_{\mathrm{fin}}(X)^k$. We say that X is (φ, T)-*stable* if

$$\widetilde{\varphi}(\vec{E}, Y) = \widetilde{\varphi}(\vec{E}, X)$$

for each $\vec{E} \in T$ and $Y \in \mathcal{G}_\infty(X)$. We say that X is φ-*stable* if X is $(\varphi, \mathcal{G}(X)_0^k)$-stable.

Alternatively, we could have defined φ-stability to be an abbreviation for the $(\varphi, \mathcal{G}_{\mathrm{fin}}(X)^k)$-stability. This merely requires some extra work to verify the continuity on $\mathcal{G}_{\mathrm{fin}}(X)^k$ rather than on $\mathcal{G}(X)_0^k$. The details are given in the final section.

Definition 4.2. A subset S of the Banach space X is said to be *plentiful* in X if for each subspace $Y \in \mathcal{G}_\infty(X)$ there is a $Z \in \mathcal{G}_\infty(Y)$ such that $Z \subseteq S$.

Clearly, if $S_1, \ldots, S_n \subseteq X$ are plentiful in X, then so is $S_1 \cap \ldots \cap S_n$.

Lemma 4.3. *If X is φ-stable, $\vec{E} \in \mathcal{G}(X)_0^k$ and $d > \widetilde{\varphi}(\vec{E}, X)$, then the set*

$$\{w \in X : \widetilde{\varphi}(\vec{E} +_i w, X) < d\}$$

is plentiful in X for $1 \le i \le k$.

Proof. Fix $Y \in \mathcal{G}_\infty(X)$ and $d > \widetilde{\varphi}(\vec{E}, X)$. Since X is φ-stable, we have $\widetilde{\varphi}(\vec{E}, Y) < d$ and hence we can find $\vec{U} = (U_1, \ldots, U_k) \in \mathcal{G}_\infty(Y)^k$ such that $\varphi(\vec{E} + \vec{U}) < d$. Now fix $i \in \{1, \ldots, k\}$. Then for each $w \in U_i$ one has

$$\widetilde{\varphi}(\vec{E} +_i w, X) \le \varphi((\vec{E} +_i w) + \vec{U}) = \varphi(\vec{E} + \vec{U}) < d.$$

This shows that the set $\{w \in X : \widetilde{\varphi}(\vec{E} +_i w, X) < d\}$ is plentiful, because $U_i \in \mathcal{G}_\infty(Y)$ and the subspace $Y \in \mathcal{G}_\infty(X)$ was arbitrary. Since i can be any number in $\{1, \ldots, k\}$, this completes the proof of the lemma.

Definition 4.4. The quantity $M(\widetilde{\varphi}; (x_i)_{i=m}^\infty)$ defined as

$$\sup\{\widetilde{\varphi}((\operatorname{span}\{x_i : m \le i \le n,\ f(i) = j\})_{j=1}^k, X)\} : n \in \mathbb{N},\ f : \mathbb{N} \to \mathbb{N}\},$$

will be called the $\widetilde{\varphi}$-*constant* of the sequence $(x_i)_{i=m}^\infty$.

Let $\mathcal{N}$ be a finite or countable set. Suppose that $(k_s)_{s\in\mathcal{N}} \in \mathbb{N}^{\mathcal{N}}$ and we are given a family $(\varphi_s)_{s\in\mathcal{N}}$ of functions $\varphi_s : \mathcal{G}_\infty(X)^{k_s} \to [1, \infty]$ for $s \in \mathcal{N}$. Fix $(n_s) \in \mathbb{N}^{\mathcal{N}}$ such that $\{s \in \mathcal{N} : n_s \le n\}$ is finite for each $n \in \mathbb{N}$ and let $(\eta_n)_{n\ge1}$ be a nonincreasing sequence in $(1, \infty)$. (If $\mathcal{N}$ is finite, then only the case where $n_s = 1$ for $s \in \mathcal{N}$ is interesting.)

Theorem 4.5. *Suppose that X is φ_s-stable and $\widetilde{\varphi}_s(\vec{0}_{k_s}, X) < d_s$ for $s \in \mathcal{N}$. Then there is a basic sequence $(x_i)_{i=1}^\infty$ in X such that $M(\widetilde{\varphi}_s; (x_i)_{i=n_s}^\infty) \le d_s$ and the basis constant of the tail sequence $(x_i)_{i=n_s}^\infty$ is $\le \eta_{n_s}$ for $s \in \mathcal{N}$.*

If $\operatorname{Card}(\mathcal{N}) = 1$, then the theorem follows readily from

Proposition 4.6. *Let X be φ-stable, $d > \widetilde{\varphi}(\vec{0}_k, X)$ and $\eta > 1$. Then there is a basic sequence $(x_i)_{i=1}^\infty$ in X with basis constant $\le \eta$ such that*

$$\widetilde{\varphi}((\operatorname{span}\{x_j : 1 \le j \le n,\ f(j) = i\})_{i=1}^k, X) < d$$

for each $n \ge 1$ and each function $f : \{1, \ldots, n\} \to \{1, \ldots, k\}$.

Proof. Note that if $Y \in \mathcal{G}(X)$ and $\dim X/Y < \infty$, then Y is plentiful in X. This will allow us to construct the sequence $(x_i)_{i=1}^\infty$ by induction, using repeatedly Lemma 4.3 and the finite intersection property of plentiful subsets of X.

We let $Y_1 = X$. If $n \ge 1$ and $Y_n, x_1, \ldots, x_{n-1}$ have been defined so that $\dim X/Y_n < \infty$, then using Lemma 4.3 we select a nonzero $x_n \in Y_n$ so that for each $f : \{1, \ldots, n\} \to \{1, \ldots, k\}$ one has

$$\widetilde{\varphi}((\operatorname{span}\{x_j : 1 \le j \le n,\ f(j) = i\})_{i=1}^k, X) < d.$$

Then we choose $Y_{n+1} \subseteq Y_n$ so that $\dim X/Y_{n+1} < \infty$ and $\|x\| \le \eta\|x + y\|$ for each $x \in \operatorname{span}\{x_1, \ldots, x_n\}$ and $y \in Y_{n+1}$. Standard arguments show that the sequence (x_i) so defined has the required properties. This completes the proof of the proposition.

The general case of Theorem 4.5 is a consequence of the following.

Proposition 4.7. *Suppose that X is φ_s-stable and $\widetilde{\varphi}_s(\vec{0}_{k_s}, X) < d_s$ for $s \in \mathcal{N}$. Let $(n_s) \in \mathbb{N}^{\mathcal{N}}$ be such that $\{s \in \mathcal{N} : n_s \leq n\}$ is finite for each $n \in \mathbb{N}$ and let $\eta_1 \geq \eta_2 \geq \ldots > 1$. Then there is a basic sequence $(x_i)_{i=1}^{\infty}$ in X such that the basis constant of the tail sequence $(x_i)_{i=n_s}^{\infty}$ is $\leq \eta_{n_s}$ for $s \in \mathcal{N}$ and*

$$\widetilde{\varphi}_s((\operatorname{span}\{x_j : n_s \leq j \leq n,\ f(j) = i\})_{i=1}^{k_s}, X) < d_s \tag{3}$$

for each $n \geq 1$, each $s \in \mathcal{N}$ and each function $f : \mathbb{N} \to \mathbb{N}$.

Indeed, a proof of this proposition can be easily obtained by adapting that of Proposition 4.6, because for each n the inductive choice of Y_n and $x_n \in Y_n$ is subject only to a finite number of conditions (pertaining to those $s \in \mathcal{N}$ such that $n_s \leq n$).

Namely, for each n there is a finite set $\mathcal{F}_n$ of functions which has the property that if (3) is satisfied for $s \in \mathcal{N}$ such that $n_s \leq n$ and for each $f \in \mathcal{F}_n$ then for this n condition (3) is automatically satisfied for each $s \in \mathcal{N}$ and $f \in \mathbb{N}^{\mathbb{N}}$.

5. Some examples of functions φ. Given a Banach space X and $k \geq 2$, we shall consider two recipes for producing families of functions $\varphi : \mathcal{G}_\infty(X)^k \to [0, \infty]$. For any function $h : X^k \to [0, \infty]$ and $\vec{U} = (U_1, \ldots, U_k) \in \mathcal{G}_\infty(X)^k$, we let

$$h^{\uparrow}(\vec{U}) = \sup\{h(x) : x \in \mathcal{U}\}, \quad h^{\downarrow}(\vec{U}) = (\inf\{h(x) : x \in \mathcal{U}\})^{-1},$$

where $\mathcal{U} = \{(u_1, \ldots, u_k) \in U_1 \times \ldots \times U_k : \|\sum_{i=1}^k u_i\| = 1\}$.

Consider the following examples of functions on X^k:

$$h_{\pm,k}(x_1, \ldots, x_k) := \max\Big\{\Big\|\sum_{i=1}^{k} \varepsilon_i x_i\Big\| : \varepsilon_i = \pm 1,\ 1 \leq i \leq k\Big\};$$

if X is a complex Banach space, then

$$h_{\mathbb{C},k}(x_1, \ldots, x_k) := \sup\Big\{\Big\|\sum_{i=1}^{k} \lambda_i x_i\Big\| : |\lambda_i| = 1,\ 1 \leq i \leq k\Big\};$$

and finally, if $\mathbf{q}$ is a 1-unconditional norm on $\mathbb{R}^k$, then

$$h_{\mathbf{q}}(x_1, \ldots, x_k) := \mathbf{q}(\|x_1\|, \ldots, \|x_k\|). \tag{4}$$

Definition 5.1. We put $\varphi_{\pm,k} = (h_{\pm,k})^{\uparrow}$, $\varphi_{\mathbb{C},k} = (h_{\mathbb{C},k})^{\uparrow}$, $\varphi_{>\mathbf{q},k} = (h_{\mathbf{q}})^{\uparrow}$ and $\varphi_{<\mathbf{q},k} = (h_{\mathbf{q}})^{\downarrow}$.

Observe that for each $(u_1, \ldots, u_k) \in U_1 \times \ldots \times U_k$ one has

$$\|\varepsilon_1 u_1 + \ldots + \varepsilon_k u_k\|) \leq \varphi_{\pm,k}(U_1, \ldots, U_k)\|u_1 + \ldots + u_k\|, \tag{5}$$

$$\|\lambda_1 u_1 + \ldots + \lambda_k u_k\|) \leq \varphi_{\mathbb{C},k}(U_1, \ldots, U_k)\|u_1 + \ldots + u_k\|, \tag{6}$$

$$\mathbf{q}(\|u_1\|,\ldots,\|u_k\|) \le \varphi_{>q,k}(U_1,\ldots,U_k)\|u_1+\ldots+u_k\|, \tag{7}$$
$$\|u_1+\ldots+u_k\| \le \varphi_{<q,k}(U_1,\ldots,U_k)\mathbf{q}(\|u_1\|,\ldots,\|u_k\|), \tag{8}$$

and the respective coefficients $\varphi(\vec{U})$ are the smallest possible.

Note that the function $\varphi_{\pm,2}$ coincides with the function φ defined by formula (1), i.e., $\varphi_{\pm,2}(U_1,U_2)=\varphi(U_1,U_2)$ for $U_1,U_2\in\mathcal{G}_\infty(X)$.

PROPOSITION 5.2. *Let* $(S,\le)=(\mathcal{G}_\infty(X),\preceq)$ *and let* $T=\mathcal{G}_{\mathrm{fin}}(X)^k$ *with the topology induced by the metric* ϱ. *If* $\varphi:\mathcal{G}_\infty(X)^k\to[0,\infty]$ *is one of the functions introduced in Definition* 5.1, *then the function* $\widetilde{\varphi}:T\times S\to[0,\infty]$ *defined by* (2) *satisfies the assumptions of Lemma* 2.1.

Actually, for any function $h:X^k\to[0,\infty]$, if either $\varphi=h^{\uparrow}$ or $\varphi=h^{\downarrow}$, then $\widetilde{\varphi}(\vec{E},\cdot)$ is nondecreasing for each $\vec{E}\in\mathcal{G}_{\mathrm{fin}}(X)^k$. Only the proof of the continuity of $\widetilde{\varphi}(\cdot,Y)$ requires some work. We shall prove a more general statement in Section 7.

6. Some special classes of basic sequences. In this section whenever we assert the existence of stable subspaces it is always an easy consequence of Proposition 5.2 and the results in Section 2, so we shall refrain from any further comments.

THEOREM 6.1. *Let* $1<M\le\infty$. *Let* X *be a Banach space such that* $\widetilde{\varphi}_{\pm,2}(\vec{0},Y)<M$ *for each* $Y\in\mathcal{G}_\infty(X)$. *Then there is a basic sequence* $(x_i)_{i=1}^\infty$ *in* X *whose real unconditionality constant is* $<M$. *If* X *is a complex Banach space*, *then the complex unconditionality constant of* (x_i) *is* $<2M$.

Clearly, Theorem 6.1 implies the Gowers dichotomy theorem, because if X fails the assumptions of Theorem 6.1 for $M=\infty$, then $\widetilde{\varphi}_{\pm,2}(\vec{0},Y)=\infty$ for some $Y\in\mathcal{G}_\infty(X)$, which implies that Y is hereditarily indecomposable.

Proof. Let $\varphi=\varphi_{\pm,2}$. Let $Y\in\mathcal{G}_\infty(X)$ be a φ-stable subspace and let $M'\in(\widetilde{\varphi}(\vec{0},Y),M)$. Fix an $\eta>1$ and apply Proposition 4.6 with $d=M'$ in order to produce a basic sequence (x_i) in Y whose basis constant is $\le\eta$ such that for each $n\ge1$ and each function $f:\{1,\ldots,n\}\to\{1,2\}$, one has

$$\widetilde{\varphi}((\mathrm{span}\{x_j:1\le j\le n,\ f(j)=i\})_{i=1}^2,Y)<M'.$$

By (5), the latter estimate implies that for each $n\ge1$, each sequence $(a_i)_{i=1}^n$ of scalars and each function $\varepsilon:\{1,\ldots,n\}\to\{1,-1\}$ one has

$$\Big\|\sum_{i=1}^n\varepsilon(i)a_ix_i\Big\|\le M'\Big\|\sum_{i=1}^n a_ix_i\Big\|.$$

This shows that the real unconditionality constant of (x_i) is $\le M'$. If X is a complex Banach space it is well known that the latter property together with the fact that (x_i) is linearly independent implies that the complex unconditionality constant of (x_i) is $\le 2M'$. This completes the proof.

REMARK 6.2. This theorem should be compared with the following easy fact. If X has an M-unconditional basis and $Y \in \mathcal{G}_\infty(X)$, then $\widetilde{\varphi}_{\pm,2}(\vec{0}_2, Y) \le M$, and if X is a complex Banach space, then also $\sup_k \widetilde{\varphi}_{\mathbb{C},k}(\vec{0}_k, Y) \le M$.

For our next theorem we need to estimate the complex unconditionality constant of a sequence by an expression that involves a bounded number of coefficients. Put $T = \{z \in \mathbb{C} : |z| = 1\}$ and $T_k = \{z \in \mathbb{C} : z^k = 1\}$.

LEMMA 6.3. *If Y is a complex Banach space and $y_1, \ldots, y_n \in Y$, then*

$$\sup_{\mu_1,\ldots,\mu_n \in T_k} \Big\| \sum_{j=1}^n \mu_j y_j \Big\| \ge \Big(\cos\frac{\pi}{k}\Big) \sup_{\lambda_1,\ldots,\lambda_n \in T} \Big\| \sum_{j=1}^n \lambda_j y_j \Big\|.$$

Proof. Put $a = \cos(\pi/k)$. Let $f \in Y^*$ be a linear functional of norm 1 and let $(\lambda_j)_{j=1}^n \in T^n$. For $1 \le j \le n$ we can find $\mu_j \in T_k$ so that $\Re(\mu_j f(y_j)) \ge a|f(y_j)|$. Then we can estimate

$$\Big\| \sum_{j=1}^n \mu_j y_j \Big\| \ge \Re f\Big(\sum_{j=1}^n \mu_j y_j\Big) \ge a \sum_{j=1}^n |f(y_j)| \ge a \Big| f\Big(\sum_{j=1}^n \lambda_j y_j\Big)\Big|.$$

This estimate completes the proof, since $\|y\| = \sup_{\|f\| \le 1} |f(y)|$ for $y \in Y$.

THEOREM 6.4. *Let X be a complex Banach space. Suppose that $M < \infty$ and $\sup_k \widetilde{\varphi}_{\mathbb{C},k}(\vec{0}_k, Y) < M$ for each $Y \in \mathcal{G}_\infty(X)$. Then there is a basic sequence $(x_i)_{i=1}^\infty$ in X whose complex unconditionality constant is $< M$.*

Proof. Using Lemma 2.2, we find $Y \in \mathcal{G}_\infty(X)$ which is $\varphi_{\mathbb{C},k}$-stable for each $k \ge 2$. Fix $M' < M$ so that $\varphi_{\mathbb{C},k}(\vec{0}_k, Y) < M'$ for each k. Then fix k so that $\cos(\pi/k) > M'/M$. Fix $\eta > 1$. Using Proposition 4.6 with $d = M'$, we obtain a basic sequence (x_i) in Y whose basis constant is $\le \eta$ such that for each $n \ge 1$ and each function $f : \{1, \ldots, n\} \to \{1, \ldots, k\}$ one has

$$\widetilde{\varphi}_{\mathbb{C}.k}((\operatorname{span}\{x_j : 1 \le j \le n,\ f(j) = i\})_{i=1}^k, Y) < M'.$$

By (6), the latter estimate implies that for each $n \ge 1$, each sequence $(a_i)_{i=1}^n$ of scalars and each function $\mu : \{1, \ldots, n\} \to T_k$ one has

$$\Big\| \sum_{i=1}^n \mu(i) a_i x_i \Big\| \le M' \Big\| \sum_{i=1}^n a_i x_i \Big\|.$$

Using this estimate and Lemma 6.3 we infer that the complex unconditionality constant of (x_i) is $\le M'/\cos(\pi/k) < M$. This completes the proof.

REMARK 6.5. Theorem 6.1 could be strengthened if one could show that if X is $\varphi_{\pm,2}$-stable and $\widetilde{\varphi}_{\pm,2}(\vec{0}_2, X) = M < \infty$, then X contains an

M-unconditional basic sequence. We are only able to show that X contains an *asymptotically* M-unconditional basic sequence (x_i), i.e., $(x_i)_{i\geq n}$ is M_n-unconditional, where $\lim_{n\to\infty} M_n = M$. This follows easily by using Proposition 4.7 with appropriate parameters.

An analogous comment can be made with regard to Theorem 6.4.

Let $\mathbb{R}^{<\omega}$ denote the linear space of all real sequences $(x_i)_{i=1}^{\infty}$ such that $\{i \in \mathbb{N} : x_i \neq 0\}$ is finite. Let X be a real or complex Banach space. If $\mathbf{q}$ is a 1-unconditional norm on $\mathbb{R}^{<\omega}$, we put

$$\Phi_{>\mathbf{q}}(X) = \sup\{\widetilde{\varphi}_{>\mathbf{q}|_{\mathbb{R}^k},k}(\vec{0}_k, Y) : k \geq 2,\, Y \in \mathcal{G}_\infty(X)\}, \tag{9}$$

$$\Phi_{<\mathbf{q}}(X) = \sup\{\widetilde{\varphi}_{<\mathbf{q}|_{\mathbb{R}^k},k}(\vec{0}_k, Y) : k \geq 2,\, Y \in \mathcal{G}_\infty(X)\}. \tag{10}$$

The most interesting special case occurs when $\mathbf{q} = \mathbf{q}_p$ for some $p \in [1, \infty]$, where $\mathbf{q}_\infty(x) := \max_i |x_i|$ and $\mathbf{q}_p(x) := (\sum_i |x_i|^p)^{1/p}$ if $1 \leq p < \infty$.

THEOREM 6.6. *Let X be a Banach space and let $\mathbf{q}$, $\mathbf{r}$ be 1-unconditional norms on $\mathbb{R}^{<\omega}$. Then, for each $\eta > 1$, there is a basic sequence $(x_j)_{j=1}^{\infty}$ in X whose basis constant is $\leq \eta$ such that for each $n \geq 1$, each function $f : \mathbb{N} \to \mathbb{N}$ and each $k \geq 2$, if $A_i = \{j \in \mathbb{N} : f(j) = i,\, k \leq j \leq n\}$ for $1 \leq i \leq k$ and $A_i = \emptyset$ for $i > k$, then for each sequence (a_j) of scalars*,

$$\mathbf{q}\Big(\Big(\Big\|\sum_{j\in A_i} a_j x_j\Big\|\Big)_i\Big) \leq \eta \Phi_{>\mathbf{q}}(X) \Big\|\sum_{j=k}^{n} a_j x_j\Big\|, \tag{11}$$

$$\Big\|\sum_{j=k}^{n} a_j x_j\Big\| \leq \eta \Phi_{<\mathbf{r}}(X) \mathbf{r}\Big(\Big(\Big\|\sum_{j\in A_i} a_j x_j\Big\|\Big)_i\Big). \tag{12}$$

Moreover, if $\Phi_{>\mathbf{q}}(X) < \infty$, then the basic sequence (x_j) can be chosen to be M-unconditional, where M is defined in Theorem 6.1 (or Theorem 6.4).

Proof. By passing to a suitable subspace of X, we may assume that X is $\varphi_{\pm,2}$-stable and for each $k \geq 2$, X is φ-stable for $\varphi \in \{\varphi_{>\mathbf{q}|_{\mathbb{R}^k},k}, \varphi_{<\mathbf{r}|_{\mathbb{R}^k},k}\}$, and $\varphi_{\mathbb{C},k}$-stable if X is a complex space. Then the sequence (x_i) can be constructed by using Proposition 4.7 with appropriate parameters. The estimate (11) follows from (3) and (7), while (12) follows from (3) and (8).

Finally, it is easy to check that if $\Phi_{>\mathbf{q}}(X) < \infty$ then the quantity M is finite. Since the estimates in the proof of Theorem 6.1 (resp. Theorem 6.4) make use of a single function $\widetilde{\varphi}_{\pm,2}$ (resp. $\widetilde{\varphi}_{\mathbb{C},k}$) and X is φ-stable for that φ, our inductive construction of the x_j's yields the M-unconditionality of the whole sequence, although the estimates (11) and (12) are obtained only for its tail sequnces $(x_i)_{i\geq k}$. This completes the proof.

Observe that if Y has a basis (x_j) satisfying the estimates (11) and (12) with some finite constants, then Y contains a sequence of subspaces of the form $Y_n = Y_{n1} + \ldots + Y_{nn}$ which admit both upper $\mathbf{r}$-estimates and lower

$\mathbf{q}$-estimates with uniformly bounded constants. It suffices to take Y_{ni} to be the closure of $\operatorname{span}(\{x_{nm+i}\}_{m=1}^{\infty})$ for $i = 1, \dots, n$.

If X is a Banach space such that $\Phi_{>\mathbf{q}}(X) < \infty$ and $\Phi_{<\mathbf{q}}(X) < \infty$, then $\mathbf{q}$ must be equivalent to a $\mathbf{q}_p$. Namely, one has

COROLLARY 6.7. *Let $\mathbf{q}$ be a 1-unconditional norm on $\mathbb{R}^{<\omega}$. If there exists a Banach space X such that $\Phi_{>\mathbf{q}}(X) < \infty$ and $\Phi_{<\mathbf{q}}(X) < \infty$, then there exist $p \in [1, \infty]$ and $C < \infty$ such that $\mathbf{q} \leq C\mathbf{q}_p$ and $\mathbf{q}_p \leq C\mathbf{q}$.*

Proof. By Theorem 6.6 (with $\mathbf{r} := \mathbf{q}$), the assumption implies that there exists a Banach space Y with an unconditional basis (x_j) which satisfies (11) and (12). Thus the conclusion can be obtained by adapting the well known argument due to M. Zippin [Z]. This completes the proof.

The preceding results motivate adopting the following definition.

DEFINITION 6.8. Let $\mathbf{m} \in \mathbb{N}^{\mathbb{N}}$. We say that a sequence $(A_i)_{i=1}^n$ of finite subsets of $\mathbb{N}$ is $\mathbf{m}$-*allowable* if $\min \bigcup_i A_i \geq m_n$ and the A_i's are mutually disjoint. A basic sequence (x_j) is said to have the $T(\mathbf{m}, p)$-*property* if there is $K < \infty$ such that for every $\mathbf{m}$-allowable sequence $(A_i)_{i=1}^n$, if (a_j) is a sequence of scalars and $y_i = \sum_{j \in A_i} a_j x_j$ for $i = 1, \dots, n$, then

$$K^{-1}\|(\|y_i\|)_{i=1}^n\|_p \leq \Big\|\sum_{i=1}^n y_i\Big\| \leq K\|(\|y_i\|)_{i=1}^n\|_p.$$

We say that the basic sequence (x_j) has the $T(p)$-*property* if it has the $T(\mathbf{m}, p)$-property for some $\mathbf{m} \in \mathbb{N}^{\mathbb{N}}$. If, in addition to the $T(p)$-property, the closed linear span of (x_j) contains no subspace isomorphic to l_p or c_0, then we say that (x_j) is a *generalized Tsirelson basic sequence*.

Clearly, if $\mathbf{m}$ is a bounded sequence, then each basis with the $T(\mathbf{m}, p)$-property is equivalent to the natural basis of l_p (if $1 \leq p < \infty$) or of c_0 (if $p = \infty$). Needless to say, for each $p \in [1, \infty]$ there exist generalized Tsirelson basic sequences with the $T(p)$-property. We refer the reader to [C-S] for many interesting results in this area. The first example of such a basic sequence was constructed by B. S. Tsirelson [T] for $p = \infty$. We should mention that only after a few years it was discovered and proved in [C-O] that Tsirelson's example was isomorphic to its modified version defined by W. B. Johnson, which was the archetype of the property defined in Definition 6.8.

Note that a special case of Theorem 6.6 yields the following corollary.

COROLLARY 6.9. *Let X be a Banach space, $1 \leq p \leq \infty$ and suppose that $\Phi_{>\mathbf{q}_p}(X) < \infty$ and $\Phi_{<\mathbf{q}_p}(X) < \infty$. Then every subspace of X contains an unconditional basic sequence with the $T(p)$-property. Moreover, if X con-*

tains no isomorphic copy of l_p *(resp. of* c_0 *if* $p=\infty$*), then* X *is saturated with generalized Tsirelson basic sequences with the* $T(p)$*-property.*

DEFINITION 6.10. A Banach space X is said to be an *asymptotic* $\mathcal{L}_p$ *space* if there exists $K<\infty$ such that for each n there is $Y_n \in \mathcal{G}_\infty(X)$ such that $\dim X/Y_n < \infty$ and for each $E \in \mathcal{G}_{\rm fin}(Y_n)$ with $\dim E = n$ there exists $F \in \mathcal{G}_{\rm fin}(Y_n)$ such that $F \supseteq E$ and the Banach–Mazur distance $d(F, l_p^{\dim F})$ is $\le K$.

PROPOSITION 6.11. *If* $1 \le p \le \infty$ *and* X *is a Banach space which has a basis with the* $T(p)$*-property, then* X *is an asymptotic* $\mathcal{L}_p$ *space.*

We omit the proof, since it can be obtained using standard methods.

7. Continuity of functions $\widetilde{\varphi}$. Let X be a Banach space, let $\varphi : \mathcal{G}_\infty(X)^k \to [0,\infty]$ and let $\widetilde{\varphi}$ be defined by (2).

We consider three conditions on φ, denoted as $(\mathcal{A})$, $(\mathcal{B})$ and $(\mathcal{C})$. We shall show that if φ satisfies $(\mathcal{A})$ and $(\mathcal{B})$, then Lemma 2.1 can be applied to $\widetilde{\varphi} : \mathcal{G}(X)_0^k \times \mathcal{G}_\infty(X) \to [0,\infty]$, hence X has φ-stable subspaces. If φ satisfies $(\mathcal{C})$ as well, then Lemma 2.1 can be applied to $\widetilde{\varphi} : \mathcal{G}_{\rm fin}(X)^k \times \mathcal{G}_\infty(X) \to [0,\infty]$, hence X has $(\varphi, \mathcal{G}_{\rm fin}(X)^k)$-stable subspaces. This reduces Proposition 5.2 to the easy verification of $(\mathcal{A})$, $(\mathcal{B})$ and $(\mathcal{C})$ (in one case $(\mathcal{C})$ fails, but Lemma 7.10 takes care of that).

DEFINITION 7.1. We say that φ satisfies *condition* $(\mathcal{A})$ if $\varphi(\vec{F}+\vec{U}) \ge \varphi(\vec{U})$ for each $\vec{F} \in \mathcal{G}_{\rm fin}(X)^k$ and $\vec{U} \in \mathcal{G}_\infty(X)^k$.

LEMMA 7.2. *If* φ *satisfies* $(\mathcal{A})$ *then, for each* $\vec{E} \in \mathcal{G}_{\rm fin}(X)^k$, *one has*

$$\widetilde{\varphi}(\vec{E}, Y) \le \widetilde{\varphi}(\vec{E}, Z), \tag{13}$$

whenever $Y, Z \in \mathcal{G}_\infty(X)$ *and* $Z \subseteq F + Y$ *for some* $F \in \mathcal{G}_{\rm fin}(X)$.

Proof. It follows from $(\mathcal{A})$ and (2) that $\widetilde{\varphi}(\vec{E}, Y) \le \widetilde{\varphi}(\vec{E}, F+Y) \le \widetilde{\varphi}(\vec{E}, Z)$.

Let $\mathrm{Gl}(X)$ denote the group of those linear isomorphisms $R : X \to X$ such that $R - I_X$ is a finite rank operator. If $\vec{U} = (U_1, \ldots, U_k) \in \mathcal{G}(X)^k$ and $R \in \mathrm{Gl}(X)$ we let $R\vec{U} = (R(U_1), \ldots, R(U_k))$.

DEFINITION 7.3. We say that φ satisfies *condition* $(\mathcal{B})$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $R \in \mathrm{Gl}(X)$ and $\vec{U} \in \mathcal{G}_\infty(X)^k$, then

$$\|R - I_X\| < \delta \ \Rightarrow\ \varphi(R\vec{U}) \le (1+\varepsilon)\varphi(\vec{U}). \tag{14}$$

Observe that if φ satisfies (14), then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $R \in \mathrm{Gl}(X)$ and $\vec{U} \in \mathcal{G}_\infty(X)^k$, then

$$\|R - I_X\| < \delta \ \Rightarrow\ (1+\varepsilon)^{-1}\varphi(\vec{U}) \le \varphi(R\vec{U}) \le (1+\varepsilon)\varphi(\vec{U}). \tag{15}$$

For, if $\delta > 0$ and $\|R - I_X\| < \delta/(1+\delta)$, then $R^{-1} \in \mathrm{Gl}(X)$ and $\|R^{-1} - I_X\| < \delta$, hence the lower estimate in (15) follows from (14) with R replaced by R^{-1}.

LEMMA 7.4. *If φ satisfies* (15) *and* (13) *then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $R \in \mathrm{Gl}(X)$, $\vec{E} \in \mathcal{G}_\infty(X)^k$ and $Y \in \mathcal{G}_\infty(X)$, then*

$$(16)\qquad \|R - I_X\| < \delta \ \Rightarrow\ (1+\varepsilon)^{-1}\widetilde{\varphi}(\vec{E}, Y) \le \widetilde{\varphi}(R(\vec{E}), Y) \le (1+\varepsilon)\widetilde{\varphi}(\vec{E}, Y).$$

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ be as in (15). Fix $R \in \mathrm{Gl}(X)$ such that $\|R - I_X\| < \delta$. Let Z be the kernel of $R - I_X$, so that $\dim X/Z < \infty$.

Fix $\vec{E} \in \mathcal{G}_\infty(X)^k$ and $Y \in \mathcal{G}_\infty(X)$. To prove the upper estimate, we may assume that $\widetilde{\varphi}(\vec{E}, Y) < \infty$. Now we fix $b \in (\widetilde{\varphi}(\vec{E}, Y), \infty)$. It follows from (13) that $\widetilde{\varphi}(\vec{E}, Y \cap Z) = \widetilde{\varphi}(\vec{E}, Y)$. Hence, by (2), we can pick $\vec{U} \in \mathcal{G}_\infty(Y \cap Z)^k$ such that $\varphi(\vec{E} + \vec{U}) < b$. Observe that $R(\vec{E} + \vec{U}) = R(\vec{E}) + \vec{U}$, because $R(E_i + U_i) = R(E_i) + U_i$ for $i = 1, \ldots, k$. Therefore,

$$\widetilde{\varphi}(R(\vec{E}), Y) \le \varphi(R(\vec{E}) + \vec{U}) = \varphi(R(\vec{E} + \vec{U})).$$

Using the upper estimate in (15), we obtain

$$\varphi(R(\vec{E} + \vec{U})) \le (1+\varepsilon)\varphi(\vec{E} + \vec{U}) < (1+\varepsilon)b.$$

Letting b tend to $\widetilde{\varphi}(\vec{E}, Y)$, we obtain the upper estimate in (16).

If δ is as in (15), then we have just also proved the upper estimate for R^{-1}. The latter yields the lower estimate in (16). This completes the proof.

Let $p : \mathcal{G}_{\mathrm{fin}}(X)^k \to [0, \infty]$ be defined as follows. If $\vec{E} \in \mathcal{G}(X)_0^k$ we let

$$p(\vec{E}) = \max\{\|P_i\| : 1 \le i \le k\},$$

where P_i is the unique linear projection of $E := E_1 + \ldots + E_k$ onto E_i such that $P_i(x) = 0$ for $x \in E_j$ with $j \ne i$. If $\vec{E} \in \mathcal{G}_{\mathrm{fin}}(X)^k \setminus \mathcal{G}(X)_0^k$ we let $p(\vec{E}) = \infty$. It is easy to verify that the function p is continuous on $\mathcal{G}_{\mathrm{fin}}(X)^k$.

DEFINITION 7.5. We say that φ satisfies *condition* $(\mathcal{C})$ if there exists $a > 0$ such that $\varphi(\vec{F} + \vec{U}) \ge a p(\vec{F})$ for each $\vec{F} \in \mathcal{G}_{\mathrm{fin}}(X)^k$ and $\vec{U} \in \mathcal{G}_\infty(X)^k$.

LEMMA 7.6. *For each $\vec{E} \in \mathcal{G}(X)_0^k$ there exists $\gamma \ge 1$ such that for each $\vec{F} \in \mathcal{G}_{\mathrm{fin}}(X)^k$, if $\varrho(\vec{F}, \vec{E}) < 1/\gamma$ then $\vec{F} = R(\vec{E})$ for some $R \in \mathrm{Gl}(X)$ such that $\|R - I_X\| \le \gamma\varrho(\vec{F}, \vec{E})$.*

Proof. Let $n = \sum_{i=1}^k \dim E_i$. The case where $n = 0$ is trivial (one may take $\gamma = 1$). Thus we may and do assume that $n > 0$. Put $t = p(\vec{E})$, $r = \varrho(\vec{F}, \vec{E})$ and let $\gamma = tn$. We shall construct an operator T on X of rank n such that $\|T\| \le \gamma r$ and $R := I_X + T$ maps isomorphically E_i onto F_i for $1 \le i \le k$. Then the assumption $\gamma r < 1$ will imply that $R \in \mathrm{Gl}(X)$.

To this end, for each j, $1 \le j \le k$, we choose an Auerbach basis in E_j, i.e., we let $(e_{ij})_{i=1}^{\dim E_j}$ be norm one vectors in E_j such that the biorthogonal functionals $(e_{ij}^*)_{i=1}^{\dim E_j}$ in E_j^* also have norm one. Let $x_{ij}^* \in X^*$ be a norm preserving extension of the linear functional $P_j^*(e_{ij}^*)$ defined on E. Then $\|x_{ij}^*\| \le \|P_j\| \le t$ and $x_{ij}^*|_{E_s} = 0$ for $s \ne j$.

Now, for each $1 \le j \le k$ and $1 \le i \le \dim E_j$, pick an element $f_{ij} \in F_j$ such that $\|f_{ij} - e_{ij}\| \le r$. Let T be the linear operator on X defined by

$$Tx = \sum_{j=1}^{k} \sum_{i=1}^{\dim E_j} x_{ij}^*(x)(f_{ij} - e_{ij}).$$

Then $\dim T(X) \le n$ and $\|T\| \le ntr = \gamma r$, since T is the sum of n rank one operators whose norms are $\le tr$.

It remains to verify that $R(E_j) = F_j$ for $1 \le j \le k$. Observe first that $R(E_j) \subseteq F_j$, because $Re_{ij} = e_{ij} + Te_{ij} = f_{ij} \in F_j$ for all elements of our Auerbach basis of E_j. Thus it suffices to observe that $\dim R(E_j) = \dim E_j = \dim F_j$. The first equality is obvious and the second one follows from a theorem of M. G. Krein, M. A. Krasnosel'skiĭ and D. P. Milman (cf. [K, p. 199]), because the condition $r < 1$ implies that $\varrho(E_j, F_j) < 1$. This completes the proof.

Proposition 7.7. *If $\varphi : \mathcal{G}_\infty(X)^k \to [0,\infty]$ satisfies condition $(\mathcal{A})$, then for each fixed $\vec{E} \in \mathcal{G}_{\mathrm{fin}}(X)^k$ the function $\widetilde{\varphi}$ is $\preceq$-nondecreasing on $\mathcal{G}_\infty(X)$.*

If φ satisfies conditions $(\mathcal{A})$ and $(\mathcal{B})$, then for each fixed $Y \in \mathcal{G}_\infty(X)$ the function $\widetilde{\varphi}$ is continuous at each $\vec{E} \in \mathcal{G}(X)_0^k$.

If φ satisfies condition $(\mathcal{C})$, then for each fixed $Y \in \mathcal{G}_\infty(X)$ the function $\widetilde{\varphi}$ is continuous at each $\vec{E} \in \mathcal{G}_{\mathrm{fin}}(X)^k \setminus \mathcal{G}(X)_0^k$.

Proof. The first assertion follows from Lemma 7.2. The second one follows from Lemmas 7.2, 7.4 and 7.6. The final assertion is proved as follows. Let $\vec{E}$ and Y be fixed. Clearly, $(\mathcal{C})$ implies that $\widetilde{\varphi}(\vec{F}, Y) \ge ap(\vec{F})$ for each $\vec{F} \in \mathcal{G}_{\mathrm{fin}}(X)^k$. Thus $\widetilde{\varphi}(\vec{E}, Y) \ge ap(\vec{E}) = \infty$ and $\liminf_{\vec{F}\to\vec{E}} \widetilde{\varphi}(\vec{F}, Y) \ge \liminf_{\vec{F}\to\vec{E}} ap(\vec{F}) = \infty$, because p is continuous. This completes the proof.

Corollary 7.8. *Let X be a Banach space and $\varphi : \mathcal{G}_\infty(X)^k \to [0,\infty]$. If φ satisfies $(\mathcal{A})$ and $(\mathcal{B})$, then X is saturated with $\widetilde{\varphi}$-stable subspaces.*

Proof. This follows from Lemma 2.1 and Proposition 7.7.

The following statement is easy to verify for each $h : X^k \to [0,\infty]$.

Proposition 7.9. *Both $h^\uparrow$ and $h^\downarrow$ have property $(\mathcal{A})$.*

If for each $x \in X^k$ one has $h(tx) = th(x)$ for each $t \ge 0$ and

$$h(Rx_1, \dots, Rx_k) \le \|R\| \, \|R^{-1}\| h(x_1, \dots, x_k)$$

for each $R \in \mathrm{Gl}(X)$, then both $h^\uparrow$ and $h^\downarrow$ have property $(\mathcal{B})$.

If there exists $a > 0$ such that for each $x \in X^k$ one has $h(x_1, \dots, x_k) \ge a\|x_i\|$ for $1 \le i \le k$, then $h^\uparrow$ has property $(\mathcal{C})$.

Thus all the functions φ introduced in Definition 5.1 *have properties* $(\mathcal{A})$ *and* $(\mathcal{B})$. *The functions* $\varphi_{\pm,k}$, $\varphi_{\mathbb{C},k}$ *and* $\varphi_{>q,k}$ *have property* $(\mathcal{C})$.

It follows from Propositions 7.7 and 7.9 that the functions $\varphi_{\pm,k}$, $\varphi_{\mathbb{C},k}$ and $\varphi_{>q,k}$ have the property asserted in Proposition 5.2. However, $\varphi_{<q,k}$ fails property $(\mathcal{C})$, because it is a bounded function. The proof of Proposition 5.2 will be complete when we have proved the following.

LEMMA 7.10. *The function* $\widetilde{\varphi}_{<q,k}$ *is a continuous function on* $\mathcal{G}_{\mathrm{fin}}(X)^k$ *for each fixed* $Y \in \mathcal{G}_\infty(X)$.

Proof. Let h denote the function $h_{\mathbf{q}}$ defined by (4). Clearly, h is a norm on X^k. Put $b = \max_{1\le i\le k} \mathbf{q}((\delta_{ij})_{j=1}^k)$ and $a = \min_{1\le i\le k} \mathbf{q}((\delta_{ij})_{j=1}^k)$. Note that for each $x = (x_1, \ldots, x_k) \in X^k$ one has the estimate

$$h(x) = \mathbf{q}(\|x_1\|, \ldots, \|x_k\|) \le b \sum_{i=1}^k \|x_i\|,$$

which implies that $\widetilde{\varphi}_{<\mathbf{q},k}(\vec{E}, Y) \ge 1/b$ for $\vec{E} \in \mathcal{G}_{\mathrm{fin}}(X)^k$ and $Y \in \mathcal{G}_\infty(X)$.

Fix a $Y \in \mathcal{G}_\infty(X)$ and put $\Psi(\vec{E}) := 1/\widetilde{\varphi}_{<\mathbf{q},k}(\vec{E}, Y)$. We shall verify that the function Ψ satisfies the condition

$$|\Psi(\vec{E}) - \Psi(\vec{E}')| \le L \sum_{i=1}^k \varrho(E_i, E_i')$$

for $\vec{E} = (E_1, \ldots, E_k)$, $\vec{E}' = (E_1', \ldots, E_k')$ in $\mathcal{G}_{\mathrm{fin}}(X)^k$, where $L = b(1 + b/a)$. Clearly, this will imply the uniform continuity of Ψ on $\mathcal{G}_{\mathrm{fin}}(X)^k$.

From now on, $\vec{E}$ and $\vec{E}'$ are fixed. It suffices to consider the case where $\Psi(\vec{E}) > \Psi(\vec{E}')$. Put for brevity $s = \Psi(\vec{E})$ and $r = \sum_{i=1}^n \varrho(E_i, E_i')$. We need to verify that $s - \Psi(\vec{E}') \le Lr$. Observe that we already know that $s \le b$.

Recall that if $\vec{U} = (U_1, \ldots, U_k) \in \mathcal{G}_\infty(X)^k$, we put

$$h^{\downarrow}(\vec{U})^{-1} = \inf\{h(u) : u \in \mathcal{U}\},$$

and if $\vec{E} = (E_1, \ldots, E_k) \in \mathcal{G}_{\mathrm{fin}}(X)^k$ and $Y \in \mathcal{G}_\infty(X)$, we put

$$\Psi(\vec{E}) = (\widetilde{\varphi}_{<\mathbf{q},k}(\vec{E}, Y))^{-1} = \sup\{h^{\downarrow}(\vec{E} + \vec{U})^{-1} : \vec{U} \in \mathcal{G}_\infty(Y)^k\}.$$

Now we fix a positive number $s' < s$. Since $\dim(E_1' + \ldots + E_k') < \infty$, for any fixed $\eta > 1$ we can find a linear subspace X_η of finite codimension in X such that if $x \in X_\eta$ and $e' \in E_i'$ for some i, then $\|e'\| \le \eta\|e' + x\|$.

Since $\Psi(\vec{E}) > s'$, one can find $\vec{U} = (U_1, \ldots, U_k) \in \mathcal{G}_\infty(Y)^k$ such that $h^{\downarrow}(\vec{E} + \vec{U})^{-1} > s'$. Replacing each U_i by $U_i \cap X_\eta$, we may assume that in addition $U_i \subseteq Y \cap X_\eta$ for each i.

Now we fix any vectors $e_1', \ldots, e_k'$ and $u_1, \ldots, u_k$ with $e_i' \in E_i'$ and $u_i \in U_i$ such that $Q' := h((e_i' + u_i)) > 0$. Then we fix for each i an element $e_i \in E_i$

such that $\|e_i - e_i'\| \le \varrho(E_i, E_i')\|e_i'\|$. We put $Q = h((e_i + u_i))$ and also

$$S' = \Big\| \sum_{i=1}^{k} (e_i' + u_i) \Big\|, \qquad S = \Big\| \sum_{i=1}^{k} (e_i + u_i) \Big\|.$$

Since $h^{\downarrow}(\vec{E} + \vec{U})^{-1} > s'$, we have $s'S \le Q$.

Using well known properties of the norms h and $\mathbf{q}$, we get

$$Q - Q' \le h((e_i - e_i')) \le \mathbf{q}(r(\|e_i'\|)) \le r\mathbf{q}(\eta(\|e_i' + u_i\|)) = r\eta Q'$$

and hence $Q \le (1 + r\eta)Q'$. Similarly, one obtains

$$S' - S \le \sum_{i=1}^{k} \|e_i - e_i'\| \le \sum_{i=1}^{k} \varrho(E_i, E_i')\|e_i'\| \le r\eta \max_{1 \le i \le k} \|e_i' + u_i\| \le a^{-1} r\eta Q'.$$

By combining the latter three estimates we obtain easily

$$s'(S' - a^{-1} r\eta Q') \le s'S \le Q \le (1 + r\eta)Q'.$$

This shows that $s'S' \le (1 + r\eta + a^{-1}s'r\eta)Q'$. Thus all ratios of the form Q'/S' are bounded from below by the number $s'/(1 + r\eta + a^{-1}s'r\eta)$. This shows that

$$\Psi(\vec{E}') \ge s'/(1 + r\eta + \eta a^{-1} s'r).$$

Since η can be any number > 1 and then s' can be any number $< s$, it follows that $\Psi(\vec{E}') \ge s/(1 + r + a^{-1}sr)$. Using this and the inequality $s \le b$, we get the estimate

$$\Psi(\vec{E}) - \Psi(\vec{E}') \le s - s/(1 + r + a^{-1}sr) \le rs(1 + s/a) \le b(1 + b/a)r = Lr.$$

This completes the proof.

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