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Asymptotic estimates for a perturbation of the linearization of an equation for compressible viscous fluid flow

by

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Abstract. We prove *a priori* estimates for a linear system of partial differential equations originating from the equations for the flow of a barotropic compressible viscous fluid under the influence of the gravity it generates. These estimates will be used in a forthcoming paper to prove the nonlinear stability of the motionless, spherically symmetric equilibrium states of barotropic, self-gravitating viscous fluids with respect to perturbations of zero total angular momentum. These equilibrium states as well as the non-stationary solutions occupy part of space, and a constant pressure is assumed on the free surface, but no surface tension.

1. Introduction. This paper contains the continuation of the analysis of systems of linear equations related to the linearization of the equations describing the flow of a barotropic fluid under the influence of its own gravity with a free surface without surface tension, begun in [8] and [9]. The linearization is carried out at a spherically symmetric stationary solution of the equations. We begin by giving a short, and not entirely rigorous, description of the original problem and its relationship with the equations studied here for the purpose of motivation.

The original equations, formulated in Euler coordinates, concern a fluid occupying a domain Ω_t at time $t \in I_\omega = [0,\omega] \cap \mathbb{R}$ with some $\omega \in (0,\infty]$. We denote the space-time region in which the flow takes place by

$$\Omega^\omega = \bigcup_{t\in I_\omega} \{(y,t) \mid y \in \Omega_t\},$$

the velocity of the flow by $v : \overline{\Omega^\omega} \to \mathbb{R}^3$, $v(t) = v(y,t)$ with $v(t) : \overline{\Omega}_t \to \mathbb{R}^3$, the density of the fluid by $\varrho : \overline{\Omega^\omega} \to \mathbb{R}$, $\varrho(t) = \varrho(y,t)$, $\varrho(t) : \overline{\Omega}_t \to \mathbb{R}$, and its pressure by $p : \overline{\Omega^\omega} \to \mathbb{R}$, $p(t) = p(y,t)$, $p(t) : \overline{\Omega}_t \to \mathbb{R}$. We also define the

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stress tensor $\mathbb{T}$ by

$$\mathbb{T}(v,p) = [\mathbb{T}_{ij}(v,p)] = [-p\delta_{ij} + \mathbb{D}_{ij}(v)]$$

with

$$\mathbb{D}(v) = [\mathbb{D}_{ij}(v)] = \mu([\nabla v] + [\nabla v]^t) + (\nu - \mu)\,\mathrm{tr}(\nabla v)E_3.$$

(In this paper the symbol ∇ only denotes the vector of space derivatives.) We assume the viscosity coefficients μ and ν satisfy $\nu \geq \mu/3 > 0$. The gravitational potential of the mass distribution is given by

$$G(y,t) = \mathcal{G}(\varrho, \Omega_t) = \mathsf{k} \int\limits_{\Omega_t} \frac{\varrho(\widetilde{y},t)}{|\widetilde{y} - y|}\, d\widetilde{y}, \tag{1}$$

where k is the gravitational constant. We also assume the equation of state $p = p(\varrho) = K\varrho^{\varkappa}$ with $\varkappa > 4/3, K > 0$ for the gas in question and that it is exposed to a positive outside pressure $\mathsf{p}_0 > 0$ on $\partial\Omega_t$. Then for $(y,t) \in \Omega^{\omega}$ we have the equations

$$\begin{aligned} \varrho(v' + v\cdot\nabla v) - \mathrm{div}(\mathbb{T}(v,p)) &= \varrho\nabla G, \\ \varrho' + \mathrm{div}(\varrho v) &= 0, \end{aligned} \tag{2}$$

while for $t \in I_\omega$, $y \in \partial\Omega_t$ we have

$$\mathbb{T}(v, p(\varrho))\cdot n = -\mathsf{p}_0 n. \tag{3}$$

Note that there is no surface tension. The domain is also moving with the flow, which means that if we follow the direction of the vector field v beginning at any point $(y,t) \in \Omega^{\omega}$ we only leave this set at $t = 0$ and $t = \omega$. It will be shown in [10] that we can describe the domains Ω_t by means of a family of diffeomorphisms $T_t : \overline{B}_R \to \overline{\Omega_t}$ which also satisfy the equations

$$\frac{dT_t}{dt}(y,t) = v(T_t(y,t),t),$$

which means that T_t follows the flow. By specifying T_0 we also describe Ω_0. Then $T_t \circ T_0^{-1}$ is identical with standard Lagrange coordinates for this problem. We can describe these transformations also in the form of the mapping $\mathbf{T} : \overline{B}_R \times I_\omega \to \mathbb{R}^3$ defined by

$$\mathbf{T}(x,t) = T_t(x) \qquad (x \in \overline{B}_R,\ t \in I_\omega). \tag{4}$$

In the main part of this paper $\mathbf{T}$ will be assumed to be given, but we will still use the two alternative ways of denoting the transformation we just introduced in (4).

For an equilibrium state without motion ($v = 0$) the equations (2) and (3) reduce to $\nabla(p(\varrho)) = \varrho\nabla G$ in Ω_t, which is then constant, and $p(\varrho) = \mathsf{p}_0$ on the spatial boundary. As was shown, e.g., in [11], given a mass $M \in (0,\infty)$ and a pressure $\mathsf{p}_0 \in (0,\infty)$ there exists exactly one number $R > 0$ and exactly one function $\varrho_e \in C^\infty([0,R])$ such that with $p_e = p(\varrho_e)$ the function $G_e(y) = \mathcal{G}(\varrho_e(|y|), B_R)$ is spherically symmetric, and for $|y| \le R$ we have

$$\nabla(p_e(|y|)) = \frac{\partial p}{\partial \varrho}(\varrho_e(|y|))\nabla(\varrho_e(|y|)) = \varrho_e(|y|)\nabla G_e(y), \qquad p_e(R) = \mathsf{p}_0,$$

and the integral of the density $\varrho_e(|y|)$ over B_R equals M. This means that $\varrho_e(|y|)$ is the density of a spherically symmetric equilibrium solution. (See [11].) From now on we will use the symbols ϱ_e, p_e, G_e interchangeably for functions of r and y, so that, e.g., $\varrho_e(|y|) = \varrho_e(y)$. The meaning will be clear from the context.

In [8] the linearization

$$\begin{aligned} \varrho_e u' - \operatorname{div}(\mathbb{D}(u)) + \varrho_e \nabla(\gamma_e \alpha) &= \varrho_e \nabla[I(\alpha,\beta)], \\ \alpha' + \operatorname{div}(\varrho_e u) &= 0, \\ \beta' &= u \cdot n \end{aligned} \tag{5}$$

of the equations (2) and (3) at these equilibrium solutions was computed. The boundary conditions are

$$\widetilde{\mathbb{T}}(u,\alpha)\cdot n = \frac{\partial p_e}{\partial r}(R)\beta n. \tag{6}$$

Here u is the variation of the velocity, α that of the density, both defined on $B_R \times I_\omega$, and β is the variation of the function describing Ω_t as a graph in polar coordinates, it is defined on $\partial B_R \times I_\omega$. Also $\gamma_e = \big(\frac{\partial p}{\partial \varrho} \circ \varrho_e\big)\varrho_e^{-1}$,

$$\widetilde{\mathbb{T}}(u,\alpha) = \mathbb{D}(u) - \gamma_e(R)\varrho_e(R)\alpha E_3$$

and

$$I(\alpha,\beta) = \mathsf{k}\bigg(\int_{B_R} \frac{\alpha(y)}{|x-y|}\,dy + \varrho_e(R)\int_{\partial B_R} \frac{\beta(y)}{|x-y|}\,d\sigma_y\bigg). \tag{7}$$

The integral operator I represents the linearization of the gravity potential. In [8] it was shown that solutions of (5) and (6) belonging to suitable spaces decay in an algebraic fashion, while the results in [5] and [7] strongly suggest that one cannot expect exponential decay for the solutions of this problem. Due to the slow decay, the application of the results about the linearization to the non-linear problem requires a very delicate analysis. To this end we include the deformation $\mathsf{d}(x,t) = \mathbf{T}(x,t) - x$ ($x \in B_R$, $t \in I_\omega$), which satisfies the equation

$$\mathsf{d}' = u. \tag{8}$$

We also need to consider suitable perturbations of the resulting system. The paper [9] is dedicated to the analysis of the homogeneous equation with homogeneous boundary conditions without any explicit dependence on time. In the present paper we will consider inhomogeneous and time-dependent equations. As this analysis strongly depends on the result of [9] and uses much the same concepts, we describe the main result of that paper in a form more appropriate for use here and introduce the notation used there to the extent that it is relevant here. This will partly be done by contrasting the problems considered in [8] with [9] and this paper. One difference between [8], [9] and this paper is that we assume here that all function spaces consist of real-valued functions only, as the direct work with analytic semigroups characteristic of the previous papers is now finished, and we can reap its benefits without getting entangled in the technical details. We unite the relevant variables in one vector function

$$\mathbf{U}(t) = [u(t), \alpha(t), \beta(t), \mathsf{d}(t)].$$

One of the conditions one needs to impose on the components of $\mathbf{U}$ in order to obtain the asymptotic estimates in [9] is that certain integral quantities, which originate from the conservation laws of the original equation, vanish. In the notation used in [9] we can express this condition by introducing the scalar product

$$\begin{aligned}&([u, \alpha, \beta, \mathsf{d}], [\widetilde{u}, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\mathsf{d}}])_{\mathfrak{H}} \\ &\qquad = \int\limits_{B_R} \varrho_e(x) u\widetilde{u}\, dx + \int\limits_{B_R} \alpha\widetilde{\alpha}\, dx + \varrho_e(R) \int\limits_{\partial B_R} \beta\widetilde{\beta}\, d\sigma + \int\limits_{B_R} \varrho_e(x) \mathsf{d}\widetilde{\mathsf{d}}\, dx\end{aligned}$$

for these functions on the space $\mathfrak{H} = L_2(B_R) \times L_2(B_R) \times L_2(\partial B_R) \times L_2(B_R)$. If $\mathcal{N}$ is the linear space spanned by the elements

$$[0, 1, 1, 0],\ [e_k, 0, 0, 0],\ [x \times e_k, 0, 0, 0],\ [0, x_k, x_k, 0] \quad (k = 1, 2, 3),$$

these conditions can be expressed in the form

$$(\mathbf{U}(t), u)_{\mathfrak{H}} = 0 \quad (u \in \mathcal{N}). \tag{9}$$

The solutions of the systems of equations considered both in [8] and [9] are defined on B_R as far as the space variables are concerned. If one poses the same problem as in [8] on a different set Ω, given as $\Omega = T(B_R)$ with a diffeomorphism $T \in W_p^2(B_R)$, and then transforms it to B_R one obtains a problem very close, but not identical with the one considered in [9]. We assume that this diffeomorphism T satisfies the conditions

$$\|T - E_{B_R}\|_{W_P^2(B_R)} \le 1, \quad \det(\nabla T) \ge 1/2, \quad \min_{x \in \partial B_R} |T(x)| \ge R/2. \tag{10}$$

For $u \in W_p^2(B_R)$ we can then define

$$\mathbb{D}_T(u) = \mathbb{D}(u \circ T^{-1}) \circ T.$$

Introducing $\mathcal{Z}_T \in W_p^1(B_R)$ by

$$\mathcal{Z}_T = [\nabla T]^{-1}, \tag{11}$$

one can easily see that

$$\mathbb{D}_T(u) = \mu(\nabla u \mathcal{Z}_T + (\nabla u \mathcal{Z}_T)^t) + (\nu - \mu)(\mathrm{tr}(\nabla u \mathcal{Z}_T))E_3. \tag{12}$$

Likewise let

$$\begin{aligned} \widetilde{\mathbb{T}}_T(u,\alpha) &= \mathbb{D}_T(u) - \gamma_e(R)\varrho_e(R)\alpha E_3, \\ (L_T)_k(u) &= \frac{1}{\varrho_e \circ T}((\mathbb{D}_T(u))_{kj})_{x_q}(\mathcal{Z}_T)_{qj} \qquad (k = 1,2,3), \\ \mathcal{L}_T(u,\alpha) &= L_T(u) - \nabla((\gamma_e \circ T)\alpha)\mathcal{Z}_T. \end{aligned} \tag{13}$$

With $m_T(x) = T(x)/|T(x)|$ for $x \in \partial B_R$ and $U = [u, \alpha, \beta, \mathsf{d}]$ we then define the operator $\mathfrak{A}_{1T}$ by

$$\mathfrak{A}_{1T}U = [\mathcal{L}_T(u,\alpha) + \nabla I(\alpha,\beta)\mathcal{Z}_T, -\mathrm{tr}(\nabla((\varrho_e \circ T)u)\mathcal{Z}_T), u \cdot m_T, u]. \tag{14}$$

The expression $u \cdot m_T$ may not seem to fit into our transformation scheme. To give a hint of why it does, we have to return to the original problem. As the flow takes boundary points to boundary points we have $T_t(\partial B_R) = \partial\Omega_t$, and for $\partial\Omega_t$ which are small perturbations of ∂B_R the ray in the direction of $T_t(x)$ only contains one point, $T_t(x)$, of $\partial\Omega_t$. Thus the function

$$\beta(x,t) = |T_t(x)| - R \tag{15}$$

can be thought of as describing $\partial\Omega_t$ as a graph in Lagrange coordinates. Differentiating (15) with respect to time we obtain $\beta' = u \cdot m_{T_t}$.

The domain of the operator $\mathfrak{A}_{1T}$ will be specified later. With

$$\mathbb{B}_T(U) = \mathbb{B}_T(u,\alpha,\beta) = \widetilde{\mathbb{T}}_T(u,\alpha) - \frac{\partial p_e}{\partial r}(R)\beta E_3$$

let

$$\mathcal{B}_T(U) = \mathcal{B}_T(u,\alpha,\beta) = \mathbb{B}_T(u,\alpha,\beta) \cdot n_T, \tag{16}$$

where $n_T(x)$ is the exterior unit normal to $\partial\Omega$ at $T(x)$. The equations (5), (6) and (8) can now be written in the form

$$\mathbf{U}_t = \mathfrak{A}_{1E_{B_R}}\mathbf{U}, \qquad \mathcal{B}_{E_{B_R}}(\mathbf{U}) = 0.$$

Let

$$\begin{aligned} \mathfrak{B}^s = \{[u,\alpha,\beta,\mathsf{d}] : u \in W_p^{s-1}(B_R),\ \alpha \in W_p^s(B_R), \\ \beta \in W_p^{s+1-1/p}(\partial B_R),\ \mathsf{d} \in W_p^{s+1}(B_R)\} \end{aligned}$$

with $p > 9$, $1/p < s \le 1$ and

$$D^s = \{[u,\alpha,\beta,\mathsf{d}] \in \mathfrak{B}^s : u \in W_p^{s+1}(B_R)\}.$$

With the norm

$$\|[u,\alpha,\beta,\mathsf{d}]\|_{\mathfrak{B}^s_{-1}} = \|u\|_{W_p^{s-1}} + \|\alpha\|_{W_p^s} + \|\beta\|_{W_p^{s-1/p}} + \|\mathsf{d}\|_{W_p^{s+1}},$$

$\mathfrak{B}^s$ is a normed vector space, as is D^s with the norm

$$\|[u,\alpha,\beta,\mathsf{d}]\|_{D^s_{-1}} = \|u\|_{W_p^{s+1}} + \|[u,\alpha,\beta,\mathsf{d}]\|_{\mathfrak{B}^s_{-1}}.$$

In addition, $\mathfrak{B}^s$ is a Banach space with

$$\|[u,\alpha,\beta,\mathsf{d}]\|_{\mathfrak{B}^s} = \|[u,\alpha,\beta,\mathsf{d}]\|_{\mathfrak{B}^s_{-1}} + \|\beta\|_{W_p^{s+1-1/p}},$$

and so is D^s with the norm

$$\|[u,\alpha,\beta,\mathsf{d}]\|_{D^s} = \|[u,\alpha,\beta,\mathsf{d}]\|_{D^s_{-1}} + \|\beta\|_{W_p^{s+1-1/p}}.$$

Specifying D^1 as the domain of $\mathfrak{A}_{1T}$ we obtain $\mathfrak{A}_{1T} : D^1 \to \mathfrak{B}^1$. The significance of the other norms will become clear in Theorem 1. Let $\mathcal{P}^c$ be the orthogonal projection from $\mathfrak{H}$ to $\mathcal{N}$ and $\mathcal{P} = E_{\mathfrak{H}} - \mathcal{P}^c$. We clearly have $\mathcal{P}^c U \in D^1$ and

$$\|\mathcal{P}^c U\|_{D^1} \le C\|U\|_{\mathfrak{H}}$$

for all $U \in \mathfrak{H}$. Also let $B = \mathcal{P}(\mathfrak{B}^1)$ and

$$\mathcal{A}_{1T} = \mathcal{P}\mathfrak{A}_{1T}.$$

This projection is a rather brute force, but effective, method for dealing with the condition $\mathbf{U}(t) \perp \mathcal{N}$. Also let

$$\Pi[u,\alpha,\beta,\mathsf{d}] = [u,\alpha,\beta,0],$$

$$D^1(\mathcal{A}_T) = \{U \in B \cap D^1 : \mathcal{B}_T(U) = 0\}, \qquad \mathcal{A}_T = \mathcal{A}_{1T}|D^1(\mathcal{A}_T).$$

With these definitions we can now formulate a version of the main result of [9] in a form suited to our purposes.

THEOREM 1. *For $s \in (1/p, 1]$ there exist numbers $C < \infty$ and $\eta > 0$ such that if $\|T - E\|_{W_p^2(B_R)} \le \eta$, then for every $U_0 \in B$ there is exactly one function $\mathbf{U} : [0,\infty) \to B$, $\mathbf{U} \in C^0([0,\infty), B) \cap C^1((0,\infty), D^1(\mathcal{A}_T))$ solving*

$$\mathbf{U}'(t) = \mathcal{A}_T\mathbf{U}(t) \quad \text{for } t > 0, \qquad \mathbf{U}(0) = U_0,$$

and it satisfies the inequality

$$\|\mathbf{U}(t)\|_{\mathfrak{B}^1} + t\|\mathbf{U}'(t)\|_{\mathfrak{B}^1} + t^2\|\Pi\mathbf{U}'(t)\|_{D^1_{-1}} + t^{2-s}\|\Pi\mathbf{U}(t)\|_{D^s_{-1}} \le C\|U_0\|_{\mathfrak{B}^1}.$$

This theorem is a direct consequence of Theorem 1.2 in [9]. As already mentioned, the purpose of this paper is to consider the case in which T is no longer static, but is replaced by a family of transformations denoted by $\mathbf{T} : \overline{B}_R \times [0,\omega] \to \mathbb{R}^3$, and at the same time to consider the non-homogeneous equation and non-homogeneous boundary conditions. To this end we need to introduce a few additional spaces and norms for various functions depending on time $t \in [0,\omega]$; these were not defined in [9]. Unfortunately, the norms we need to consider are somewhat complicated.

DEFINITION 2. Let $\omega \in [1,\infty)$. For the transformations $\mathbf{T}$ we introduce the space

$$\mathsf{B}^\omega = C^0([0,\omega], W_p^2(B_R)) \cap C^{2/3}([0,\omega], C^1(\overline{B}_R))$$

and the weighted norm

$$\|\mathbf{T}\|_{\mathsf{B}^\omega} = \sup_{t\in[0,\omega-1]} [\|\mathbf{T}\|_{C^0([t,t+1],W_p^2)} + (1+t)^{1/3}\|\mathbf{T}\|_{C^{2/3}([t,t+1],C^1)}].$$

Let $B_1^\omega = L_p((0,\omega),\mathfrak{B}^1)$, $B_2^\omega = W_p^{1-1/p,1/2-1/2p}(\partial B_R \times (0,\omega))$ and

$$\mathfrak{W}^\omega = \{\mathbf{U} = [u,\alpha,\beta,\mathsf{d}] : \mathbf{U} \in L_p((0,\omega),D^1),\ \mathbf{U}' \in L_p((0,\omega),\mathfrak{B}^1)\}$$

(for the meaning of $\mathbf{U}'$ see Section 2). Then let

$$\|F\|_{B_1^\omega} = \sup_{t\in[0,\omega-1]} (1+t)^{4/3}\Big[\int_t^{t+1} \|F(\tau)\|_{\mathfrak{B}^1}^p\,d\tau\Big]^{1/p},$$

$$\|g\|_{B_2^\omega} = \sup_{t\in[0,\omega-1]} (1+t)^{4/3}\|g\|_{W_p^{1-1/p,1/2-1/2p}(\partial B_R\times[t,t+1])}.$$

Defining the norm

$$\|\mathbf{U}\|_{\mathfrak{W}_1^\omega} = \sup_{t\in[0,\omega-1]} \Big[\int_t^{t+1} (\|\mathbf{U}'(\tau)\|_{\mathfrak{B}^1}^p + \|\mathbf{U}(\tau)\|_{D^1}^p)\,d\tau\Big]^{1/p}$$

and the two seminorms

$$[\mathbf{U}]_1 = \sup_{t\in[0,\omega-1]} (1+t)\Big[\int_t^{t+1} (\|\mathbf{U}'(\tau)\|_{\mathfrak{B}^1}^p + \|\varPi\mathbf{U}(\tau)\|_{D_{-1}^1}^p)\,d\tau\Big]^{1/p},$$

$$[\mathbf{U}]_2 = \sup_{t\in[0,\omega-1]} (1+t)^{4/3}\Big[\int_t^{t+1} \|\varPi\mathbf{U}'(\tau)\|_{\mathfrak{B}_{-1}^1}^p\,d\tau\Big]^{1/p} + \sup_{t\in[0,\omega]} (1+t)^{4/3}\|\varPi\mathbf{U}(t)\|_{D_{-1}^{2/3}},$$

we can now conclude with the final norm

$$\|\mathbf{U}\|_{\mathfrak{W}^\omega} = [\mathbf{U}]_1 + [\mathbf{U}]_2 + \|\mathbf{U}\|_{\mathfrak{W}_1^\omega}.$$

It may be helpful to make a few remarks about the spaces $\mathfrak{W}^\omega$. It is not hard to see that as sets they are identical with the collection of all $\mathbf{U} = [u,\alpha,\beta,\mathsf{d}]$ such that $u \in W_p^{2,1}(B_R\times(0,\omega))$, $\alpha \in W_p^1((0,\omega),W_p^1(B_R))$, $\beta \in W_p^1((0,\omega),W_p^{2-1/p}(\partial B_R))$, $\mathsf{d} \in W_p^1((0,\omega),W_p^2(B_R))$. It is also obvious that the weights used in defining the norm of $\mathfrak{W}^\omega$ require certain decay properties for functions with a bounded norm. Using $s = 2/3$ one can see from Theorem 1 that solutions of the equation considered there satisfy such estimates at least for $t \ge 1$. The specific value $s = 2/3$ and the exponent $4/3$ are somewhat arbitrary. We could choose any $s \in [2/3,1)$ and then would have to replace $4/3$ by $2-s$. It is necessary, though, to have $2-s > 1$ in order to ensure the boundedness of certain integrals involving these functions as t goes to infinity.

Replacing the fixed transformation T by the time dependent $\mathbf{T}$ and using the notation introduced in (4) and (11), we define

$$\mathcal{Z}_{\mathbf{T}}(t,x) = \mathcal{Z}_{T_t}(x) \quad (t \in [0,\omega],\, x \in \overline{B}_R)$$

and in complete analogy

$$(\mathcal{A}_{1\mathbf{T}}\mathbf{U})(t) = \mathcal{A}_{1T_t}\mathbf{U}(t)$$

for $t \in I_\omega$. The meaning of $\mathcal{B}_{\mathbf{T}}\mathbf{U}$ as well as a number of other similar expressions should now be clear. Note that

$$\{U = [u,\alpha,\beta,\mathsf{d}] \in \mathfrak{B}^1 \mid u \in W_p^{2-2/p}(B_R)\} = \mathfrak{B}^1 \cap D^{1-2/p}.$$

Our main result in this paper is

Theorem 3. *There exist numbers $\eta > 0$ and $C < \infty$ with the following properties. Let $\omega \in [1,\infty)$, $\mathbf{T} \in \mathsf{B}^\omega$ and $\widetilde{\mathbf{T}}(x,t) = \mathbf{T}(x,\omega)$ $(x \in \overline{B}_R,\, t \in [0,\omega])$. If*

$$\|\mathbf{T} - \widetilde{\mathbf{T}}\|_{\mathsf{B}^\omega} + \|\mathbf{T}(t) - E_{B_R}\|_{W_p^2} \le \eta \quad (t \in [0,\omega]),$$

then for $F \in B_1^\omega$, $g \in B_2^\omega, U_0 \in B \cap D^{1-2/p}$ with $\mathcal{B}_{T_0}(U_0) = g(0)$ there is exactly one function $\mathbf{U} \in \mathfrak{W}^\omega$ with $\mathbf{U}(t) \in B$ for $t \in I_\omega$ such that

$$\mathcal{B}_{\mathbf{T}}(\mathbf{U}) = g, \tag{17}$$

$\mathbf{U}(0) = U_0$ and

$$\mathbf{U}' = \mathcal{A}_{1\mathbf{T}}\mathbf{U} + F \tag{18}$$

for almost all $t \in I_\omega$, where the time derivative is meant in the sense of distributions (*see Section* 2). *This function also satisfies the inequality*

$$\|\mathbf{U}\|_{\mathfrak{W}^\omega} \le C(\|F\|_{B_1^\omega} + \|g\|_{B_2^\omega} + \|U_0\|_{\mathfrak{B}^1} + \|U_0\|_{D^{1-2/p}}).$$

Note in particular that although the theorem only makes a statement about $\omega < \infty$, the constant C is independent of ω.

In the application of this theorem in [10], g and F themselves depend on the vector $\mathbf{U}$ in such a way that

$$\|F(\mathbf{U})\|_{B_1^\omega} + \|g(\mathbf{U})\|_{B_2^\omega} \le C\|\mathbf{U}\|_{\mathfrak{W}^\omega}^{4/3}.$$

This then allows us to estimate $\|\mathbf{U}\|_{\mathfrak{W}^\omega}$ for small solutions. There is, however, the additional complication that the $\mathbf{U}$ describing the flow does not actually belong to B, but one can split it up into a component in B and one in $\mathcal{N}$. The former can then be estimated by using Theorem 3, and the latter by means of the conservation laws.

The proof of Theorem 3 proceeds by combining the local maximum regularity estimates proved in Section 4 with asymptotic estimates for semigroups taken from [9] by means of an abstract theorem proved in Section 5.

I am indebted to W. Zajączkowski from the Institute of Mathematics of the Polish Academy of Sciences for many crucial discussions.

2. Notation and function spaces. We denote generic constants by C. Let $\operatorname{tr}(A)$ denote the trace of the matrix A, A^t its transpose, while E_n is the $n \times n$ unit matrix. For arbitrary $R > 0$ and $y \in \mathbb{R}^3$ let $B_R(y) = \{x \in \mathbb{R}^3 : |x - y| < R\}$ and $B_R = B_R(0)$. We also use the summation convention that any index occurring at least twice in an expression is to be summed over its natural range.

Unless otherwise stated all our vectors are column vectors. Exceptions are the function vector U and the gradient of a vector or scalar function. Taking the transformation T as an example, we have $T(x) = [T_1(x), T_2(x), T_3(x)]^t$ and

$$\nabla T = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} & \frac{\partial T_2}{\partial x_3} \\ \frac{\partial T_3}{\partial x_1} & \frac{\partial T_3}{\partial x_2} & \frac{\partial T_3}{\partial x_3} \end{bmatrix}.$$

Thus, if T and $\widehat{T}$ are two such transformations, we have $\nabla(T \circ \widehat{T}) = ((\nabla T) \circ \widehat{T})\nabla\widehat{T}$ wherever such a composition makes sense.

The norm of any Banach space $\mathcal{B}$ is denoted by $\|\cdot\|_{\mathcal{B}}$. Let $S \subset \mathbb{R}^m$ be an open set or its closure. For any Banach space $\mathcal{B}$ and $\delta \in [0,1)$ let $C^\delta(S,\mathcal{B})$ be the space of all continuous functions $f : S \to \mathcal{B}$ such that

$$\|f\|_{C^\delta(S,\mathcal{B})} = \sup_{x,x' \in S,\, x \neq x'} \frac{\|f(x) - f(x')\|_{\mathcal{B}}}{|x - x'|^\delta} + \sup_{x \in S} \|f(x)\|_{\mathcal{B}} < \infty.$$

The set $C^\delta(S,\mathcal{B})$ is a Banach space. Let $C_0^\delta(S,\mathcal{B})$ contain all elements of $C^\delta(S,\mathcal{B})$ with compact support in S. If Ω is open, $L_p(\Omega,\mathcal{B})$ is the closure of $C_0^0(\Omega,\mathcal{B})$ with respect to the norm

$$\Big(\int_\Omega \|u(x)\|_{\mathcal{B}}^p \, dx\Big)^{1/p}.$$

Throughout the paper we use primes to denote time derivatives. This is to be understood in the sense of distributions if necessary. If, e.g., $\mathbf{U} \in L_p((0,\omega),\mathcal{B})$ the statement $\mathbf{U}' \in L_p((0,\omega),\mathcal{B})$ means that there is a function $\mathbf{V} \in L_p((0,\omega),\mathcal{B})$ such that for any $\varphi \in C_0^\infty((0,\omega))$ we have

$$\int_0^\omega \varphi'(t)\mathbf{U}(t)\,dt = -\int_0^\omega \varphi(t)\mathbf{V}(t)\,dt, \tag{19}$$

and $\mathbf{U}'$ then denotes this function $\mathbf{V}$. Then $W_p^1((0,\omega),\mathcal{B})$ consists of the functions $\mathbf{U} \in L_p((0,\omega),\mathcal{B})$ such that $\mathbf{U}' \in L_p((0,\omega),\mathcal{B})$. For this and other spaces for time-dependent functions also see Section 5.9.2 in [3].

We define $C^k(S)$ for $(k = 1,2,3,\ldots)$ as the spaces of all real scalar or vector functions which have continuous derivatives up to order k in S. This means that these are Banach spaces with the usual norms only in the case that S is compact. We denote by $C_0^k(S)$ the space of functions in $C^k(S)$ with compact support in S. This notation is used if the target space of a mapping is finite-dimensional, and it will be clear from the context what it is.

For real $s \geq 0$ we define $W_p^s(\Omega)$ as in [12, 4.2.1, equation (3)], and $W_{p0}^s(\Omega)$ is the closure in $W_p^s(\Omega)$ of the set of infinitely differentiable functions with compact support. We can define a seminorm for $u \in W_p^s(O)$ $(0 < s < 1)$, where O is now contained in $\mathbb{R}^m$, by

$$[u]_{W_p^s(O)} = \Big[\int_O\int_O \frac{|u(x)-u(y)|^p}{|x-y|^{m+sp}}\,dx\,dy\Big]^{1/p}, \tag{20}$$

while $[u]_{W_p^1(O)} = \|\nabla u\|_{L_p}$ and $[u]_{W_p^0(O)} = 0$. Then $\|u\|_{L_p(O)} + [u]_{W_p^s(O)}$ is a norm for $W_p^s(O)$ according to [12, equation (8), Section 4.4.1] for $0 \leq s \leq 1$. For $\nu < 0$ we define $W_p^{-\nu}(\Omega)$ as the dual space of $W_{q0}^{\nu}(\Omega)$ with $1/p + 1/q = 1$.

For anisotropic spaces on the product of two open sets $O_1 \times O_2$ of dimensions m_1, m_2 we define, for $0 < s_1 < 1$,

$$[u]_{W_p^{s_1,0}(O_1\times O_2)} = \Big[\int_{O_2}\Big(\int_{O_1}\int_{O_1} \frac{|u(x,z)-u(y,z)|^p}{|x-y|^{m_1+s_1p}}\,dx\,dy\Big)dz\Big]^{1/p}, \tag{21}$$

while $[u]_{W_p^{1,0}(O_1\times O_2)} = \|\nabla_x u(x,y)\|_{L_p}$ and $[u]_{W_p^{0,0}(O_1\times O_2)} = 0$. Also for $0 < s_2 < 1$ let

$$[u]_{W_p^{0,s_2}(O_1\times O_2)} = \Big[\int_{O_1}\Big(\int_{O_2}\int_{O_2} \frac{|u(z,x)-u(z,y)|^p}{|x-y|^{m_2+s_2p}}\,dx\,dy\Big)dz\Big]^{1/p}, \tag{22}$$

$[u]_{W_p^{0,1}(O_1\times O_2)} = \|\nabla_y u(x,y)\|_{L_p}$, and $[u]_{W_p^{s_1,s_2}(O_1\times O_2)} = [u]_{W_p^{s_1,0}(O_1\times O_2)} + [u]_{W_p^{0,s_2}(O_1\times O_2)}$. Then $\|u\|_{L_p(O_1\times O_2)} + [u]_{W_p^{s_1,s_2}(O_1\times O_2)}$ is a norm for the space $W_p^{s_1,s_2}(O_1 \times O_2)$. Such anisotropic spaces will be used with $O_2 = I$, where I is an open interval in $\mathbb{R}$. It is easy to see that, e.g., $W_p^{2,1}(B_R \times I)$ is isomorphic to $\{U \in L_p(I, W_p^2(B_R)) : U' \in L_p(I, L_p(B_R))\}$ with the norm

$$\Big[\int_I \|U(t)\|_{W_p^2}^p dt + \int_I \|U'(t)\|_{L_p}^p dt\Big]^{1/p}.$$

This and other similar relationships will be used extensively.

Function spaces for functions on any compact differentiable manifold with or without boundary can easily be defined using a finite collection of charts covering the manifold.

Where no confusion can arise, we will often omit the domain of definition of the functions in the notation for these spaces.

3. Continuity properties. In this section we assume $\mathbf{T}, \widehat{\mathbf{T}} : \overline{B}_R \times [0,\omega] \to \mathbb{R}^3$ and $\mathbf{T}, \widehat{\mathbf{T}} \in \mathsf{B}^\omega$, and that both T_t and $\widehat{T}_t$ satisfy condition (10) for all $t \in [0,\omega]$. Likewise let $T, \widehat{T} : \overline{B}_R \to \mathbb{R}^3$, $T, \widehat{T} \in W_p^2(B_R)$, satisfy (10).

LEMMA 4. *For every $\omega < \infty$ there exists a constant $C(\omega) < \infty$ such that for $\mathbf{U} \in \mathfrak{W}^\omega$,*

$$\|\mathbf{U}\|_{\mathfrak{W}_1^\omega} \le \|\mathbf{U}\|_{\mathfrak{W}^\omega} \le C(\omega)\|\mathbf{U}\|_{\mathfrak{W}_1^\omega}.$$

Also $\mathfrak{W}^\omega$ is a Banach space with both norms.

Proof. All constants C in this proof may depend on ω. Let $\mathbf{U} = [u, \alpha\,\beta, \mathsf{d}]$. The estimate for $\|\mathbf{U}\|_{\mathfrak{W}^\omega}$ from below is obvious due to the definition of $\|\mathbf{U}\|_{\mathfrak{W}^\omega}$. The estimate from above is also immediate except for the inequality

$$\|\Pi\mathbf{U}(t)\|_{D_{-1}^{2/3}} \le C\|\mathbf{U}\|_{\mathfrak{W}_1^\omega}$$

for $t \in [0, \omega]$. The only part of this statement requiring any thought is

$$\|u(t)\|_{W_p^{5/3}} \le C\|\mathbf{U}\|_{\mathfrak{W}_1^\omega}$$

for $t \in [0, \omega]$. Now this is also easy to see as

$$\|u(t)\|_{W_p^{5/3}} \le C\|u(t)\|_{W_p^{2-2/p}} \le C\|u\|_{W_p^{2,1}(B_R\times(0,\omega))} \le C\|\mathbf{U}\|_{\mathfrak{W}_1^\omega} \tag{23}$$

by Theorem 1.8.2 in [12], owing to the fact that $5/3 \le 2 - 2/p$. The last claim is straightforward.

LEMMA 5. *Assume O_k $(k = 1, 2)$ are open sets with C^1 boundaries whose closures are compact, lying either in $\mathbb{R}^{m_k}$ or in C^2 manifolds of dimension m_k. Then there exists a constant C such that for $s_1, s_2 \in [0, 1]$ we have, for any $p \in (1, \infty)$ and $u, v \in W_p^{s_1,s_2}(O_1 \times O_2) \cap L_\infty(O_1 \times O_2)$,*

$$[uv]_{W_p^{s_1,s_2}} \le C[\|u\|_{L_\infty}[v]_{W_p^{s_1,s_2}} + [u]_{W_p^{s_1,s_2}}\|v\|_{L_\infty}],$$
$$\|uv\|_{W_p^{s_1,s_2}} \le C[\|u\|_{L_\infty}\|v\|_{W_p^{s_1,s_2}} + \|u\|_{W_p^{s_1,s_2}}\|v\|_{L_\infty}].$$

If $s \in [0, 1]$ and $u, v \in W_p^s(O_1) \cap L_\infty(O_1)$ then

$$\|uv\|_{W_p^s} \le C[\|u\|_{L_\infty}\|v\|_{W_p^s} + \|u\|_{W_p^s}\|v\|_{L_\infty}].$$

If $sp > m_1$ then also

$$\|uv\|_{W_p^{s+1}} \le C[\|u\|_{W_p^{s+1}}\|v\|_{W_p^s} + \|u\|_{W_p^s}\|v\|_{W_p^{s+1}}]$$

for $u, v \in W_p^{s+1}(O_1)$.

Proof. It is easy to derive the claim for subsets of manifolds once it is proved for subsets of $\mathbb{R}^{m_k}$. The first three inequalities are easy to see from (20)–(22). For the last one note that as W_p^s is embedded into L_∞ we have

$$\|uv\|_{W_p^s} \le C\|u\|_{W_p^s}\|v\|_{W_p^s},$$

and therefore

$$\begin{aligned}\|uv\|_{W_p^{s+1}} &\le C(\|uv\|_{W_p^s} + \|\nabla(uv)\|_{W_p^s}) = C(\|uv\|_{W_p^s} + \|v\nabla u\|_{W_p^s} + \|u\nabla v\|_{W_p^s})\\ &\le C(\|u\|_{W_p^s}\|v\|_{W_p^s} + \|v\|_{W_p^s}\|\nabla u\|_{W_p^s} + \|u\|_{W_p^s}\|\nabla v\|_{W_p^s}),\end{aligned}$$

proving the claim.

The following three lemmas contain refinements of estimates given in [9].

LEMMA 6. *There exists a constant $C < \infty$ such that if $u \in W_p^2(B_R)$, then*

$$\|\mathbb{D}_T(u) - \mathbb{D}_{\widehat{T}}(u)\|_{W_p^1} + \|L_T(u) - L_{\widehat{T}}(u)\|_{L_p} \\ \le C(\|T - \widehat{T}\|_{W_p^2}\|u\|_{W_p^{5/3}} + \|T - \widehat{T}\|_{C^1(\overline{B}_R)}\|u\|_{W_p^2}).$$

Proof. For all $k, m, n, q \in \{1, 2, 3\}$ we have, by Lemma 3.1 in [9], the Sobolev embedding theorem and some elementary calculations,

$$\left\|\nabla\left[\frac{\partial u_k}{\partial x_m}(\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}})_{nq}\right]\right\|_{L_p} \\ \le \left\|\left(\nabla\frac{\partial u_k}{\partial x_m}\right)(\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}})_{nq}\right\|_{L_p} + \left\|\frac{\partial u_k}{\partial x_m}\nabla(\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}})_{nq}\right\|_{L_p} \\ \le C\|u\|_{W_p^2}\|\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}}\|_{C^0(\overline{B}_R)} + C\|u\|_{C^1(\overline{B}_R)}\|\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}}\|_{W_p^1} \\ \le C\|u\|_{W_p^2}\|T - \widehat{T}\|_{C^1(\overline{B}_R)} + C\|u\|_{W_p^{5/3}}\|T - \widehat{T}\|_{W_p^2}.$$

From this our claim follows easily.

LEMMA 7. *There exists a constant $C < \infty$ such that if $u \in W_p^2(B_R)$ and $\alpha \in W_p^1(B_R)$, then*

$$\|\mathcal{L}_T(u, \alpha) - \mathcal{L}_{\widehat{T}}(u, \alpha)\|_{L_p} \\ \le C(\|T - \widehat{T}\|_{C^1}(\|u\|_{W_p^2} + \|\alpha\|_{W_p^1}) + \|T - \widehat{T}\|_{W_p^2}\|u\|_{W_p^{5/3}}).$$

Proof. This inequality follows easily from Lemmas 5 and 6.

LEMMA 8. *There exists a constant $C < \infty$ such that if $U \in D^1$, then*

$$\|\mathfrak{A}_{1T}U - \mathfrak{A}_{1\widehat{T}}U\|_{\mathfrak{B}^1} \le C\|T - \widehat{T}\|_{W_p^2}\|\mathit{\Pi} U\|_{D_{-1}^{2/3}} + C\|T - \widehat{T}\|_{C^1}\|\mathit{\Pi} U\|_{D_{-1}^1}.$$

Proof. Let $U = [u, \alpha, \beta, \mathsf{d}]$. The estimate for the first component follows from Lemma 7 here together with Lemma 3.1 of [9] and inequality (27) of [8]. The last component of the difference is zero. We also infer, using Lemma 5, that

$$\|u \cdot (m_T - m_{\widehat{T}})\|_{W_p^{2-1/p}(\partial B_R)} \\ \le C\|u\|_{W_p^{1-1/p}}\|m_T - m_{\widehat{T}}\|_{W_p^{2-1/p}} + C\|u\|_{W_p^{2-1/p}}\|m_T - m_{\widehat{T}}\|_{W_p^{1-1/p}} \\ \le C\|u\|_{W_p^1(B_R)}\|T - \widehat{T}\|_{W_p^2(B_R)} + C\|u\|_{W_p^2(B_R)}\|T - \widehat{T}\|_{W_p^1(B_R)},$$

and the proof of the estimate for the remaining component is similar to the proof of Lemma 6.

THEOREM 9. *There exists a constant $C<\infty$ independent of ω such that if $\mathbf{U}\in\mathfrak{W}^\omega$ then*

$$\|\mathfrak{A}_{1\mathbf{T}}\mathbf{U}-\mathfrak{A}_{1\widehat{\mathbf{T}}}\mathbf{U}\|_{B_1^\omega}\le C\|\mathbf{T}-\widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}\|\mathbf{U}\|_{\mathfrak{W}^\omega}.$$

Proof. By Lemma 8, for $t\in[0,\omega-1]$,

$$\Big[\int_t^{t+1}\|\mathfrak{A}_{1T_\tau}\mathbf{U}(\tau)-\mathfrak{A}_{1\widehat{T}_\tau}\mathbf{U}(\tau)\|^p_{\mathfrak{B}^1}\,d\tau\Big]^{1/p}$$
$$\le C\Big[\int_t^{t+1}\|T_\tau-\widehat{T}_\tau\|^p_{W_p^2}\|\varPi\mathbf{U}(\tau)\|^p_{D_{-1}^{2/3}}\,d\tau\Big]^{1/p}$$
$$+C\Big[\int_t^{t+1}\|T_\tau-\widehat{T}_\tau\|^p_{C^1}\|\varPi\mathbf{U}(\tau)\|^p_{D_{-1}^1}\,d\tau\Big]^{1/p}.$$

Now

$$\Big[\int_t^{t+1}\|T_\tau-\widehat{T}_\tau\|^p_{W_p^2}\|\varPi\mathbf{U}(\tau)\|^p_{D_{-1}^{2/3}}\,d\tau\Big]^{1/p}$$
$$\le\max_{t\le\tau\le t+1}[\|T_\tau-\widehat{T}_\tau\|_{W_p^2}\|\varPi\mathbf{U}(\tau)\|_{D_{-1}^{2/3}}]$$
$$\le C\|\mathbf{T}-\widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}(1+t)^{-4/3}[\mathbf{U}]_2\le C\|\mathbf{T}-\widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}(1+t)^{-4/3}\|\mathbf{U}\|_{\mathfrak{W}^\omega}$$

and

$$\Big[\int_t^{t+1}\|T_\tau-\widehat{T}_\tau\|^p_{C^1(\overline{B}_R)}\|\varPi\mathbf{U}(\tau)\|^p_{D_{-1}^1}\,d\tau\Big]^{1/p}$$
$$\le\max_{t\le\tau\le t+1}\|T_\tau-\widehat{T}_\tau\|_{C^1}\Big[\int_t^{t+1}\|\varPi\mathbf{U}(\tau)\|^p_{D_{-1}^1}\,d\tau\Big]^{1/p}$$
$$\le C\|\mathbf{T}-\widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}(1+t)^{-1/3}[\mathbf{U}]_1(1+t)^{-1}$$
$$\le C\|\mathbf{T}-\widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}(1+t)^{-1/3}(1+t)^{-1}\|\mathbf{U}\|_{\mathfrak{W}^\omega},$$

and therefore

$$(1+t)^{4/3}\Big[\int_t^{t+1}\|\mathfrak{A}_{1T_\tau}\mathbf{U}(\tau)-\mathfrak{A}_{1\widehat{T}_\tau}\mathbf{U}(\tau)\|^p_{\mathfrak{B}^1}\,d\tau\Big]^{1/p}\le C\|\mathbf{T}-\widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}\|\mathbf{U}\|_{\mathfrak{W}^\omega}.$$

This proves our claim.

Now we deal with the boundary conditions.

LEMMA 10. *There exists a constant $C<\infty$ such that for $t\in[0,\omega-1]$,*

$$\|\mathcal{Z}_{\mathbf{T}}-\mathcal{Z}_{\widehat{\mathbf{T}}}\|_{W_p^{1-1/p,1/2-1/2p}(\partial B_R\times(t,t+1))}+\|n_{\mathbf{T}}-n_{\widehat{\mathbf{T}}}\|_{W_p^{1-1/p,1/2-1/2p}(\partial B_R\times(t,t+1))}\le C\|\mathbf{T}-\widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}.$$

Proof. Using the formula

$$n_{T_t}(x) = \frac{1}{|x^t \mathcal{Z}_{T_t}(x)|}\, x^t \mathcal{Z}_{T_t}(x), \tag{24}$$

from Lemma 3.2 in [9] and the definition of $\mathcal{Z}_{\mathbf{T}}$ this easily follows as

$$\begin{aligned}\|\mathcal{Z}_{\mathbf{T}} - \mathcal{Z}_{\widehat{\mathbf{T}}}\|_{C^{2/3}([t,t+1],C^0(\overline{B}_R))} + \|n_{\mathbf{T}} - n_{\widehat{\mathbf{T}}}\|_{C^{2/3}([t,t+1],C^0(\overline{B}_R))}\\ \le C\|\mathbf{T} - \widehat{\mathbf{T}}\|_{C^{2/3}([t,t+1],C^1(B_R))},\end{aligned}$$

which allows us to estimate the fractional time derivatives in these norms, while estimates for the space derivatives directly follow from Lemmas 3.1 and 3.3 of [9].

Theorem 11. *There is a constant C independent of ω such that for $\mathbf{U} \in \mathfrak{W}^\omega$,*

$$\|\mathcal{B}_{\mathbf{T}}(\mathbf{U}) - \mathcal{B}_{\widehat{\mathbf{T}}}(\mathbf{U})\|_{B_2^\omega} \le C\|\mathbf{T} - \widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}\|\mathbf{U}\|_{\mathfrak{W}^\omega}.$$

Proof. Let $t \in [0, \omega - 1]$. All spaces in the following sequences of inequalities consist of functions on $\partial B_R \times [t, t+1]$, unless otherwise stated, and during the proof let $\delta = 1 - 1/p$. As usual let $\mathbf{U} = [u, \alpha, \beta, \mathsf{d}]$.

First observe

$$\mathbb{B}_{\mathbf{T}}(\mathbf{U}) \cdot n_{\mathbf{T}} - \mathbb{B}_{\widehat{\mathbf{T}}}(\mathbf{U}) \cdot n_{\widehat{\mathbf{T}}} = (\mathbb{D}_{\mathbf{T}}(u) - \mathbb{D}_{\widehat{\mathbf{T}}}(u)) \cdot n_{\mathbf{T}} + \mathbb{B}_{\widehat{\mathbf{T}}}(\mathbf{U}) \cdot (n_{\mathbf{T}} - n_{\widehat{\mathbf{T}}}).$$

Also

$$\begin{aligned}\|(\mathbb{D}_{\mathbf{T}}(u) - \mathbb{D}_{\widehat{\mathbf{T}}}(u)) \cdot n_{\mathbf{T}}\|_{W_p^{\delta,\delta/2}} &\le C\|\mathbb{D}_{\mathbf{T}}(u) - \mathbb{D}_{\widehat{\mathbf{T}}}(u)\|_{W_p^{\delta,\delta/2}}\|n_{\mathbf{T}}\|_{W_p^{\delta,\delta/2}}\\ &\le C\|\mathbb{D}_{\mathbf{T}}(u) - \mathbb{D}_{\widehat{\mathbf{T}}}(u)\|_{W_p^{\delta,\delta/2}},\end{aligned}$$

as $\|n_{\mathbf{T}}\|_{W_p^{\delta,\delta/2}}$ is bounded by Lemma 10, and this space is embedded into L_∞, allowing us to use Lemma 5. Now, using Lemma 5 again as well as Lemma 10 and Theorem 1.8.2 in [12], we obtain

$$\begin{aligned}&\|\mathbb{D}_{\mathbf{T}}(u) - \mathbb{D}_{\widehat{\mathbf{T}}}(u)\|_{W_p^{\delta,\delta/2}}\\ &\quad\le C(\|\nabla u\|_{W_p^{\delta,\delta/2}}\|\mathcal{Z}_{\mathbf{T}} - \mathcal{Z}_{\widehat{\mathbf{T}}}\|_{C^0} + \|\nabla u\|_{C^0}\|\mathcal{Z}_{\mathbf{T}} - \mathcal{Z}_{\widehat{\mathbf{T}}}\|_{W_p^{\delta,\delta/2}})\\ &\quad\le C(\|u\|_{W_p^{2,1}(B_R\times(t,t+1))}\|\mathcal{Z}_{\mathbf{T}} - \mathcal{Z}_{\widehat{\mathbf{T}}}\|_{C^0}\\ &\qquad + \sup_{t\le\tau\le t+1}\|u(\tau)\|_{W_p^{5/3}(B_R)}\|\mathcal{Z}_{\mathbf{T}} - \mathcal{Z}_{\widehat{\mathbf{T}}}\|_{W_p^{\delta,\delta/2}})\\ &\quad\le C[\mathbf{U}]_1(1+t)^{-1}\|\mathbf{T} - \widehat{\mathbf{T}}\|_{C^0([t,t+1],C^1(\overline{B}_R))} + C[\mathbf{U}]_2(1+t)^{-4/3}\|\mathbf{T} - \widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}\\ &\quad\le C((1+t)^{-1}(1+t)^{-1/3}\|\mathbf{T} - \widehat{\mathbf{T}}\|_{\mathsf{B}^\omega} + (1+t)^{-4/3}\|\mathbf{T} - \widehat{\mathbf{T}}\|_{\mathsf{B}^\omega})\|\mathbf{U}\|_{\mathfrak{W}^\omega}\\ &\quad\le C(1+t)^{-4/3}\|\mathbf{T} - \widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}\|\mathbf{U}\|_{\mathfrak{W}^\omega}.\end{aligned}$$

Also by the lemmas mentioned above,

$$\begin{aligned}
&\|\mathbb{B}_{\widehat{\mathbf{T}}}(\mathbf{U})\cdot(n_{\mathbf{T}}-n_{\widehat{\mathbf{T}}})\|_{W_p^{\delta,\delta/2}}\\
&\quad\le C\|\mathbb{B}_{\widehat{\mathbf{T}}}(\mathbf{U})\|_{L_\infty}\|n_{\mathbf{T}}-n_{\widehat{\mathbf{T}}}\|_{W_p^{\delta,\delta/2}}+C\|\mathbb{B}_{\widehat{\mathbf{T}}}(\mathbf{U})\|_{W_p^{\delta,\delta/2}}\|n_{\mathbf{T}}-n_{\widehat{\mathbf{T}}}\|_{L_\infty}\\
&\quad\le C[\mathbf{U}]_2(1+t)^{-4/3}\|\mathbf{T}-\widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}+C[\mathbf{U}]_1(1+t)^{-1}\|\mathbf{T}-\widehat{\mathbf{T}}\|_{C^0([t,t+1],C^1(\overline{B}_R))}\\
&\quad\le C(1+t)^{-4/3}\|\mathbf{T}-\widehat{\mathbf{T}}\|_{\mathsf{B}^\omega}\|\mathbf{U}\|_{\mathfrak{W}^\omega}.
\end{aligned}$$

Then our claim easily follows.

4. Maximum regularity estimates. In the following let $\mathbf{T}\in\mathsf{B}^\omega$, and also assume that T_t satisfies condition (10) as well as

$$\|\mathbf{T}-E_{\overline{B}_R}\|_{C^{2/3}([0,\omega],C^1(\overline{B}_R))}\le 1.$$

For the relationship between $\mathbf{T}$ and T_t recall (4).

THEOREM 12. *Let $\widetilde{\omega}\in(0,\omega]$. Then there exists a constant $C(\omega)$ independent of $\widetilde{\omega}$ such that if*

$F_1\in L_p(B_R\times(0,\widetilde{\omega}))$, $u_0\in W_p^{2-2/p}(B_R)$, $g\in W_p^{1-1/p,1/2-1/2p}(\partial B_R\times(0,\widetilde{\omega}))$ *with $g(0)=\mathbb{D}_{T_0}(u_0)\cdot n_{T_0}$ are given, then there is exactly one solution $u\in W_p^{2,1}(B_R\times(0,\widetilde{\omega}))$ of the equation*

$$u'-L_{\mathbf{T}}(u)=F_1$$

with the boundary condition $\mathbb{D}_{\mathbf{T}}(u)\cdot n_{\mathbf{T}}=g$ and the initial value $u(0)=u_0$. This solution satisfies the estimate

$$\begin{aligned}
&\|u\|_{W_p^{2,1}(B_R\times(0,\widetilde{\omega}))}\\
&\le C(\omega)(\|g\|_{W_p^{1-1/p,1/2-1/2p}(\partial B_R\times(0,\widetilde{\omega}))}+\|F_1\|_{L_p(B_R\times(0,\widetilde{\omega}))}+\|u_0\|_{W_p^{2-2/p}(B_R)}).
\end{aligned}$$

Proof. This claim follows directly from Theorem 5.4 of [6], by using some remarks in that treatise about less regular coefficients.

Now we want to proceed to the entire operator.

THEOREM 13. *There exists a constant $C(\omega)$ such that if $F\in B_1^\omega$, $U_0\in\mathfrak{B}^1\cap D^{1-2/p}$ and $g\in W_p^{1-1/p,1/2-1/2p}(\partial B_R\times(0,\omega))$ with $g(0,x)=\mathcal{B}_{T_0}(U_0)(x)$ are given, then there is exactly one solution $\mathbf{U}\in\mathfrak{W}^\omega$ of the equation*

$$\mathbf{U}'=\mathfrak{A}_{1\mathbf{T}}\mathbf{U}+F$$

with the boundary condition

$$\mathcal{B}_{\mathbf{T}}(\mathbf{U})=g$$

and the initial value $\mathbf{U}(0)=U_0$. Also $\mathbf{U}(t)\in D^1$ for almost all $t\in[0,\omega]$, and

$$\|\mathbf{U}\|_{\mathfrak{W}^\omega}\le C(\omega)(\|g\|_{B_2^\omega}+\|F\|_{B_1^\omega}+\|U_0\|_{\mathfrak{B}^1}+\|u_0\|_{W_p^{2-2/p}(B_R)}).$$

Proof. Let $\mathbf{U} = [u, \alpha, \beta, \mathsf{d}]$ and $U_0 = [u_0, \alpha_0, \beta_0, \mathsf{d}_0]$. Note that in view of Lemma 4 we can replace $\|\mathbf{U}\|_{\mathfrak{W}^\omega}$ in the estimate by $\|\mathbf{U}\|_{\mathfrak{W}_1^\omega}$, as our constant is allowed to depend on ω.

The last three components of the equation $\mathbf{U}' = \mathfrak{A}_{1\mathbf{T}}\mathbf{U} + F$ are

$$\alpha' = (\mathrm{tr}(\nabla((\varrho_e \circ \mathbf{T})u)\mathcal{Z}_{\mathbf{T}})) + F_2,$$
$$\beta' = u \cdot m_{\mathbf{T}} + F_3, \qquad \mathsf{d}' = u + F_4.$$

Together with the initial values $\alpha(0) = \alpha_0, \beta(0) = \beta_0, \mathsf{d}(0) = \mathsf{d}_0$ the function u therefore determines $\alpha, \beta, \mathsf{d}$. The first component of the equation can be written as

$$u' - L_{\mathbf{T}}(u) = (\nabla((\gamma_e \circ \mathbf{T})\alpha) - \nabla I(\alpha, \beta))\mathcal{Z}_{\mathbf{T}} + F_1.$$

Together with the boundary conditions

$$\mathbb{D}_{\mathbf{T}}(u) \cdot n_{\mathbf{T}} = g + \gamma_e(R)\varrho_e(R)\alpha n_{\mathbf{T}} + \frac{\partial p_e}{\partial r}(R)\beta n_{\mathbf{T}}$$

and the initial condition $u(0) = u_0$ Theorem 12 allows us to conclude that given α and β with $\alpha(0) = \alpha_0$, $\beta(0) = \beta_0$ there is exactly one function $u \in W^{2,1}(B_R \times (0, \omega))$ satisfying these equations. For sufficiently small ω the mapping taking u to the next u in this cycle is a contraction in $W_p^{2,1}(B_R \times (0, \omega))$. It is possible and sometimes necessary to choose $\omega < 1$ for this part of the argument. Verifying this takes some work, but is completely straightforward. Thus we can prove existence and uniqueness of such solutions by the Banach fixed point theorem. Then we can continue the construction to any longer interval by a finite number of steps, also obtaining the estimate we claimed.

THEOREM 14. *There exists a number $\eta > 0$ and a constant $C(\omega)$ such that if $\|T_t - E_{B_R}\|_{W_p^2} \le \eta$, then the following is true. For $F \in B_1^\omega$ with $F(t) \in B$ for almost all $t, U_0 \in B \cap D^{1-2/p}$, and $g \in W_p^{1-1/p,1/2-1/2p}(\partial B_R \times (0, \omega))$ with $g(0) = \mathcal{B}_{T_0}(U_0)$ there is exactly one solution $\mathbf{U} \in \mathfrak{W}^\omega$ of the equation*

$$\mathbf{U}' = \mathcal{A}_{1\mathbf{T}}\mathbf{U} + F \tag{25}$$

with the boundary condition

$$\mathcal{B}_{\mathbf{T}}(\mathbf{U}) = g \tag{26}$$

and the initial value $\mathbf{U}(0) = U_0$. Also $\mathbf{U}(t) \in B$ for $t \in [0, \omega]$, and

$$\|\mathbf{U}\|_{\mathfrak{W}^\omega} \le C(\omega)(\|g\|_{B_2^\omega} + \|F\|_{B_1^\omega} + \|U_0\|_{\mathfrak{B}^1} + \|U_0\|_{D^{1-2/p}}).$$

Proof. Let $\mathbf{U}(t) = [u(t), \alpha(t)\,\beta(t), \mathsf{d}(t)]$ and let $\mathcal{N}_1$ be the space spanned by $[0, x_k, x_k, 0]$ $(k = 1, 2, 3)$ and $[0, 1, 1, 0]$, and with $v_k^1 = e_k$ and $v_k^2 = x \times e_k$ $(k = 1, 2, 3)$ let $\mathcal{N}_2$ be the space spanned by $[v_k^m, 0, 0, 0]$ for $m = 1, 2$, $k = 1, 2, 3$. These spaces are perpendicular to each other. Then let $\mathcal{P}_m^c : \mathfrak{H} \to \mathcal{N}_k$ be the orthogonal projectors, resulting in $\mathcal{P}^c = \mathcal{P}_1^c + \mathcal{P}_2^c$. In order to solve equations (25) and (26) we initially solve the equations

$$\mathbf{U}' = \mathfrak{A}_{1\mathbf{T}}\mathbf{U} - \mathcal{P}^c(\mathfrak{A}_{1\mathbf{T}} - \mathfrak{A}_{1E_{B_R}})\mathbf{U} - \mathcal{P}_2^c\mathfrak{A}_{1E_{B_R}}\mathbf{U} + F, \tag{27}$$

$$\mathcal{B}_{\mathbf{T}}(\mathbf{U}) = g \tag{28}$$

instead. Now Lemma 3.10 in [9] states that for $U \in D^1$ we have

$$\begin{aligned}(\mathfrak{A}_{1E_{B_R}}U, [0,1,1,0])_{\mathfrak{H}} &= 0,\\ (\mathfrak{A}_{1E_{B_R}}U, [0,x_k,x_k,0])_{\mathfrak{H}} &= (U, [e_k,0,0,0])_{\mathfrak{H}}\end{aligned} \tag{29}$$

and

$$(\mathfrak{A}_{1E_{B_R}}U, [v_k^m,0,0,0])_{\mathfrak{H}} = \int\limits_{\partial B_R} \mathcal{B}_{E_{B_R}}(U)\cdot v_k^m\, d\sigma$$

for $m = 1,2, k = 1,2,3$. Therefore

$$\mathcal{P}_2^c\mathfrak{A}_{1E_{B_R}}U = \mathfrak{M}(\mathcal{B}_{E_{B_R}}(U)),$$

where $\mathfrak{M}$ is a bounded linear mapping from $L_1(\partial B_R)$ to D^1 and

$$\|\mathfrak{M}(h)\|_{D^1} \le C\|h\|_{L_1(\partial B_R)}$$

for $h \in L_1(\partial B_R)$. Thus equation (27) can be written as

$$\begin{aligned}\mathbf{U}'(t) &= \mathfrak{A}_{1\mathbf{T}}\mathbf{U}(t) + F(t) + \mathfrak{M}(g)(t)\\ &\quad - \mathcal{P}^c(\mathfrak{A}_{1\mathbf{T}} - \mathfrak{A}_{1E_{B_R}})\mathbf{U}(t) + \mathfrak{M}((\mathcal{B}_{E_{B_R}} - \mathcal{B}_{\mathbf{T}})(\mathbf{U}(t))).\end{aligned} \tag{30}$$

Now the expression $F(t) + \mathfrak{M}(g)(t)$ is given, and, at least if we fix $\omega = 1$, the expressions in the second line of (30) are small perturbations if η is sufficiently small. This gives us the existence and uniqueness of solutions of equation (30) with boundary condition (28) using the Banach fixed point theorem, and we obtain an estimate for this solution as well. Now (27) is equivalent to

$$\mathbf{U}' = \mathcal{A}_{1\mathbf{T}}\mathbf{U} + \mathcal{P}_1^c\mathfrak{A}_{1E_{B_R}}\mathbf{U} + F,$$

thus for $v \in \mathcal{N}$ we have, as $F(t) \in B$ for almost all t,

$$0 = (F+\mathcal{A}_{1\mathbf{T}}\mathbf{U}, v)_{\mathfrak{H}} = (\mathbf{U}'-\mathcal{P}_1^c\mathfrak{A}_{1E_{B_R}}\mathbf{U}, v)_{\mathfrak{H}} = \frac{d}{dt}(\mathbf{U}, v)_{\mathfrak{H}} - (\mathcal{P}_1^c\mathfrak{A}_{1E_{B_R}}\mathbf{U}, v)_{\mathfrak{H}}.$$

Thus

$$\frac{d}{dt}(\mathbf{U}, v)_{\mathfrak{H}} = (\mathfrak{A}_{1E_{B_R}}\mathbf{U}, \mathcal{P}_1^c v)_{\mathfrak{H}}.$$

For $v \in \mathcal{N}_2$ this implies $0 = (U_0, v)_{\mathfrak{H}} = (\mathbf{U}(t), v)_{\mathfrak{H}}$, so $\mathbf{U}(t) \perp \mathcal{N}_2$, and by (29) then $(\mathfrak{A}_{1E_{B_R}}\mathbf{U}, \mathcal{P}_1^c v)_{\mathfrak{H}} = 0$ as well. Thus we have $\mathbf{U}(t) \perp \mathcal{N}$, and $\mathcal{P}_1^c\mathfrak{A}_{1E_{B_R}}\mathbf{U} = 0$, which means we have actually solved our original problem for $\omega = 1$. For larger values of ω we have to use a suitable partition of the interval $[0,\omega]$.

5. Asymptotic estimates for an abstract Green operator. The abstract theory developed here will be applied in the next section.

Let $\omega \in (0,\infty]$ and let $\mathcal{I}$ be an arbitrary set. For any $f : [0,\omega] \cap \mathbb{R} \to \mathcal{I}$ we define $f^t : [0,\omega - t] \cap \mathbb{R} \to \mathcal{I}$ for $t \in [0,\omega] \cap \mathbb{R}$ by $f^t(\tau) = f(t+\tau)$ for $\tau \in [0,\omega - t] \cap \mathbb{R}$. This notation will only be used for objects in this section, and of course also where the results of this section are applied. As there are no matrices in this section, it is hoped that no confusion with the transpose will arise. This definition immediately carries over to classes of functions consisting of all functions agreeing almost everywhere in $[0,\omega] \cap \mathbb{R}$.

Let $\mathcal{I}_1, \mathcal{I}_2$ be two Banach spaces, let $\omega \in [1,\infty]$ and

$$\widehat{\mathcal{S}}_1^\omega = \{u : [0,\omega] \cap \mathbb{R} \to \mathcal{I}_1\}/\backsim, \qquad \widehat{\mathcal{S}}_2^\omega = \{u : [0,\omega] \cap \mathbb{R} \to \mathcal{I}_2\}/\backsim,$$

where two functions are considered equivalent in the sense of $\backsim$ if they agree almost everywhere. We also assume $\mathcal{S}_k \subset \widehat{\mathcal{S}}_k^1$ $(k = 1,2)$ are Banach spaces with norms $\|\cdot\|_{\mathcal{S}_k}$. For $f \in \widehat{\mathcal{S}}_k^{\widehat{\omega}}$ with $\widehat{\omega} \in [1,\omega]$ the statement $f \in \mathcal{S}_k$ is used instead of $f|[0,1] \in \mathcal{S}_k$ and likewise $\|f\|_{\mathcal{S}_k}$ means $\|f|[0,1]\|_{\mathcal{S}_k}$, a practice we follow throughout this section. Then we define

$$\mathcal{S}_k^\omega = \{f \in \widehat{\mathcal{S}}_k^\omega : f^t \in \mathcal{S}_k \text{ for } t \in [0,\omega-1] \text{ and } \sup_{0\le t\le \omega-1} \|f^t\|_{\mathcal{S}_k} < \infty\} \tag{31}$$

and

$$\|f\|_{\mathcal{S}_k^\omega} = \sup_{1\le t\le \omega-1} \|f^t\|_{\mathcal{S}_k}. \tag{32}$$

Now $\mathcal{S}_1^\omega, \mathcal{S}_2^\omega$ are Banach spaces and $\mathcal{S}_k^1 = \mathcal{S}_k$. We also assume that there exists a constant C_1 such that if $f \in \mathcal{S}_1$ and $t \in [0,1]$ then

$$\|f(t)\|_{\mathcal{I}_1} \le C_1 \|f\|_{\mathcal{S}_1} \tag{33}$$

for all $t \in [0,1]$, and that if $f \in \mathcal{S}_k$ and $\varphi \in C^1([0,1])$ and $(\varphi f)(t) = \varphi(t) f(t)$ then $\varphi f \in \mathcal{S}_k$ and

$$\|\varphi f\|_{\mathcal{S}_k} \le C_1 \|\varphi\|_{C^1} \|f\|_{\mathcal{S}_k} \quad (k = 1,2). \tag{34}$$

We assume there is a bounded linear operator $\mathcal{C} : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathcal{I}_2$ such that if $u_0 \in \mathcal{I}_1$ and $f \in \mathcal{S}_2$, then the function class $t \mapsto \mathcal{C}(u_0, f(t))$ contains a representative which is a continuous function from $[0,1]$ to $\mathcal{I}_2$, and we denote this representative by $\mathcal{C}(u_0, f)$. Let us define

$$\mathcal{D} = \{(u_0, f) \in \mathcal{I}_1 \times \mathcal{S}_2 : \mathcal{C}(u_0, f)(0) = 0\}.$$

We assume there exists a linear operator $\Gamma : \mathcal{D} \to \mathcal{S}_1$, referred to as a *Green operator* here, and a constant C_1 with

$$\|\Gamma(u_0, f)\|_{\mathcal{S}_1} \le C_1 (\|u_0\|_{\mathcal{I}_1} + \|f\|_{\mathcal{S}_2}) \tag{35}$$

for all $(u_0, f) \in \mathcal{D}$, which also has the property that for $(u_0, f) \in \mathcal{D}$,

$$\Gamma(u_0, f)(0) = u_0 \tag{36}$$

and

$$\mathcal{C}(\Gamma(u_0,f)(t),f)(t)=0 \tag{37}$$

for $t\in[0,1]$. Finally, we make the following consistency assumption.

If $\tau\in(0,1)$ and $f:[0,1+\tau]\to\mathcal{I}_2, u_0\in\mathcal{I}_1$ are such that $f,f^\tau\in\mathcal{S}_2$ and $(u_0,f)\in\mathcal{D}$, then for $t\in[0,1-\tau]$ we have

$$\Gamma(\Gamma(u_0,f)(\tau),f^\tau)(t)=\Gamma(u_0,f)(t+\tau). \tag{38}$$

Note that (37) implies the left-hand side of the equation is well-defined.

Now we will define a Green operator for the interval $[0,\infty)$, which we will also denote by Γ. For the definition of $\mathcal{S}_k^\infty$ see (31).

Definition 15. Let

$$\mathcal{D}^\infty=\{(u_0,f)\in\mathcal{I}\times\mathcal{S}_2^\infty:\mathcal{C}(u_0,f)(0)=0\}.$$

We define $\Gamma:\mathcal{D}^\infty\to\mathcal{S}_1^\infty$ as follows. For any $t\in[0,\omega]$ and $(u_0,f)\in\mathcal{D}^\infty$ let $0=t_0<\dots<t_{n-1}<t_n=t$ with $t_k-t_{k-1}<1$. Then for $k=1,\dots,n$ let $u_k=\Gamma(u_{k-1},f^{t_{k-1}})(t_k-t_{k-1})$, defined inductively, and finally $\Gamma(u_0,f)=u_n$.

It is not clear whether Γ is well-defined. This, together with some of the properties of this operator, is the content of the next lemma.

Lemma 16. *The operator $\Gamma:\mathcal{D}^\infty\to\mathcal{S}_1^\infty$ is well-defined and linear, and for $s,t\ge 0$ we have $\Gamma(\Gamma(u_0,f)(s),f^s)(t)=\Gamma(u_0,f)(s+t)$. Given C_1 (see (35)) and $\omega\in[1,\infty)$ there is a constant $C(\omega,C_1)$ such that if $(u_0,f)\in\mathcal{D}^\infty$, then*

$$\|\Gamma(u_0,f)\|_{\mathcal{S}_1^\omega}\le C(\omega,C_1)(\|u_0\|_{\mathcal{I}_1}+\|f\|_{\mathcal{S}_2^\omega}). \tag{39}$$

This constant is an increasing function of C_1 and ω. Also for $t\ge 1$ the value $\Gamma(u_0,f)(t)$ is independent of $f(s)$ for $s>t$.

Proof. First we prove that the definition for Γ we just gave is independent of the partition $0=t_0<t_1<\dots<t_n=t$. As an initial step we consider only partitions for which n is the same. Then, for given u_0, t and f, this procedure at least defines a mapping from

$$\mathfrak{T}_n=\{(t_1,\dots,t_{n-1})\in\mathbb{R}^{n-1}\mid 0<t_{k+1}-t_k<1 \text{ for } k=0,\dots,n-1 \text{ with } t_0=0,\ t_n=t\}$$

to $\mathcal{I}_1$. Now let $(t_1,\dots,t_{n-1}),(\widetilde{t}_1,\dots,\widetilde{t}_{n-1})\in\mathfrak{T}_n$ and $t_k=\widetilde{t}_k$ for $k\ne m$, while $t_m\ne\widetilde{t}_m$. Let u_k and $\widetilde{u}_k$ be the elements of $\mathcal{I}_1$ produced by these two partitions. Without restriction we may assume $\widetilde{t}_m<t_m$. Then $u_k=\widetilde{u}_k$ for $k<m$. Now

$$\widetilde{u}_m=\Gamma(u_{m-1},f^{t_{m-1}})(\widetilde{t}_m-t_{m-1}) \quad\text{and}\quad u_m=\Gamma(u_{m-1},f^{t_{m-1}})(t_m-t_{m-1}).$$

Owing to (38) we have

$$\begin{aligned}\Gamma(\widetilde{u}_m, f^{\widetilde{t}_m})(\tau) &= \Gamma(\Gamma(u_{m-1}, f^{t_{m-1}})(\widetilde{t}_m - t_{m-1}), f^{\widetilde{t}_m})(\tau) \\ &= \Gamma(u_{m-1}, f^{t_{m-1}})(\tau + \widetilde{t}_m - t_{m-1})\end{aligned}$$

for $0 \leq \tau \leq 1 - \widetilde{t}_m + t_{m-1}$. Now with $\tau = t_m - \widetilde{t}_m$ we have $\tau \leq 1 - \widetilde{t}_m + t_{m-1}$, and therefore

$$\Gamma(\widetilde{u}_m, f^{\widetilde{t}_m})(t_m - \widetilde{t}_m) = \Gamma(u_{m-1}, f^{t_{m-1}})(t_m - t_{m-1}) = u_m.$$

Using (38) again, we get

$$\begin{aligned}u_{m+1} &= \Gamma(u_m, f^{t_m})(t_{m+1} - t_m) \\ &= \Gamma(\Gamma(\widetilde{u}_m, f^{\widetilde{t}_m})(t_m - \widetilde{t}_m), f^{t_m})(t_{m+1} - t_m) \\ &= \Gamma(\widetilde{u}_m, f^{\widetilde{t}_m})(t_{m+1} - \widetilde{t}_m) = \widetilde{u}_{m+1}.\end{aligned}$$

From then on all u_k will of course be equal. So we have now proved that if we have two elements of $\mathfrak{T}_n$ differing in only one component, then the procedure we claim defines Γ gives the same result. Now $\mathfrak{T}_n$ is open, thus for any point $(t_1, \ldots, t_{n-1})$ in $\mathfrak{T}_n$ there exists a $\delta > 0$ such that

$$\{(\tau_1, \ldots, \tau_{n-1}) \in \mathbb{R}^{n-1} \mid |t_k - \tau_k| < \delta,\ k = 1, \ldots, n-1\} \subset \mathfrak{T}_n.$$

It is clear that we can reach any point in this neighborhood from the center by moving in the direction of the coordinate axes only, so our procedure gives us the same value as at $(t_1, \ldots, t_{n-1})$ throughout this set. Thus the set of all points where the procedure leads to a certain value is an open subset of $\mathfrak{T}_n$. As $\mathfrak{T}_n$ is also connected, there cannot be more than one result of this procedure, as otherwise we would be able to split $\mathfrak{T}_n$ up into at least two open sets. Due to (38) we can always add points to any partition without changing the result. This proves that the result of the procedure is independent of the partition. Also one immediately sees, using a partition containing s as one of its points, that for $s, t \geq 0$ we have

$$\Gamma(\Gamma(u_0, f)(s), f^s)(t) = \Gamma(u_0, f)(s + t).$$

With this it is easy to prove the linearity and the estimate (39) by induction over an upper bound for ω, and the last remark is obvious by construction. Now all our claims are proved.

For $\omega \in [0, \infty], a_2 \geq 0$ let us define

$$\|f\|_{\mathcal{S}_2^{\omega,a_2}} = \sup_{0 \leq t \leq \omega - 1} (1 + t)^{a_2} \|f^t\|_{\mathcal{S}_2}, \tag{40}$$

and

$$\mathcal{S}_2^{\omega,a_2} = \{f \in \mathcal{S}_2^{\omega} \mid \|f\|_{\mathcal{S}_2^{\omega,a_2}} < \infty\}.$$

Then we obtain

THEOREM 17. *Let* $[\cdot]_*$ *be a seminorm defined on* $\mathcal{S}_1$. *Assume that there are constants* $C_2, \widehat{C}_2$ *and a number* $a_1 \geq 0$ *such that for* $u \in \mathcal{S}_1$,

$$[u]_* \leq \widehat{C}_2 \|u\|_{\mathcal{S}_1}, \tag{41}$$

and for $(u_0, 0) \in \mathcal{D}^\infty$ *and* $t \in [0, \infty)$,

$$[(\Gamma(u_0, 0))^t]_* \leq C_2(1+t)^{-a_1} \|u_0\|_{\mathcal{I}_1}. \tag{42}$$

Then for $a_2 > 1$ *with* $a_2 \geq a_1$ *there exists a constant* C_3, *which only depends on* C_1, C_2, $\widehat{C}_2$, a_1 *and* a_2, *such that for all* $(u_0, f) \in \mathcal{D}^\infty$ *with* $f \in \mathcal{S}_2^{\infty, a_2}$,

$$[(\Gamma(u_0, f))^t]_* \leq C_3(1+t)^{-a_1}(\|f\|_{\mathcal{S}_2^{\infty,a_2}} + \|u_0\|_{\mathcal{I}_1}) \tag{43}$$

for $t \in [0, \infty)$.

Proof. Let $\varphi \in C^\infty(\mathbb{R})$ be such that $0 \leq \varphi(t) \leq 1$ for $t \in \mathbb{R}$, $\varphi(t) = 0$ for $|t| \geq 1, \varphi(t) = 1$ for $|t| \leq 1/2$, $\varphi'(t)t \leq 0$ for all $t \in \mathbb{R}$, and $|\varphi'| \leq 2$. Then it is easy to see that

$$1 \leq \sum_{p=-\infty}^{\infty} \varphi(t-p) \leq 3$$

for all $t \in \mathbb{R}$, and this sum is 1-periodic and belongs to $C^\infty(\mathbb{R})$. Now let

$$\Psi(t) = \varphi(t)\Big(\sum_{p=-\infty}^{\infty} \varphi(t-p)\Big)^{-1},$$

$\Psi_k(t) = \Psi(t-k)$ and $f_k(t) = \Psi_k(t) f(t)$. Also letting $u_k = 0$ for $k \geq 1$ we have

$$f(t) = \sum_{0 \leq k \leq t+1} f_k(t),$$

and $f_k(0) = 0$ and $\mathcal{C}(u_k, f_k(0)) = 0$ for $k \geq 1$. As $\Psi_0(0) = 1$ we have $(u_k, f_k) \in \mathcal{D}^\infty$ even for $k \geq 0$. Now, owing to the linearity of Γ and Lemma 16,

$$\Gamma(u_0, f)(t) = \sum_{0 \leq k \leq t+1} \Gamma(u_k, f_k)(t)$$

and $\Gamma(u_k, f_k)(t) = 0$ for $t \leq k-1$. Therefore

$$(\Gamma(u_0, f))^t|[0,1] = \sum_{0 \leq k \leq t+2} (\Gamma(u_k, f_k))^t|[0,1]$$

and

$$[(\Gamma(u_0, f))^t]_* \leq \sum_{0 \leq k \leq t+2} [(\Gamma(u_k, f_k))^t]_*. \tag{44}$$

We first consider the case $k-2 \leq t \leq k+2 \leq 5$. Then, remembering (31) for the definition of $\mathcal{S}_k^5, \mathcal{S}_k^6$, by Lemma 16 and (34) we have

$$\begin{aligned}\|(\Gamma(u_k, f_k))^t\|_{\mathcal{S}_1} &\le \|\Gamma(u_k, f_k)\|_{\mathcal{S}_1^6}\\ &\le C(\|f_k\|_{\mathcal{S}_2^6} + \|u_k\|_{\mathcal{I}_1}) \le C(\|f\|_{\mathcal{S}_2^6} + \|u_0\|_{\mathcal{I}_1})\\ &\le C(\|f\|_{\mathcal{S}_2^{\infty,a_2}} + \|u_0\|_{\mathcal{I}_1}).\end{aligned}$$

If $k-2 \le t \le k+2 > 5$ then for $\tau \ge k-2$,

$$\begin{aligned}\Gamma(u_k, f_k)(\tau) = \Gamma(0, f_k)(\tau) &= \Gamma(\Gamma(0, f_k)(k-2), f_k^{k-2})(\tau - k + 2)\\ &= \Gamma(0, f_k^{k-2})(\tau - k + 2).\end{aligned}$$

Therefore, again using Lemma 16 and (34),

$$\begin{aligned}\|(\Gamma(u_k, f_k))^t\|_{\mathcal{S}_1} = \|\Gamma(0, f_k^{k-2})^{t-k+2}\|_{\mathcal{S}_1} &\le \|\Gamma(0, f_k^{k-2})\|_{\mathcal{S}_1^5} \le C\|f_k^{k-2}\|_{\mathcal{S}_2^5}\\ &\le C\|f^{k-2}\|_{\mathcal{S}_2^5} \le C(1+k)^{-a_2}\|f\|_{\mathcal{S}_2^{a_2,\infty}}.\end{aligned}$$

Putting these two cases together we find that for $k-2 \le t \le k+2$,

$$\|(\Gamma(u_k, f_k))^t\|_{\mathcal{S}_1} \le C(2+k)^{-a_2}(\|f\|_{\mathcal{S}_2^{a_2,\infty}} + \|u_0\|_{\mathcal{I}_1}) \tag{45}$$

with a fixed constant C.

Now we consider the case $t \ge k+2$. Then

$$(\Gamma(u_k, f_k))^t = (\Gamma(\Gamma(u_k, f_k)(k+2), f_k^{k+2}))^{t-k-2}.$$

Now $f_k^{k+2} = 0$, and therefore by (42),

$$\begin{aligned}[(\Gamma(u_k, f_k))^t]_* &= [(\Gamma(\Gamma(u_k, f_k)(k+2), 0))^{t-k-2}]_*\\ &\le C(1 + t - k - 2)^{-a_1}\|\Gamma(u_k, f_k)(k+2)\|_{\mathcal{I}_1}.\end{aligned}$$

Thus, keeping in mind that $t \ge k+2$, we obtain

$$\begin{aligned}[(\Gamma(u_k, f_k))^t]_* &\le C(t+5-k)^{-a_1}\|\Gamma(u_k, f_k)(k+2)\|_{\mathcal{I}_1}\\ &\le C(t+5-k)^{-a_1}\|\Gamma(u_k, f_k)^{k+1}\|_{\mathcal{S}_1}.\end{aligned}$$

Combining this with (45) and (41), we have

$$[(\Gamma(u_k, f_k))^t]_* \le C(t+5-k)^{-a_1}(2+k)^{-a_2}(\|f\|_{\mathcal{S}_2^{\infty,a_2}} + \|u_0\|_{\mathcal{I}_1})$$

for $t \ge k-2$. Using (44), we therefore even obtain

$$[(\Gamma(u, f))^t]_* \le C(\|f\|_{\mathcal{S}_2^{\infty,a_2}} + \|u_0\|_{\mathcal{I}_1})S(t)$$

with

$$S(t) = \sum_{0 \le k \le t+2} (5 + t - k)^{-a_1}(k+2)^{-a_2}.$$

For $\tau \in [k, k+1]$ we have

$$(5 + t - k)^{-a_1}(k+2)^{-a_2} \le (4 + t - \tau)^{-a_1}(1+\tau)^{-a_2}$$

and for this reason

$$S(t) \le \sum_{0\le k\le t+2} \int_k^{k+1} (4+t-\tau)^{-a_1}(1+\tau)^{-a_2}\,d\tau$$
$$\le \int_0^{t+3} (4+t-\tau)^{-a_1}(1+\tau)^{-a_2}\,d\tau.$$

Substituting $\tau = \sigma(t+5)-1$ and letting $\delta = (t+5)^{-1}$, we get, as $a_1 \le a_2$ and $0 \le \sigma \le 1$,

$$S(t) \le \delta^{a_1+a_2-1} \int_\delta^{1-\delta} (1-\sigma)^{-a_1}\sigma^{-a_2}\,d\sigma \le \delta^{a_1+a_2-1} \int_\delta^{1-\delta} (1-\sigma)^{-a_2}\sigma^{-a_2}\,d\sigma$$

and for reasons of symmetry, and as $a_2 > 1$,

$$\int_\delta^{1-\delta} (1-\sigma)^{-a_2}\sigma^{-a_2}\,d\sigma = 2\int_\delta^{1/2} (1-\sigma)^{-a_2}\sigma^{-a_2}d\sigma \le 2^{1+a_2}\int_\delta^{1/2} \sigma^{-a_2}d\sigma$$
$$\le \frac{2^{1+a_2}}{a_2-1}\,\delta^{1-a_2}.$$

Thus

$$S(t) \le C\delta^{a_1+a_2-1}\delta^{1-a_2} = C\delta^{a_1} = C(t+5)^{-a_1}.$$

This proves our theorem.

6. The proof of Theorem 3. First we prove the following theorem, which implies Theorem 3 in case $\mathbf{T}$ does not depend on time.

THEOREM 18. *There exist numbers $\eta > 0$ and $C < \infty$ with the following properties. Assume that $T \in W_p^2(B_R)$ and*

$$\|T - E_{B_R}\|_{W_p^2} \le \eta,$$

and for $\omega \in [1,\infty)$ let $\mathbf{T}(x,t) = T(x)$ $(x \in \overline{B}_R,\ t \in [0,\omega])$. Then for $F \in B_1^\omega$, $g \in B_2^\omega$, $U_0 \in B \cap D^{1-2/p}$ with $\mathcal{B}_T(U_0) = g(0)$ there is exactly one function $\mathbf{U} \in \mathfrak{W}^\omega$ with $\mathbf{U}(t) \in B$ for $t \in [0,\omega]$ which solves the initial boundary value problem given by the equations

$$\mathbf{U}' = \mathcal{A}_{1\mathbf{T}}\mathbf{U} + F, \qquad \mathcal{B}_{\mathbf{T}}(\mathbf{U}) = g,$$

which are equations (18) *and* (17), *and $\mathbf{U}(0) = U_0$. This function also satisfies the inequality*

$$\|\mathbf{U}\|_{\mathfrak{W}^\omega} \le C(\|F\|_{B_1^\omega} + \|g\|_{B_2^\omega} + \|U_0\|_{\mathfrak{B}^1} + \|U_0\|_{D^{1-2/p}})$$

with a constant C independent of ω.

Proof. The existence and uniqueness immediately follow from Theorem 14, therefore we only need to prove the estimate.

We obtain this result by combining Theorem 1, which, as we saw, is an immediate consequence of Theorem 1.2 of [9], with Theorems 14 and 17. In applying this latter theorem to our problem we use

$$\mathcal{S}_2 = \{(F,g) \mid F \in B_1^1, g \in B_2^1\} = B_1^1 \times B_2^1$$

and $\mathcal{I}_2 = B \times L_p(\partial B_R)$ with the norms

$$\|(F,g)\|_{\mathcal{S}_2} = \|F\|_{B_1^1} + \|g\|_{B_2^1}, \qquad \|(F,g)(t)\|_{\mathcal{I}_2} = \|F(t)\|_{\mathfrak{B}^1} + \|g(t)\|_{L_p}.$$

It is easy to verify that $\mathcal{S}_2$ has property (34), and that if $(F,g) \in \mathcal{S}_2$, then $(F,g)(t) \in \mathcal{I}_2$ for almost all t, and $\mathcal{S}_2^\omega = B_1^\omega \times B_2^\omega$ for $\omega \in [1,\infty)$. Also it is easy to see that there is a constant $C \in (0,\infty)$ independent of ω such that

$$C^{-1}(\|F\|_{B_1^\omega} + \|g\|_{B_2^\omega}) \le \|(F,g)\|_{\mathcal{S}_2^{4/3,\omega}} \le C(\|F\|_{B_1^\omega} + \|g\|_{B_2^\omega}).$$

(For the definition of $\mathcal{S}_2^{4/3,\omega}$ see (40).) Unfortunately the use of Theorem 17 requires $(F,g) \in \mathcal{S}_2^{4/3,\infty}$. Now, if we are given $(F,g) \in \mathcal{S}_2^{4/3,\omega}$ we can extend (F,g) to infinity by defining

$$\widetilde{F}(t) = \begin{cases} F(t) & \text{for } t \le \omega, \\ 0 & \text{for } t > \omega. \end{cases}$$

and with a function $\varphi \in C^\infty(\mathbb{R})$ with $0 \le \varphi(t) \le 1$, $\varphi'(t) \le 0$ and $\varphi(t) = 1$ for $t \le 0$, $\varphi(t) = 0$ for $t \ge 1$ we can define

$$\widetilde{g}(t) = \begin{cases} g(t) & \text{for } t \le \omega, \\ g(2\omega - t)\varphi(t-\omega) & \text{for } \omega < t \le 2\omega, \\ 0 & \text{for } t > 2\omega. \end{cases}$$

Then one can easily verify

$$\|(\widetilde{F},\widetilde{g})\|_{\mathcal{S}_2^{4/3,\infty}} \le C\|(F,g)\|_{\mathcal{S}_2^{4/3,\omega}}$$

and $(\widetilde{F},\widetilde{g})(t) = (F,g)(t)$ for $t \le \omega$. Furthermore, let

$$\mathcal{S}_1 = \{\mathbf{U} \in \mathfrak{W}_1^1 \mid \mathbf{U}(t) \in B \text{ for } t \in [0,1]\}, \qquad \mathcal{I}_1 = B \cap D^{1-2/p}$$

with the norms

$$\|\mathbf{U}\|_{\mathcal{S}_1} = \|\mathbf{U}\|_{\mathfrak{W}_1^1}, \qquad \|U\|_{\mathcal{I}_1} = \|U\|_{\mathfrak{B}^1} + \|u\|_{W_p^{2-2/p}}.$$

Using the inequalities

$$\max_{0\le\tau\le1} \|\mathbf{U}(\tau)\|_{\mathfrak{B}^1} \le C\Big[\int_0^1 (\|\mathbf{U}'(\tau)\|_{\mathfrak{B}^1}^p + \|\mathbf{U}(\tau)\|_{\mathfrak{B}^1}^p)\,d\tau\Big]^{1/p} \le C\|\mathbf{U}\|_{\mathfrak{W}_1^1}$$

and inequality (23), we can easily verify condition (33). Condition (34) for $\mathcal{S}_1$ is also easy to see. We define $\mathcal{C}(U,(F,g)) = (0, \mathcal{B}_T(U) - g)$. For $(F,g) \in \mathcal{S}_2$ and $U_0 \in \mathcal{I}_1$ with $\mathcal{C}(U_0,(F,g)) = 0$ by Theorem 14 we have a solution $\mathbf{U}$ on $[0,1]$ of the equation $\mathbf{U}' = \mathcal{A}_{1\mathbf{T}}\mathbf{U} + F$ with initial value $\mathbf{U}(0) = U_0$ and boundary condition $\mathcal{B}_{\mathbf{T}}(\mathbf{U}) = g$. Then we define $\Gamma(U_0,(F,g)) = \mathbf{U}$.

It is easy to see that $\mathcal{C}(\mathbf{U}(t),(F(t),g(t)))$ is a continuous function from $[0,1]$ to $\mathcal{I}_2$, as the first component of the pair is zero. Conditions (35)–(37) follow from Theorem 14. Condition (38) is due to the uniqueness of that solution. The same fact implies that the abstract Γ we constructed in Lemma 16 is identical with the solution operator that one obtains from Theorem 14 for $\omega > 1$.

Now we apply Theorem 17 several times with $a_2 = 4/3$. In the first application let $a_1 = 4/3$ and

$$[\mathbf{U}]_* = \Big[\int_0^1 \|\Pi\mathbf{U}'(\tau)\|^p_{\mathfrak{B}^1_{-1}}\,d\tau\Big]^{1/p} + \sup_{0\le\tau\le1}\|\Pi\mathbf{U}(\tau)\|_{D^{2/3}_{-1}}.$$

Using Lemma 4 we see that $[\mathbf{U}]_* \le C\|\mathbf{U}\|_{\mathfrak{W}^1_1}$, and by Theorem 1 we deduce that if $U_0 \in \mathcal{I}_1$ and $\mathbf{U} = \Gamma(U_0,0)$, then with $s = 2/3$,

$$t^{4/3}(\|\Pi\mathbf{U}'(t)\|_{D^1_{-1}} + \|\Pi\mathbf{U}(t)\|_{D^{2/3}_{-1}}) \le C\|U_0\|_{\mathfrak{B}^1}$$

for $t \ge 1$. Therefore

$$(1+t)^{4/3}[\mathbf{U}^t]_* \le C(\|U_0\|_{\mathfrak{B}^1} + \|u\|_{W_p^{2-2/p}})$$

for $t \ge 1$, while for $t \le 1$ this follows directly from Theorem 14. Thus Theorem 17 implies that for $t \ge 0$,

$$\begin{aligned}(1+t)^{4/3}[\Gamma(U_0,(\widetilde{F},\widetilde{g}))^t]_* &\le C(\|U_0\|_{\mathfrak{B}^1} + \|u\|_{W_p^{2-2/p}} + \|(\widetilde{F},\widetilde{g})\|_{\mathcal{S}_2^{4/3,\infty}})\\ &\le C(\|U_0\|_{\mathfrak{B}^1} + \|u\|_{W_p^{2-2/p}} + \|(F,g)\|_{\mathcal{S}_2^{4/3,\omega}})\\ &\le C(\|U_0\|_{\mathfrak{B}^1} + \|u\|_{W_p^{2-2/p}} + \|F\|_{B_1^\omega} + \|g\|_{B_2^\omega}).\end{aligned}$$

Thus

$$\begin{aligned}[\Gamma(U_0,(F,g))]_2 &= [\Gamma(U_0,(\widetilde{F},\widetilde{g}))]_2\\ &\le C(\|U_0\|_{\mathfrak{B}^1} + \|u\|_{W_p^{2-2/p}} + \|F\|_{B_1^\omega} + \|g\|_{B_2^\omega}),\end{aligned}$$

as for $t \le \omega$ we have $\Gamma(U_0,(F,g))(t) = \Gamma(U_0,(\widetilde{F},\widetilde{g}))$. We omit the details of the completely analogous argument which gives us with $a_1 = 1$ and

$$[\mathbf{U}]_* = \Big[\int_0^1 (\|\mathbf{U}'(\tau)\|^p_{\mathfrak{B}^1} + \|\Pi\mathbf{U}(\tau)\|^p_{D^1_{-1}})\,d\tau\Big]^{1/p}$$

that

$$[\Gamma(U_0,(F,g))]_1 \le C(\|U_0\|_{\mathfrak{B}^1} + \|u\|_{W_p^{2-2/p}} + \|F\|_{B_1^\omega} + \|g\|_{B_2^\omega})$$

and finally for $a_1 = 0$ and

$$[\mathbf{U}]_* = \|\mathbf{U}\|_{\mathfrak{W}^1_1}$$

that

$$\|\Gamma(U_0,(F,g))\|_{\mathfrak{W}^\omega_1} \le C(\|U_0\|_{\mathfrak{B}^1} + \|u\|_{W_p^{2-2/p}} + \|F\|_{B_1^\omega} + \|g\|_{B_2^\omega}).$$

Putting these inequalities together, we conclude that if $F \in B_1^\omega$, $g \in B_2^\omega$, $U_0 \in B \cap D^{1-2/p}$, then for $\omega \in [1,\infty)$,

$$\|\mathbf{U}\|_{\mathfrak{W}^\omega} = \|\mathbf{U}\|_{\mathfrak{W}_1^\omega} + [\mathbf{U}]_1 + [\mathbf{U}]_2 \le C(\|U_0\|_{\mathfrak{B}^1} + \|u\|_{W_p^{2-2/p}} + \|F\|_{B_1^\omega} + \|g\|_{B_2^\omega})$$

with $\mathbf{U} = \Gamma(U_0,(F,g))$. This completes the proof of Theorem 18.

Now we can proceed to the case of variable $\mathbf{T}$ in order to prove Theorem 3 in its generality. Again Theorem 14 implies the existence and uniqueness of the solution. To obtain the estimate we can rewrite our equation

$$\mathbf{U}' = \mathcal{A}_{1\mathbf{T}}\mathbf{U} + F,$$

using $\widetilde{\mathbf{T}}(t,x) = T_\omega(x)$, as

$$\mathbf{U}' = \mathcal{A}_{1\widetilde{\mathbf{T}}}\mathbf{U} + (\mathcal{A}_{1\mathbf{T}} - \mathcal{A}_{1\widetilde{\mathbf{T}}})\mathbf{U} + F$$

and the boundary conditions as

$$\mathcal{B}_{\widetilde{\mathbf{T}}}(\mathbf{U}) = \mathcal{B}_{\widetilde{\mathbf{T}}}(\mathbf{U}) - \mathcal{B}_{\mathbf{T}}(\mathbf{U}) + g.$$

Using Theorem 18, we obtain

$$\|\mathbf{U}\|_{\mathfrak{W}^\omega} \le C(\|F\|_{B_1^\omega} + \|g\|_{B_2^\omega} + \|(\mathcal{A}_{1\mathbf{T}} - \mathcal{A}_{1\widetilde{\mathbf{T}}})\mathbf{U}\|_{B_1^\omega} + \|\mathcal{B}_{\widetilde{\mathbf{T}}}(\mathbf{U}) - \mathcal{B}_{\mathbf{T}}(\mathbf{U})\|_{B_2^\omega}).$$

By Theorem 9 we have

$$\|(\mathcal{A}_{1\mathbf{T}} - \mathcal{A}_{1\widetilde{\mathbf{T}}})\mathbf{U}\|_{B_1^\omega} \le C\|\mathbf{T} - \widetilde{\mathbf{T}}\|_{\mathsf{B}^\omega}\|\mathbf{U}\|_{\mathfrak{W}^\omega} \le C\eta\|\mathbf{U}\|_{\mathfrak{W}^\omega},$$

while Theorem 11 implies

$$\|\mathcal{B}_{\widetilde{\mathbf{T}}}(\mathbf{U}) - \mathcal{B}_{\mathbf{T}}(\mathbf{U})\|_{B_2^\omega} \le C\|\mathbf{T} - \widetilde{\mathbf{T}}\|_{\mathsf{B}^\omega}\|\mathbf{U}\|_{\mathfrak{W}^\omega} \le C\eta\|\mathbf{U}\|_{\mathfrak{W}^\omega}.$$

Combining these estimates we get

$$\|\mathbf{U}\|_{\mathfrak{W}^\omega} \le C(\|F\|_{B_1^\omega} + \|g\|_{B_2^\omega}) + C\eta\|\mathbf{U}\|_{\mathfrak{W}^\omega}.$$

For $C\eta \le 1/2$ this concludes the proof of Theorem 3.

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