Linear combinations of partitions of unity with restricted supports

by

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Abstract. Given a locally finite open covering $\mathcal{C}$ of a normal space X and a Hausdorff topological vector space E, we characterize all continuous functions $f : X \to E$ which admit a representation $f = \sum_{C \in \mathcal{C}} a_C \varphi_C$ with $a_C \in E$ and a partition of unity $\{\varphi_C : C \in \mathcal{C}\}$ subordinate to $\mathcal{C}$.

As an application, we determine the class of all functions $f \in C(|\boldsymbol{\mathcal{P}}|)$ on the underlying space $|\boldsymbol{\mathcal{P}}|$ of a Euclidean complex $\boldsymbol{\mathcal{P}}$ such that, for each polytope $P \in \boldsymbol{\mathcal{P}}$, the restriction $f|_P$ attains its extrema at vertices of P. Finally, a class of extremal functions on the metric space $([-1, 1]^m, d_\infty)$ is characterized, which appears in approximation by so-called controllable partitions of unity.

1. The representation theorem and first conclusions. Let X be a topological space. A system $\{\varphi_\iota : \iota \in \mathcal{I}\}$ of continuous real-valued functions is called a *partition of unity* on X if $\varphi_\iota \geq 0$ for all $\iota \in \mathcal{I}$, $\sum_{\iota \in \mathcal{I}} \varphi_\iota \equiv 1$, and the open covering $\{\varphi_\iota^{-1}((0, 1]) : \iota \in \mathcal{I}\}$ is locally finite. For a locally finite open covering $\mathcal{C} = \{C_\iota : \iota \in \mathcal{I}\}$ of X, a partition of unity $\{\varphi_\iota : \iota \in \mathcal{I}\}$ is said to be *subordinate* to $\mathcal{C}$ if $\varphi_\iota^{-1}((0, 1]) \subseteq C_\iota$ for all $\iota \in \mathcal{I}$ (cf. [10], p. 352).

For a fixed locally finite open covering $\mathcal{C}$ of X and for a Hausdorff topological vector space E, we want to characterize all continuous functions $f : X \to E$ which admit a representation $f = \sum_{C \in \mathcal{C}} a_C \varphi_C$ with vectors $a_C \in E$ and a partition of unity $\{\varphi_C : C \in \mathcal{C}\}$ subordinate to $\mathcal{C}$. The following theorem solves this question for normal spaces X. We shall see that the claim of this very general theorem can even be used for characterizing normal spaces among T_1-spaces. Besides that, we shall present an application concerning continuous real-valued functions on polytopes and on Euclidean complexes in $\mathbb{R}^m$ as they appear in optimization theory (see Section 2).

A further application deals with continuous real-valued functions on compact metric spaces (X, d) and their approximation by linear combinations of partitions of unity related to the entropy numbers of (X, d) (see Section 3).

In the theorem we use the symbol $\mathrm{conv}(\cdot)$ for the convex hull.

THEOREM 1. *Let X be a normal space, $\mathcal{C}$ a locally finite open covering of X, E a Hausdorff topological vector space, $f : X \to E$ a continuous function, and $a_C \in E$ for $C \in \mathcal{C}$. Then the following are equivalent:*

(i) *There exists a partition of unity $\{\varphi_C : C \in \mathcal{C}\}$ subordinate to $\mathcal{C}$ such that*
$$f = \sum_{C \in \mathcal{C}} a_C \varphi_C.$$

(ii) *For all $x \in X$,*
$$f(x) \in \mathrm{conv}\{a_C : x \in C \in \mathcal{C}\}.$$

Proof. The implication (i)$\Rightarrow$(ii) is evident (cf. [10], p. 354).

For the proof of the converse we assume $\mathcal{C}$ to be well ordered. The partition functions φ_{C_0}, $C_0 \in \mathcal{C}$, will be obtained by transfinite induction as continuous selections of set-valued functions $\Phi_{C_0} : X \to 2^{\mathbb{R}}$ using a selection theorem from Michael's paper [5]. The beginning of the induction will be treated in the same way as the inductive steps.

We define
$$\Phi_{C_0}(x) = \Big\{\mu_{C_0} : \text{there exist } \{\mu_C \geq 0\}_{C \in \mathcal{C}} \text{ such that } \sum_{C \in \mathcal{C}} \mu_C = 1, \ \mu_C = 0$$
$$\text{if } x \notin C, \ \sum_{C \in \mathcal{C}} a_C \mu_C = f(x), \text{ and } \mu_C = \varphi_C(x) \text{ for all } C < C_0 \Big\},$$

where the functions φ_C, $C < C_0$, are supposed to be already known. Note that for fixed x all but finitely many values μ_C vanish. Obviously, $\Phi_{C_0}(x)$ is convex and compact. If C_0 is the smallest element of $\mathcal{C}$, then $\Phi_{C_0}(x) \neq \emptyset$ by assumption (ii). Otherwise one obtains $\Phi_{C_0}(x) \neq \emptyset$, because $\Phi_C(x) \neq \emptyset$ for all $C < C_0$ according to the induction hypothesis. Now Theorem 3.1$'$ of [5] says that Φ_{C_0} admits a continuous selection φ_{C_0} if Φ_{C_0} is lower semicontinuous (l.s.c.).

Let $x_0 \in X$ be fixed. We choose an open neighbourhood U of x_0 which intersects only finitely many sets from $\mathcal{C}$, say $C_1, \ldots, C_n$, and which, moreover, satisfies

(1)
$$U \subseteq \bigcap_{x_0 \in C \in \mathcal{C}} C.$$

If $C_0 \notin \{C_1, \ldots, C_n\}$ then Φ_{C_0} is l.s.c. at x_0, since then $\Phi_{C_0}(x) = \{0\}$ for all $x \in U$. Now let us assume that $C_0 \in \{C_1, \ldots, C_n\}$, say $C_1 < \ldots < C_{k+1} = C_0 < C_{k+2} < \ldots < C_n$. We use the subsimplex $S(x) = \{(\mu_1, \ldots, \mu_n) \in S : \mu_i = 0 \text{ for } x \notin C_i\}$ of the simplex $S = \{(\mu_1, \ldots, \mu_n) : \mu_i \geq 0, \sum_{i=1}^n \mu_i = 1\}$,

the linear map $\pi(\mu_1, \ldots, \mu_n) = (\sum_{i=1}^{n} a_{C_i}\mu_i; \mu_1, \ldots, \mu_k)$, the continuous map $g(x) = (f(x); \varphi_{C_1}(x), \ldots, \varphi_{C_k}(x))$, and the set-valued function

$$\Psi(x) = \{(\mu_1, \ldots, \mu_n) \in S(x) : \pi(\mu_1, \ldots, \mu_n) = g(x)\} = S(x) \cap \pi^{-1}(g(x))$$

in order to rewrite $\Phi_{C_0}(x)$, $x \in U$, as

$$\Phi_{C_0}(x) = \{\mu_{k+1} : \text{there exists } (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$$
$$\text{with } (\mu_1, \ldots, \mu_n) \in \Psi(x)\}.$$

Now it suffices to show that Ψ is l.s.c. at x_0, since this obviously implies that Φ_{C_0} is l.s.c. at x_0.

Let an open set $W \subseteq \mathbb{R}^n$ be fixed such that $\Psi(x_0) \cap W \neq \emptyset$. We must find a neighbourhood V of x_0 such that $\Psi(x) \cap W \neq \emptyset$ for all $x \in V$. For every simplex $S(x)$, the restriction of the linear map $\pi : \mathbb{R}^n \to E \times \mathbb{R}^k$ to $S(x)$ is an open map onto its image $\pi(S(x))$. Hence there exists an open set $G_{S(x)} \subseteq E \times \mathbb{R}^k$ such that

$$\pi(S(x) \cap W) = \pi(S(x)) \cap G_{S(x)}.$$

Moreover, $g(x_0) \in G_{S(x)}$. Indeed, we can choose a vector $(\mu_1, \ldots, \mu_n) \in \Psi(x_0) \cap W$. Hence $g(x_0) = \pi(\mu_1, \ldots, \mu_n) \in \pi(S(x_0) \cap W)$ by the definition of $\Psi(x_0)$. The inclusion (1) yields $S(x_0) \subseteq S(x)$ for all $x \in U$. Hence $g(x_0) \in \pi(S(x) \cap W) \subseteq G_{S(x)}$.

Since there exist only finitely many simplices $S(x)$, $x \in U$, the set $V = U \cap g^{-1}(\bigcap_{x \in U} G_{S(x)})$ is an open neighbourhood of x_0. For all $x \in V$, we trivially have $g(x) \in G_{S(x)}$ and, moreover, $g(x) \in \pi(S(x))$, for $\Psi(x) \neq \emptyset$. Accordingly, $g(x) \in \pi(S(x)) \cap G_{S(x)} = \pi(S(x) \cap W)$ and finally $\Psi(x) \cap W = S(x) \cap \pi^{-1}(g(x)) \cap W \neq \emptyset$. Hence Ψ is l.s.c. at x_0.

Now Theorem 3.1$'$ of [5] yields a continuous selection φ_{C_0} of Φ_{C_0}. It follows from the definition of the multifunctions Φ_C, $C \in \mathcal{C}$, that the resulting functions φ_C form a partition of unity with the properties claimed in (i). ∎

As a special case we consider peaked coverings $\mathcal{C}$ of X. A covering $\mathcal{C}$ is called *peaked* if, for all $C_0 \in \mathcal{C}$, the set $C_0 \setminus \bigcup_{C \in \mathcal{C} \setminus \{C_0\}} C$ is non-empty. All points $x \in C_0 \setminus \bigcup_{C \in \mathcal{C} \setminus \{C_0\}} C$ are called *peaks* of the covering set C_0.

The motivation for this name is given by the concept of a peaked partition of unity $\{\varphi_\iota : \iota \in \mathcal{I}\}$ introduced in [6], which is defined by the property that, for all $\iota \in \mathcal{I}$, there exists a point $x \in X$ with $\varphi_\iota(x) = 1$. Clearly, a peaked partition of unity gives rise to a peaked covering $\{\varphi_\iota^{-1}((0, 1]) : \iota \in \mathcal{I}\}$. Conversely, any partition of unity $\{\varphi_\iota : \iota \in \mathcal{I}\}$ subordinate to a peaked covering $\{C_\iota : \iota \in \mathcal{I}\}$ is peaked.

COROLLARY 1. *Let X be a normal space, $\mathcal{C}$ a locally finite peaked open covering of X with peaks $x_C \in C$, E a Hausdorff topological vector space, and $f : X \to E$ a continuous function. Then the following are equivalent:*

(i) *There exists a partition of unity $\{\varphi_C : C \in \mathcal{C}\}$ subordinate to $\mathcal{C}$ and vectors $a_C \in E$ such that*

$$f = \sum_{C \in \mathcal{C}} a_C \varphi_C.$$

(ii) *For all $x \in X$,*

$$f(x) \in \operatorname{conv}\{f(x_C) : x \in C \in \mathcal{C}\}.$$

Proof. The points $x_C \in C$ being peaks we get $f(x_C) = a_C$ from (i). By Theorem 1, condition (i) implies that $f(x) \in \operatorname{conv}\{a_C : x \in C \in \mathcal{C}\} = \operatorname{conv}\{f(x_C) : x \in C \in \mathcal{C}\}$ for all $x \in X$. This proves the implication (i)$\Rightarrow$(ii). The converse direction (ii)$\Rightarrow$(i) can easily be seen by applying Theorem 1 to the vectors $a_C = f(x_C)$. ∎

Let us point out that the normality of X cannot be weakened in Theorem 1 as well as in Corollary 1. In fact, the implications (ii)$\Rightarrow$(i) from Theorem 1 and Corollary 1 can be used to characterize normal spaces among T_1-spaces.

THEOREM 2. *Let X be a T_1-space. Then the following are equivalent*:

(α) *X is normal.*

(β) *The implication* (ii)$\Rightarrow$(i) *from Theorem 1 holds true for all locally finite open coverings $\mathcal{C}$ of X, all Hausdorff topological vector spaces E, all continuous functions $f : X \to E$, and all vectors $a_C \in E$, $C \in \mathcal{C}$.*

(γ) *The implication* (ii)$\Rightarrow$(i) *from Corollary 1 holds true for all locally finite peaked open coverings $\mathcal{C}$ of X with peaks $x_C \in C$, all Hausdorff topological vector spaces E, and all continuous functions $f : X \to E$.*

The equivalences remain valid if E is replaced by the particular space $\mathbb{R}$.

Proof. The implication (α)$\Rightarrow$(β) is given by Theorem 1. (β)$\Rightarrow$(γ) was shown in the proof of Corollary 1. To prove (γ)$\Rightarrow$(α), let $A, B \subseteq X$ be two disjoint closed non-empty sets. Applying Corollary 1 to $\mathcal{C} = \{X \setminus A, X \setminus B\}$, $E = \mathbb{R}$, and $f \equiv 1$ one obtains a partition function $\varphi = \varphi_{X \setminus B}$ such that $\varphi|_A \equiv 1$ and $\varphi|_B \equiv 0$. ∎

We close this section with a corollary on partitions of unity which is well-known from the literature (cf. [10], p. 353). However, Theorem 1 gives rise to a very simple proof. One only has to apply the implication (ii)$\Rightarrow$(i) to the constant function $f \equiv 1$ and the reals $a_C = 1$.

COROLLARY 2. *For any locally finite open covering $\mathcal{C}$ of a normal space X, there exists a partition of unity $\{\varphi_C : C \in \mathcal{C}\}$ on X subordinate to $\mathcal{C}$.*

2. A question from optimization theory. Another application of Theorem 1 concerns a question from optimization theory. We consider an m-dimensional polytope $P \subseteq \mathbb{R}^m$. A *polytope* is meant to be the convex hull

of a finite set of points. The *face lattice* $\mathcal{F}(P)$ of P is the set of all k-faces of P, $-1 \leq k \leq m$. A *k-face* of P with $0 \leq k \leq m-1$ is a k-dimensional intersection of P with a supporting hyperplane of P. Besides that, the empty set $\emptyset$ and the polytope P itself are considered as the (-1)-face and the m-face of P, respectively. For example, the face lattice $\mathcal{F}(S)$ of a square $S \subseteq \mathbb{R}^2$ consists of $\emptyset$ (the (-1)-face), the four singletons $\{v\}$ formed by the *vertices* $v \in \mathrm{vert}(S)$ (the 0-faces), the four edges (the 1-faces), and S (the 2-face). Note that all faces of a polytope P are polytopes themselves.

The following simple fact is widely used in optimization theory: Let the continuous real-valued function $f \in C(P)$ be convex (resp. concave) and let $F \in \mathcal{F}(P)$ be a face of P. Then the restriction $f|_F$ attains its maximum (resp. minimum) at a vertex of F.

We want to tackle a related problem: Given a polytope P, we ask for a characterization of all functions $f \in C(P)$ such that, for all faces $F \in \mathcal{F}(P)$, $f|_F$ attains its minimum and its maximum at vertices of F, that is,

$$\min f(F), \max f(F) \in f(\mathrm{vert}(F)).$$

Before giving the solution we generalize the problem in so far as we consider Euclidean complexes instead of single polytopes.

A *Euclidean complex* $\boldsymbol{\mathcal{P}}$ in $\mathbb{R}^m$ is a non-empty collection of polytopes $P \subseteq \mathbb{R}^m$ satisfying the following three conditions (cf. [9], p. 313):

(C1) For all $P \in \boldsymbol{\mathcal{P}}$, all faces of P belong to $\boldsymbol{\mathcal{P}}$:

$$P \in \boldsymbol{\mathcal{P}} \ \Rightarrow \ \mathcal{F}(P) \subseteq \boldsymbol{\mathcal{P}}.$$

(C2) For all $P, Q \in \boldsymbol{\mathcal{P}}$, the intersection $P \cap Q$ is a face of both P and Q:

$$P, Q \in \boldsymbol{\mathcal{P}} \ \Rightarrow \ P \cap Q \in \mathcal{F}(P) \cap \mathcal{F}(Q).$$

(C3) The system $\boldsymbol{\mathcal{P}}$ is locally finite in the so-called *underlying space*

$$|\boldsymbol{\mathcal{P}}| = \bigcup_{P \in \boldsymbol{\mathcal{P}}} P$$

of $\boldsymbol{\mathcal{P}}$. (Clearly, the topology of $|\boldsymbol{\mathcal{P}}|$ is induced by the usual topology of $\mathbb{R}^m$.)

Euclidean complexes generalize the concept of a polytope in the sense that the face lattice of any polytope is a Euclidean complex. A useful reference concerning Euclidean complexes is [4], Chapter 2. Although the complexes considered there are supposed to be finite and the emphasis is even on the particular case where all polytopes are simplices, many arguments of that exposition apply to the general case.

The set $\mathrm{vert}(\boldsymbol{\mathcal{P}})$ of the *vertices of the complex* $\boldsymbol{\mathcal{P}}$ contains all vertices of polytopes from $\boldsymbol{\mathcal{P}}$. The *relative interior* $\mathrm{relint}(P)$ of a polytope P is meant to be the interior of P with respect to the topology of the affine space spanned
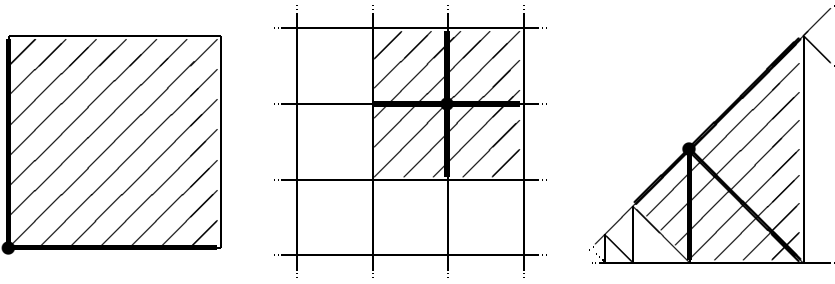
Fig. 1

by P. For a vertex $v \in \mathrm{vert}(\mathcal{P})$, the *star* of v is defined by

$$\mathrm{st}(v) = \bigcup_{P \in \mathcal{P},\, v \in \mathrm{vert}(P)} \mathrm{relint}(P).$$

Examples of stars are illustrated in Figure 1.

LEMMA. *A Euclidean complex $\mathcal{P}$ in $\mathbb{R}^m$ has the following properties*:

(a) *The underlying space $|\mathcal{P}|$ can be expressed as the disjoint union*

$$|\mathcal{P}| = \bigcup_{P \in \mathcal{P}} \mathrm{relint}(P).$$

(b) *For all $P \in \mathcal{P}$, $v \in \mathrm{vert}(\mathcal{P})$, and $x \in |\mathcal{P}|$,*

$$x \in \mathrm{relint}(P) \;\Rightarrow\; (x \in \mathrm{st}(v) \;\Leftrightarrow\; v \in \mathrm{vert}(P)).$$

(c) *The system $\{\mathrm{st}(v) : v \in \mathrm{vert}(\mathcal{P})\}$ is a locally finite peaked open covering of the topological space $|\mathcal{P}|$ with peaks $v \in \mathrm{st}(v)$.*

Proof. Part (a) can be obtained by following the proof of Theorem 2.2.2 from [4].

To prove (b), assume that $x \in \mathrm{relint}(P)$. First suppose that $x \in \mathrm{st}(v) = \bigcup_{Q \in \mathcal{P},\, v \in \mathrm{vert}(Q)} \mathrm{relint}(Q)$. Then, by (a), $P \in \{Q \in \mathcal{P} : v \in \mathrm{vert}(Q)\}$ and hence $v \in \mathrm{vert}(P)$. The opposite implication is a trivial consequence of the definition of $\mathrm{st}(v)$ and of the inclusion $x \in \mathrm{relint}(P)$.

Now we turn to (c). Applying the proofs of Assertions 2.3.12 and 2.3.15 from [4] one shows that $\{\mathrm{st}(v) : v \in \mathrm{vert}(\mathcal{P})\}$ is an open covering of $|\mathcal{P}|$.

The peak property of a star $\mathrm{st}(v)$ can be seen as follows: By (a), the only polytope $P \in \mathcal{P}$ with $v \in \mathrm{relint}(P)$ is $P = \{v\}$. By (b), we obtain $v \in \mathrm{st}(w)$ if and only if $w \in \mathrm{vert}(\{v\})$, i.e. $w = v$. Thus v is a peak of $\mathrm{st}(v)$.

Finally, we fix a point $x \in |\mathcal{P}|$ in order to show that $\{\mathrm{st}(v) : v \in \mathrm{vert}(\mathcal{P})\}$ is locally finite. By (C3), there exists a neighbourhood $U \subseteq |\mathcal{P}|$ of x intersecting at most finitely many of the relatively open polytopes $\mathrm{relint}(P)$, $P \in \mathcal{P}$. We assume that $U \cap \mathrm{relint}(P) \neq \emptyset$ for $P \in \{P_1, \ldots, P_l\}$ only. By (b), all the sets $\mathrm{st}(v)$ covering at least one point $y \in U$ must satisfy

$v \in \mathrm{vert}(P_1) \cup \ldots \cup \mathrm{vert}(P_l)$. Accordingly, U intersects only finitely many sets from $\{\mathrm{st}(v) : v \in \mathrm{vert}(\boldsymbol{\mathcal{P}})\}$. ∎

Given a Euclidean complex $\boldsymbol{\mathcal{P}}$, we have to characterize all functions $f \in C(|\boldsymbol{\mathcal{P}}|)$ with

$$\min f(P), \max f(P) \in f(\mathrm{vert}(P)) \quad \text{for all } P \in \boldsymbol{\mathcal{P}}.$$

These functions turn out to be linear combinations of partitions of unity subordinate to the covering $\{\mathrm{st}(v) : v \in \mathrm{vert}(\boldsymbol{\mathcal{P}})\}$.

THEOREM 3. *Let $\boldsymbol{\mathcal{P}}$ be a Euclidean complex in $\mathbb{R}^m$ and let $f \in C(|\boldsymbol{\mathcal{P}}|)$. Then the following are equivalent*:

(i) *The function f can be represented as*

$$f = \sum_{v \in \mathrm{vert}(\boldsymbol{\mathcal{P}})} \lambda_v \varphi_v$$

with coefficients $\lambda_v \in \mathbb{R}$ and a partition of unity $\{\varphi_v : v \in \mathrm{vert}(\boldsymbol{\mathcal{P}})\}$ on $|\boldsymbol{\mathcal{P}}|$ subordinate to the covering $\{\mathrm{st}(v) : v \in \mathrm{vert}(\boldsymbol{\mathcal{P}})\}$.

(ii) *For all polytopes $P \in \boldsymbol{\mathcal{P}}$,*

$$\min f(P), \max f(P) \in f(\mathrm{vert}(P)).$$

Proof. The above preparations make the proof rather easy, although the implication from the global condition (i) to the more or less local one (ii) is not as trivial as in Theorem 1. We apply Corollary 1 to the locally finite peaked open covering $\{\mathrm{st}(v) : v \in \mathrm{vert}(\boldsymbol{\mathcal{P}})\}$ and the peaks $v \in \mathrm{st}(v)$. Accordingly, condition (i) is equivalent to

(2) $\quad \forall x \in |\boldsymbol{\mathcal{P}}| : f(x) \in \mathrm{conv}\{f(v) : x \in \mathrm{st}(v)\}.$

The arguments we shall give below show that the following six claims are equivalent to (i) as well:

(3) $\quad \forall x \in |\boldsymbol{\mathcal{P}}|, \ \forall P \in \boldsymbol{\mathcal{P}} : x \in \mathrm{relint}(P) \ \Rightarrow \ f(x) \in \mathrm{conv}\{f(v) : x \in \mathrm{st}(v)\},$

(4) $\quad \forall x \in |\boldsymbol{\mathcal{P}}|, \ \forall P \in \boldsymbol{\mathcal{P}} : x \in \mathrm{relint}(P) \ \Rightarrow \ f(x) \in \mathrm{conv}(f(\mathrm{vert}(P))),$

(5) $\quad \forall x \in |\boldsymbol{\mathcal{P}}|, \ \forall P \in \boldsymbol{\mathcal{P}} : x \in P \ \Rightarrow \ f(x) \in \mathrm{conv}(f(\mathrm{vert}(P))),$

(6) $\quad \forall P \in \boldsymbol{\mathcal{P}} : f(P) \subseteq \mathrm{conv}(f(\mathrm{vert}(P))),$

(7) $\quad \forall P \in \boldsymbol{\mathcal{P}} : \min f(P), \max f(P) \in \mathrm{conv}(f(\mathrm{vert}(P))),$

(8) $\quad \forall P \in \boldsymbol{\mathcal{P}} : \min f(P), \max f(P) \in f(\mathrm{vert}(P)).$

Indeed, (2)⇔(3) and (3)⇔(4) are consequences of claims (a) and (b), respectively, from the lemma. (4)⇔(5) is due to the continuity of f. The equivalences (5)⇔(6) and (6)⇔(7) are obvious. The final step (7)⇔(8) is based on the trivial inclusion $\mathrm{vert}(P) \subseteq P$.

Hence the assertion (i)⇔(ii) is shown, since (8) coincides with (ii). ∎

3. An approximation class on $([-1,1]^m, d_\infty)$**.** In this section we shall sketch the approximation by so-called controllable partitions of unity (see [11], [8], [7]). This method of approximating real-valued continuous functions on compact metric spaces gave rise to the fairly general considerations of the present paper. Theorem 3 will be applied to describe a particular approximation class on the cube $([-1,1]^m, d_\infty)$.

Let (X, d) be a compact metric space. Then the *kth entropy number* of a subset $S \subseteq X$ is given by

$$\varepsilon_k(S) = \inf\{\varepsilon > 0 : S \text{ can be covered by } k \text{ balls of radius } \varepsilon\}$$

(cf. [3], p. 7). A finite partition of unity $\{\varphi_1, \ldots, \varphi_k\}$ on X is called *controllable* if either $k = 1$, i.e. $\{\varphi_1\} = \{\mathbf{1}_X\}$ is the trivial partition of unity, or the partition satisfies the geometric uniformity condition

$$\varepsilon_1(\text{supp}(\varphi_i)) < \varepsilon_{k-1}(X) \quad \text{for } 1 \le i \le k$$

restricting the size of the supports $\text{supp}(\varphi_i) = \overline{\{x \in X : \varphi_i(x) \neq 0\}}$ of the single functions φ_i. One uses linear combinations of controllable partitions of unity from the classes

$$\Phi_n(X) = \Big\{ \sum_{i=1}^k \lambda_i \varphi_i : \lambda_i \in \mathbb{R}, \ \{\varphi_1, \ldots, \varphi_k\} \text{ is a controllable}$$
$$\text{partition of unity on } X \text{ with cardinality } k \le n \Big\},$$

$n = 1, 2, \ldots$, in order to define a sequence of approximation numbers $a_n(f)$ for functions $f \in C(X)$ by

$$a_n(f) = \inf\{\|f - \varphi\| : \varphi \in \Phi_n(X)\},$$

$\|\cdot\|$ denoting the maximum norm of $C(X)$. These approximation quantities give rise to the Jackson type estimate

$$a_n(f) \le \omega(f, \varepsilon_n(X))$$

for all $f \in C(X)$ and $n \in \{1, 2, \ldots\}$, where $\omega(f, \delta) = \sup\{|f(x) - f(y)| : d(x, y) \le \delta\}$ is the modulus of continuity.

Now we turn to approximation on the cube $[-1, 1]^m$ equipped with the maximum metric d_∞. In [11] it is shown that controllable partitions of unity of cardinality $k \ge 2$ on a space X exist if and only if the jump condition $\varepsilon_k(X) < \varepsilon_{k-1}(X)$ is satisfied. The cube $[-1, 1]^m$ has the entropy numbers $\varepsilon_{l^m}([-1,1]^m) = \varepsilon_{l^m+1}([-1,1]^m) = \ldots = \varepsilon_{(l+1)^m-1}([-1,1]^m) = 1/l$ for $l \ge 1$ (cf. [1], [2]). Thus there exist controllable partitions of unity exactly for the cardinalities $k = 1, 2^m, 3^m, \ldots$ In particular, the approximation classes $\Phi_n([-1,1]^m)$, $1 \le n \le 2^m - 1$, coincide and consist of the constant functions only. The first interesting class is $\Phi_{2^m}([-1,1]^m)$ formed by all linear combinations of controllable partitions of unity of cardinality 2^m. We shall characterize all functions $f \in C([-1,1]^m)$ belonging to the closure $\overline{\Phi_{2^m}([-1,1]^m)}$,

which means that $a_{2^m}(f) = 0$. In [8] this characterization is already given for the one-dimensional case $m = 1$.

THEOREM 4. *Let f be a continuous real-valued function on the compact metric space $([-1,1]^m, d_\infty)$. Then the following are equivalent:*

(i) $f \in \overline{\Phi_{2^m}([-1,1]^m)}$.

(ii) *For all faces $Q \in \mathcal{F}([-1,1]^m)$,*
$$\min f(Q), \max f(Q) \in f(\text{vert}(Q)).$$

Proof. First we point out that a partition of unity Φ of cardinality 2^m on $[-1,1]^m$ is controllable if and only if there exists some $\varepsilon > 0$ such that the partition has the structure
$$\Phi = \{\varphi_v : v \in \text{vert}([-1,1]^m)\} \quad \text{with } \text{supp}(\varphi_v) \subseteq \overline{B_{2-\varepsilon}(v)},$$
$B_{2-\varepsilon}(v)$ denoting the open ball of radius $2 - \varepsilon$ centred at v. Indeed, Φ must satisfy the uniformity condition $\varepsilon_1(\text{supp}(\varphi)) < \varepsilon_{2^m-1}([-1,1]^m) = 1$ for all $\varphi \in \Phi$. That is, there exists some $\varepsilon > 0$ such that $\varepsilon_1(\text{supp}(\varphi)) \leq 1 - \varepsilon/2$ or, in other words, each support $\text{supp}(\varphi)$ is contained in a closed subcube with edges of length $2-\varepsilon$. Thus each vertex $v \in [-1,1]^m$ belongs to exactly one of the 2^m supports, say $v \in \text{supp}(\varphi_v)$ and $\Phi = \{\varphi_v : v \in \text{vert}([-1,1]^m)\}$. The condition of being contained in a cube with edges of length $2 - \varepsilon$ amounts to $\text{supp}(\varphi_v) \subseteq \overline{B_{2-\varepsilon}(v)}$ for $v \in \text{vert}([-1,1]^m)$.

Next we observe that the covering $\mathcal{C} = \{\text{st}(v) : v \in \text{vert}([-1,1]^m)\}$ of the cube $[-1,1]^m$ induced by the face lattice $\mathcal{F}([-1,1]^m)$ in the sense of the previous section is given by $\text{st}(v) = B_2(v)$, $v \in \text{vert}([-1,1]^m)$. Hence a partition of unity $\{\varphi_v : v \in \text{vert}([-1,1]^m)\}$ is subordinate to $\mathcal{C}$ if and only if
$$\varphi_v^{-1}((0,1]) \subseteq B_2(v) \quad \text{for } v \in \text{vert}([-1,1]^m).$$

Now let us verify the implication (i)$\Rightarrow$(ii). We have already seen that any controllable partition of unity of cardinality 2^m is subordinate to $\mathcal{C}$. Hence, by Theorem 3, as any function $\varphi \in \Phi_{2^m}([-1,1]^m)$ is a linear combination of a partition of unity of this kind, it satisfies the extremality condition
$$\min \varphi(Q), \max \varphi(Q) \in \varphi(\text{vert}(Q)) \quad \text{for all } Q \in \mathcal{F}([-1,1]^m).$$
It can easily be checked that this remains true if one passes from elements φ of the class $\Phi_{2^m}([-1,1]^m)$ to functions from the closure $\overline{\Phi_{2^m}([-1,1]^m)}$.

The proof of the converse direction (ii)$\Rightarrow$(i) again uses Theorem 3. We assume that f satisfies condition (ii). Then it must admit a representation
$$f = \sum_{v \in \text{vert}([-1,1]^m)} \lambda_v \varphi_v$$
with $\lambda_v \in \mathbb{R}$ and a partition of unity $\{\varphi_v : v \in \text{vert}([-1,1]^m)\}$ subordinate to $\mathcal{C}$, i.e. $\varphi_v^{-1}((0,1]) \subseteq B_2(v)$. Let $0 < \varepsilon < 1$. We consider the metric

projection $P_\varepsilon : [-1,1]^m \to [-1+\varepsilon, 1-\varepsilon]^m$ given by

$$P_\varepsilon(x) = y \iff d_2(x,y) = \min\{d_2(x,z) : z \in [-1+\varepsilon, 1-\varepsilon]^m\},$$

d_2 denoting the Euclidean distance. Moreover, let $D_\varepsilon : [-1+\varepsilon, 1-\varepsilon]^m \to [-1,1]^m$ be the dilatation mapping y onto $y/(1-\varepsilon)$. Now a new partition of unity $\{\varphi_v^{(\varepsilon)} : v \in \mathrm{vert}([-1,1]^m)\}$ can be defined by

$$\varphi_v^{(\varepsilon)} = \varphi_v D_\varepsilon P_\varepsilon \quad \text{for } v \in \mathrm{vert}([-1,1]^m).$$

Then obviously

$$\mathrm{supp}(\varphi_v^{(\varepsilon)}) = \overline{P_\varepsilon^{-1} D_\varepsilon^{-1}(\varphi_v^{-1}((0,1]))} \subseteq \overline{P_\varepsilon^{-1} D_\varepsilon^{-1}(B_2(v))}$$

$$= \overline{P_\varepsilon^{-1}([-1+\varepsilon, 1-\varepsilon]^m \cap B_{2-\varepsilon}(v))} = \overline{B_{2-\varepsilon}(v)}.$$

Therefore the partition of unity $\{\varphi_v^{(\varepsilon)} : v \in \mathrm{vert}([-1,1]^m)\}$ is controllable and $\varphi_\varepsilon = \sum_{v \in \mathrm{vert}([-1,1]^m)} \lambda_v \varphi_v^{(\varepsilon)}$ belongs to $\Phi_{2^m}([-1,1]^m)$. We obtain $\varphi_\varepsilon = f D_\varepsilon P_\varepsilon$ and

$$\|f - \varphi_\varepsilon\| \leq \omega(f, 2\varepsilon),$$

since $d_\infty(x, D_\varepsilon P_\varepsilon(x)) \leq d_\infty(x, P_\varepsilon(x)) + d_\infty(P_\varepsilon(x), D_\varepsilon P_\varepsilon(x)) \leq \varepsilon + \varepsilon = 2\varepsilon$ for $x \in [-1,1]^m$.

Consequently, f can be written as a limit $f = \lim_{\varepsilon \downarrow 0} \varphi_\varepsilon$ of functions $\varphi_\varepsilon \in \Phi_{2^m}([-1,1]^m)$. This is our claim. ∎

One can read Theorem 4 as an approximation-theoretic characterization of extremal functions on the cube. Besides that, Theorem 4 shows that the first non-trivial approximation class $\Phi_{2^m}([-1,1]^m)$ on the cube is surprisingly large and serves as an example of a non-linear approximation class.

Note that Theorem 4 is essentially based on the choice of the metric d_∞ on the cube, which reflects the polyhedral structure of $[-1,1]^m$. We conjecture that an analogue of the above theorem can be proved for all polytopes P. Then the extremal functions $f \in C(P)$ would arise as the functions with $a_k(f) = 0$ where k denotes the number of vertices of P. The crucial step to such a generalization of Theorem 4 seems to be the definition of an appropriate metric d_P on P. For instance, if P is the unit ball of a Banach space and if $d_{\|\cdot\|}$ is the metric on P induced by the norm, then the above-mentioned analogue of Theorem 4 need not be true for the compact space $(P, d_{\|\cdot\|})$ in general.

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