On the range of the derivative of a real-valued function with bounded support

by

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Abstract. We study the set \( f'(X) = \{ f'(x) : x \in X \} \) when \( f : X \to \mathbb{R} \) is a differentiable bump. We first prove that for any \( C^2 \)-smooth bump \( f : \mathbb{R}^2 \to \mathbb{R} \) the range of the derivative of \( f \) must be the closure of its interior. Next we show that if \( X \) is an infinite-dimensional separable Banach space with a \( C^p \)-smooth bump \( b : X \to \mathbb{R} \) such that \( \|b^{(p)}\|_\infty \) is finite, then any connected open subset of \( X^* \) containing 0 is the range of the derivative of a \( C^p \)-smooth bump. We also study the finite-dimensional case which is quite different. Finally, we show that in infinite-dimensional separable smooth Banach spaces, every analytic subset of \( X^* \) which satisfies a natural linkage condition is the range of the derivative of a \( C^1 \)-smooth bump. We then find an analogue of this condition in the finite-dimensional case.

1. Introduction. A bump is a function from a Banach space \( X \) to \( \mathbb{R} \) with a bounded nonempty support. In this paper we study the set \( f'(X) = \{ f'(x) : x \in X \} \), which is the range of the derivative of \( f \), when \( f \) is a Fréchet differentiable bump. More precisely we will try to find necessary or sufficient conditions for a subset \( A \) of \( X^* \) to be the range of the derivative of a bump.

D. Azagra and M. Jiménez-Sevilla proved in [2] that Rolle’s theorem fails in infinite dimensions. As a consequence, they deduce that there is a \( C^1 \)-smooth Lipschitz bump on \( l_2 \) such that the range of its derivative has an empty interior. However it can be shown by using Ekeland’s Variational Principle ([4]) that \( 0 \in \operatorname{int}(f'(X)) \) even if \( f \) is only Gateaux differentiable. Thus, if \( f \) is a \( C^1 \)-smooth bump on \( \mathbb{R}^n \), then \( f'(\mathbb{R}^n) \) is a compact neighbourhood of 0.

Let us introduce some notations. The symbol \( \mathbb{N} \) means the set \( \{1, 2, \ldots\} \). We write \( B(x, r) \) for the closed ball of centre \( x \) and radius \( r \), and \( S(x, r) \) for the sphere of centre \( x \) and radius \( r \). Sometimes \( B_X \) is used for \( B(0, 1) \). For a function \( f : X \to \mathbb{R} \), the support of \( f \) is \( \operatorname{supp}(f) = \{ x \in X : f(x) \neq 0 \} \). As said before, \( f \) is called a bump if its support is nonempty and bounded. Recall that a function \( f : X \to \mathbb{R} \) is said to be Fréchet differentiable at

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$x_0 \in X$ if there exists $f'(x_0)$ in $X^*$ such that

$$\lim_{y \to 0} \frac{f(x_0 + y) - f(x_0) - f'(x_0)(y)}{\|y\|} = 0.$$  

$f'(x_0)$ is then called the derivative of $f$ at $x_0$. The set $f'(X) = \{f'(x) : x \in X\}$ is the range of the derivative of $f$. We will be concerned only with Fréchet differentiability.

Let us recall some notations for multiindices. The symbol $\mathbb{N}^{<\mathbb{N}}$ stands for the set of finite sequences of natural numbers. If $\sigma = (q_1, \ldots, q_k) \in \mathbb{N}^{<\mathbb{N}}$, then $k$ is called the length of $\sigma$ and we write $k = |\sigma|$. If $k \geq 2$ we define $\sigma_\prec = (q_1, \ldots, q_{k-1})$. For $j \in \{1, \ldots, k\}$, $\sigma(j) = q_j$ and $\sigma^j = (\sigma(1), \ldots, \sigma(j))$. For $\tau = (r_1, \ldots, r_m) \in \mathbb{N}^{<\mathbb{N}}$, $\sigma^\tau = (q_1, \ldots, q_k, r_1, \ldots, r_m)$. The symbol $\mathbb{N}^\mathbb{N}$ denotes the set of infinite sequences of natural numbers. For $\sigma = (q_j)_{j \geq 1} \in \mathbb{N}^\mathbb{N}$ and $j \in \mathbb{N}$, $\sigma^j = q_j$ and $\sigma^j = (\sigma(1), \ldots, \sigma(j))$.

Now we describe our main results and the organization of the paper.

The goal in Section 2 is to try to answer the following question of [3]: If $f : \mathbb{R}^n \to \mathbb{R}$ is a $C^1$-smooth bump, is $f'(\mathbb{R}^n)$ equal to the closure of its interior? We give a partial answer when $n = 2$ and $f$ is $C^2$-smooth in Theorem 2.1. Notice that in infinite dimensions, $f'(X)$ has no reason to be closed and $\text{int}(f'(X))$ can be empty (see [5]).

Section 3 is devoted to finding sufficient conditions for a connected open set to be the range of the derivative of a bump. We recall that $f'(X)$ is connected if $f$ is a Fréchet differentiable bump. This extension of Darboux’s theorem is proved by J. Malý in [7]. However $f'(X)$ is not always simply connected (see [3]). In finite dimensions we prove that any connected open subset of $\mathbb{R}^n$ containing 0 is the range of the derivative of a Fréchet differentiable bump (Theorem 3.1). We then extend this result to the case when $X$ is an infinite-dimensional separable Banach space with a $C^p$-smooth bump $b : X \to \mathbb{R}$ such that $\|b^{(p)}\|_\infty$ is finite (Theorem 3.6).

In Section 4, we find a sufficient condition for an analytic subset of $X^*$ to be the range of the derivative of a $C^1$-smooth bump when $X^*$ is separable (Proposition 4.2). We then exhibit analytic sets, neither closed nor open, which are the range of the derivative of a $C^1$-smooth bump (Theorem 4.4). We obtain an analogue of Proposition 4.2 in finite dimensions in Theorem 4.6. Finally, we study the relationship between Theorem 4.6 and a result of [3].

2. The range of the derivative of a $C^n$-bump. In this section we focus on the case $X = \mathbb{R}^n$ with $n \geq 2$. Our main result is

**Theorem 2.1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$-smooth bump. Then $f'(\mathbb{R}^2)$ is equal to the closure of its interior.
Before proceeding with the proof of this result we recall that the range of the derivative of a $C^1$-smooth bump on $\mathbb{R}^n$ is a connected compact neighbourhood of the origin. We now show other properties which, applied to the case $n = 2$, will allow us to prove Theorem 2.1.

**Proposition 2.2.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^n$-smooth function. If $f' = 0$ on a compact connected set $K$, then $f$ is constant on $K$.

**Proof.** If $C$ is the set of critical points of $f$, Sard’s Theorem shows that $f(C)$ is of Lebesgue measure 0. Since $K$ is a compact connected subset of $C$, $f(K)$ is a compact interval of $\mathbb{R}$ of measure 0, and hence a single point. ■

We need a result on connectedness.

**Lemma 2.3.** Let $C$ be a connected compact subset of $\mathbb{R}^n$ and $G$ the unbounded connected component of $\mathbb{R}^n \setminus C$. Then $\partial G$, the boundary of $G$, is connected.

This follows from [6, §52.III.6 and §52.I.9].

**Proposition 2.4.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^n$-smooth bump and $z \in \partial(f'(\mathbb{R}^n))$. Then $\mathbb{R}^n \setminus f'^{-1}(z)$ is connected.

**Proof.** Assume that $\mathbb{R}^n \setminus f'^{-1}(z)$ is not connected. Since $z \neq 0$, $f'^{-1}(z)$ is bounded and thus $\mathbb{R}^n \setminus f'^{-1}(z)$ has a bounded nonempty connected component, which we call $B$. If we denote by $G$ the unbounded connected component of $\mathbb{R}^n \setminus \bar{B}$, Lemma 2.3 asserts that $\partial G$ is connected. We put $g(x) = f(x) - \langle z, x \rangle$ for $x \in \mathbb{R}^n$. Since $\partial G \subset \partial B$ (see [6, §44.III.3]), $g'(x) = 0$ for all $x$ in $\partial G$. Proposition 2.2 implies that $g$ is constant, equal to some $C$ on $\partial G$. We define $h(x) = 0$ if $x \in G$ and $h(x) = g(x) - C$ if $x \notin G$. Then supp $h$ is bounded and nonempty, since $h'(x) = f'(x) - z \neq 0$ if $x \in B$. Clearly $h$ is $C^1$, so $h$ is a $C^1$-smooth bump, and hence $0 \in \text{int}(h'(\mathbb{R}^n))$. But $h'(\mathbb{R}^n) \subset f'(\mathbb{R}^n) - z$, so $z \in \text{int}(f'(\mathbb{R}^n))$. This contradicts the fact that $z \in \partial(f'(\mathbb{R}^n))$. ■

**Proposition 2.5.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^n$-smooth bump. Then $f'(\mathbb{R}^n)$ cannot be the union of compact sets $A$ and $B$ such that $0 \notin B \subset A$ and $A \cap B$ is a totally disconnected subset of $\partial(f'(\mathbb{R}^n))$.

**Proof.** We suppose that $f'(\mathbb{R}^n) = A \cup B$ with $A$ and $B$ as in the statement. Let $K = f'^{-1}(B)$. Then $K$ is compact, since $B$ is closed and $0 \notin B$. Let $x_0 \in K$ be so that $f'(x_0) \notin A \cap B$. We denote by $C$ the connected component of $x_0$ in $K$ and by $G$ the unbounded connected component of $\mathbb{R}^n \setminus C$. Then $\partial G \subset \partial C \subset \partial K$ ([6, §44.III.3]) and $\partial G$ is connected (Lemma 2.3). Thus $f'(\partial G)$ is a connected subset of $A \cap B$ and hence $f'(\partial G)$ is a single point, called $y$. Proposition 2.4 asserts that $\mathbb{R}^n \setminus f'^{-1}(y)$ is connected. Recall that $0 \notin B$, hence $y \neq 0$ and $\mathbb{R}^n \setminus f'^{-1}(y)$ is unbounded. Since $f'(x_0) \notin A \cap B$, $x_0 \in \mathbb{R}^n \setminus f'^{-1}(y)$. So it is possible to join $x_0$ to infinity with a continuous
path staying in $\mathbb{R}^n \setminus f'^{-1}(y)$. This is absurd, because such a path must cross $\partial G$ which is included in $f'^{-1}(y)$. ■

**Corollary 2.6.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$-smooth bump. Let $y \in f'(\mathbb{R}^2)$. Then there is $\alpha > 0$ such that for all $0 < \varepsilon < \alpha$, the set $f'(\mathbb{R}^2) \cap S(y, \varepsilon)$ contains a nontrivial arc of a circle.

**Proof.** Let $y \in f'(\mathbb{R}^2)$. If $y = 0$ the conclusion is obvious. If $y \neq 0$, let $\varepsilon \in [0, \|y\|/2]$. If $S(y, \varepsilon) \cap \text{int}(f'(\mathbb{R}^2)) \neq \emptyset$ the result follows. Otherwise, $S(y, \varepsilon) \cap f'(\mathbb{R}^2) \subset \partial(f'(\mathbb{R}^2))$. We define $A = f'(\mathbb{R}^2) \cap \{z : \|z - y\| \geq \varepsilon\}$ and $B = f'(\mathbb{R}^2) \cap \{z : \|z - y\| \leq \varepsilon\}$. The sets $A$ and $B$ are both compact, $0 \notin B$ and $y \in B \setminus A$. By Proposition 2.5, $f'(\mathbb{R}^2) \cap S(y, \varepsilon) = A \cap B$ cannot be a totally disconnected subset of $\partial(f'(\mathbb{R}^2))$. So $f'(\mathbb{R}^2) \cap S(y, \varepsilon)$ has a nontrivial connected component. It is easy to see that a closed connected subset of $S(y, \varepsilon)$ is an arc. ■

**Proof of Theorem 2.1.** We set $K = f'(\mathbb{R}^2)$. As $K$ is closed, $\overline{\text{int}K} \subset K$. To show the other inclusion, let $y \in K$. For our $f$ and $y$ we find $\alpha > 0$ by Corollary 2.6. We fix $0 < \beta < \alpha$. For $q \in \mathbb{N}$ and $k \in \{1, \ldots, 2q\}$ we define

$$U_k(q) = \{y + t(\cos \theta, \sin \theta) : t \in [0, \beta], \theta \in [(k-1)\pi/q, k\pi/q]\},$$

$$F_{q,k} = \{\varepsilon \in [0, \beta] : U_k(q) \cap S(y, \varepsilon) \subset K\}.$$ 

Thanks to Corollary 2.6,

$$[0, \beta] = \bigcup_{q \in \mathbb{N}} \bigcup_{k=1}^{2q} F_{q,k}.$$ 

Furthermore each $F_{q,k}$ is closed. Indeed, let $(\varepsilon_j)_j$ be a sequence in $F_{q,k}$ which has a limit $\varepsilon$. Then $\varepsilon \in [0, \beta]$. Let $z \in U_k(q) \cap S(y, \varepsilon)$ and $\theta \in [(k-1)\pi/q, k\pi/q]$ so that $z = y + \varepsilon_j(\cos \theta, \sin \theta)$. Then $z_j = y + \varepsilon_j(\cos \theta, \sin \theta)$ is a sequence in $K$ which converges to $z$. Thus $z \in K$ and $U_k(q) \cap S(y, \varepsilon) \subset K$. So $\varepsilon \in F_{q,k}$ and $F_{q,k}$ is closed.

By Baire’s theorem, there are $q_0 \in \mathbb{N}$ and $k_0 \in \{1, \ldots, 2q_0\}$ such that $F_{q_0,k_0}$ has a nonempty interior. Thus

$$U_{k_0}(q_0) \cap \{y + t(\cos \theta, \sin \theta) : t \in \text{int}F_{q_0,k_0}, \theta \in [0, 2\pi]\}$$

is an open subset of $K \cap B(y, \beta)$. Since $\beta$ can be taken arbitrarily small, $y \in \overline{\text{int}K}$. ■

3. Connected open subsets of $X^*$ and ranges of derivative. First we study the finite-dimensional case. Our main result is

**Theorem 3.1.** Let $U$ be a connected open subset of $\mathbb{R}^n$ containing 0. Then there is a differentiable bump $f : \mathbb{R}^n \to \mathbb{R}$ such that $f'(\mathbb{R}^n) = U$.

We first recall some tools introduced in [3].
Definition 3.2. Let \((y, a) \in (\mathbb{R}^n)^2\) and \(0 < \varepsilon < \|y\|\). We define
\[
D_\varepsilon(y) = \{(1 - t)u + \sqrt{t}y : t \in [0, 1], \|u\| \leq \varepsilon\}.
\]
The set \(T(a, y, \varepsilon) = a + D_\varepsilon(y - a)\) is called the drop with centre \(a\), vertex \(y\), and thickness \(\varepsilon\).

We also introduce the notion of stationary images.

Definition 3.3. Let \(g : X \to Y\) be a mapping and \(y \in Y\). We call \(y\) a stationary image of \(g\) if there is a nonempty open subset \(\Omega\) of \(X\) such that \(g(\Omega) = \{y\}\).

The following lemma is proved in [3].

Lemma 3.4. For every \(y \in \mathbb{R}^n \setminus \{0\}\) and every \(0 < \varepsilon < \|y\|\) there exists a \(C^1\)-smooth bump \(g : \mathbb{R}^n \to \mathbb{R}\) such that \(g'(\mathbb{R}^n) = D_\varepsilon(y)\) and \(y\) is a stationary image of \(g'\).

Lemma 3.5. Let \(q \in \mathbb{N}\) and \(T_1, \ldots, T_q\) be drops with \(T_i = T(a_i, y_i, \varepsilon_i)\), \(a_{i+1} = y_i\) for all \(i\) in \(\{1, \ldots, q - 1\}\) and \(a_1 = 0\). Then there exists a \(C^1\)-smooth bump \(g : \mathbb{R}^n \to \mathbb{R}\) such that
\[
g'(\mathbb{R}^n) = T_1 \cup \ldots \cup T_q.
\]

Proof. The proof is a simple induction. We want to show that the following holds for every \(q \in \mathbb{N}\): “For every \(T_1, \ldots, T_q\) as in the lemma there is a \(C^1\)-smooth bump \(g\) such that \(g'(\mathbb{R}^n) = T_1 \cup \ldots \cup T_q\) and \(y_q\) is a stationary image of \(g'\).”

If \(q = 1\) this is Lemma 3.4. Suppose that the property is true for some \(q \geq 1\). Consider a finite set \(T_1, \ldots, T_{q+1}\) of drops with \(T_i = T(a_i, y_i, \varepsilon_i)\), \(a_1 = 0\), \(a_{i+1} = y_i\) for \(1 \leq i \leq q\). There are a \(C^1\)-smooth bump \(g : \mathbb{R}^n \to \mathbb{R}\), \(x_0 \in X\) and \(r > 0\) such that \(g'(\mathbb{R}^n) = \bigcup_{1 \leq i \leq q} T_i\) and \(g'(x) = y_q\) for all \(x\) in \(B(x_0, r)\). We apply Lemma 3.4 with the drop \(T_{q+1} - a_{q+1} = T(0, y_{q+1} - y_q, \varepsilon_{q+1})\). It gives a \(C^1\)-smooth bump \(h\) so that \(h'(\mathbb{R}^n) = T_{q+1} - y_q\) and \(y_{q+1} - y_q\) is a stationary image of \(h'\). Let \(M\) be large enough to ensure that \(\text{supp}(h) \subset B(0, M)\). Define \(b(x) = g(x) + (2M)^{-1}rh(2Mr^{-1}(x - x_0))\) for \(x \in \mathbb{R}^n\). The function \(b\) is a \(C^1\)-smooth bump, \(y_{q+1}\) is a stationary image of \(b'\), and
\[
b'(\mathbb{R}^n) = g'(\mathbb{R}^n) \cup (y_q + h'(\mathbb{R}^n)) = \bigcup_{1 \leq i \leq q+1} T_i. \quad \blacksquare
\]

Now we can prove Theorem 3.1. The idea is the following: Lemma 3.5 allows us to write any finite union of drops as the range of the derivative of a smooth bump. We cover \(U\) by a countable sequence of such sets. We show that the bumps can be taken in such a way that the series is convergent, differentiable, and that the range of its derivative is \(U\).
**Proof of Theorem 3.1.**

**Step 1:** *U is covered by a countable sequence of good finite unions of drops.*

Consider the following set:

\[ W = \{ y \in U : \text{there are } q \in \mathbb{N} \text{ and } q \text{ drops} \} \]

\[ T_1 = T(a_1, y_1, \varepsilon_1), \ldots, T_q = T(a_q, y_q, \varepsilon_q) \text{ in } U \text{ such that} \]

\[ a_1 = 0, \ y_q = y \text{ and } a_{i+1} = y_i \text{ for all } 1 \leq i \leq q - 1. \]

We are going to show that \( W = U \). Since \( U \) is connected, it is sufficient to prove that \( W \) is a closed open nonempty subset of \( U \). Of course \( 0 \in W \), so \( W \neq \emptyset \). Let \( y \in W \) and \( \varepsilon > 0 \) with \( B(y, \varepsilon) \subset U \). If \( z \in B(y, \varepsilon/2) \), then \( T(y, z, \|z - y\|/10) \subset U \), so \( z \in W \) and \( W \) is open. We take a sequence \((z_k)\) in \( W \) which has a limit \( z \) in \( U \). There is \( \varepsilon > 0 \) with \( B(z, 2\varepsilon) \subset U \). Find \( k > 0 \) so that \( z_k \in B(z, \varepsilon) \). Then \( T(z_k, z, \|z - z_k\|/10) \subset U \), thus \( z \in W \). Therefore \( W \) is a closed subset of \( U \). Hence \( W = U \).

If \( y \in U = W \), there exist \( q \) drops \( T_1 = T(a_1, y_1, \varepsilon_1), \ldots, T_q = T(a_q, y_q, \varepsilon_q) \) in \( U \) such that \( a_1 = 0, \ y_q = y \) and \( a_{i+1} = y_i \) for all \( 1 \leq i \leq q - 1 \). We take \( \varepsilon_y > 0 \) such that \( B(y, 2\varepsilon_y) \subset U \) and \( w_y \) in \( B(y, \varepsilon_y) \). We define \( P_y = T_1 \cup \ldots T_q \cup T(y, w_y, \|w_y - y\|/10) \). Then

\[ U = \bigcup_{y \in U} \text{int } P_y. \]

By Lindelöf’s theorem ([8]), there exists a countable sequence \((y_k)_{k \in \mathbb{N}}\) in \( U \) such that

\[ U = \bigcup_{k \geq 1} \text{int } P_{y_k}. \]

**Step 2:** *There is a differentiable bump \( f \) such that each \( P_{y_k} \) is in \( f'(\mathbb{R}^n) \).*

According to Lemma 3.5, for all \( k \in \mathbb{N} \), there is a \( C^1 \)-smooth bump \( f_k \) with \( f'_k(\mathbb{R}^n) = P_{y_k} \). After a possible homothety we can suppose that

\[ \|f_k\|_{\infty} \leq 1. \]

Let \( M_k \geq 1 \) be such that \( \text{supp}(f_k) \subset B(0, M_k) \). We define

\[ x_k = (2^{-1} + \ldots + 2^{-k}, 0, \ldots, 0), \quad b_k(x) = 8^{-k}M_k^{-1}f_k(8^k M_k(x - x_k)). \]

Then \( b'_k(\mathbb{R}^n) = P_{y_k} \) and \( \text{supp}(b_k) \subset B(x_k, 8^{-k}) = S_k \). If \( k \neq j \), then \( S_k \cap S_j = \emptyset \) and \( \bigcup_{k \in \mathbb{N}} S_k \subset B(0, 2) \). We denote by \( x_\infty \) the point \((1, 0, \ldots, 0)\). The function

\[ f = \sum_{k \geq 1} b_k \]

is obviously \( C^1 \) on \( \mathbb{R}^n \setminus \{x_\infty\} \). Let \( x \in \mathbb{R}^n \) and \( k \geq 1 \). If \( x \not\in S_k \), then \( b_k(x) = 0 \). If \( x \in S_k \), then \( |b_k(x)| \leq 8^{-k}M_k^{-1}\|f_k\|_{\infty} \leq 8^{-k} \) and \( \|x - x_\infty\| \geq \)
1 - ((2^{-1} + \ldots + 2^{-k}) + 8^{-k}) \geq 2^{-k-1}. Thus \( |b_k(x)| \leq 4\|x - x_\infty\|^2 \) and
\[
\frac{|f(x) - f(x_\infty)|}{\|x - x_\infty\|} \leq \sup_k |b_k(x)| \leq 4\|x - x_\infty\|,
\]
so \( f \) is differentiable at \( x_\infty \) and \( f'(x_\infty) = 0 \). Therefore \( f \) is a differentiable bump on \( \mathbb{R}^n \) and
\[
f'(\mathbb{R}^n) = \bigcup_{k \in \mathbb{N}} P_{y_k} = U. \quad \blacksquare
\]

We remark that \( f \) is not \( C^1 \)-smooth because if it were, \( U \) would be closed. \( f \) is nevertheless \( C^1 \)-smooth on \( \mathbb{R}^n \setminus \{x_\infty\} \).

We now obtain similar results in infinite dimensions. Our main result is

**Theorem 3.6.** Let \( X \) be an infinite-dimensional Banach space with a separable dual. Let \( p \in \mathbb{N} \) be such that there exists a \( C^p \)-smooth bump \( b : X \to \mathbb{R} \) with \( \|b^{(p)}\|_\infty \) finite. Let \( U \) be a connected open subset of \( X^* \) containing 0. Then there is a \( C^p \)-smooth bump \( f : X \to \mathbb{R} \) such that \( f'(X) = U \).

Until the end of this section, \( X \) is as in Theorem 3.6. Notice that the separability of \( X^* \) implies that there exists indeed \( p \geq 1 \) and a \( C^p \)-smooth bump \( b : X \to \mathbb{R} \) such that \( \|b^{(p)}\|_\infty \) is finite ([4, p. 58]). We remark that the mean value theorem implies that \( \|b^{(j)}\|_\infty \) is finite for all \( j \in \{0, \ldots, p\} \).

In [1], it was proved that there is a \( C^1 \)-smooth bump such that the range of its derivative is equal to \( X^* \). Theorem 3.6 is an improvement of this result. We now establish results which will be used to prove Theorem 3.6.

**Lemma 3.7.** There is a \( C^p \)-smooth bump \( F : X \to \mathbb{R} \) such that \( B_{X^*} \subset F'(X) \) and \( \|F^{(p)}\|_\infty \) is finite.

**Proof.**

**Step 1:** There is a \( C^p \)-smooth bump \( f \) so that \( f(x) = 1 \) for all \( x \in 2B_X \) and \( \|f^{(p)}\|_\infty \) is finite.

After maybe a translation and multiplication by \(-1\), we can suppose \( b(0) > 0 \). We take a \( C^\infty \)-smooth bump \( \varphi : \mathbb{R} \to \mathbb{R} \) such that \( 0 \leq \varphi \leq 1 \), \( \varphi(t) = 1 \) if \( t \in [2^{-1}b(0), 2^{-1}3b(0)] \), and \( \varphi(0) < 1 \). By the continuity of \( b \) there is \( \delta > 0 \) such that \( b(x) \in [2^{-1}b(0), 2^{-1}3b(0)] \) if \( x \in \delta B_X \). We put \( f(x) = (1 - \varphi(0))^{-1}(\varphi(b(\delta x/2)) - \varphi(0)) \) and the result follows.

**Step 2:** There is a \( C^p \)-smooth bump \( f_0 \) such that the stationary images of \( f_0' \) are dense in \( B_{X^*} \) and \( \|f_0^{(p)}\|_\infty \) is finite.

Since \( X^* \) is separable, there is a dense sequence \((y_k^*)_{k \geq 1} \) in \( B_{X^*} \). Let \( M > 1 \) be so large that \( \text{supp}(f) \subset MB_X \) and \( \|f^{(j)}\|_\infty < M \) for all \( j \in \{0, \ldots, p\} \). Fix now a sequence \((x_k)_{k \geq 1} \) in \( X \) so that \( \|x_k - x_q\| \geq 2M + 1 > 3 \)
if \( k \neq q \) and \( \| x_k \| < 4M + 3 \). We define

\[
  f_0(x) = \sum_{k \geq 1} \langle y_k^*, x \rangle f(x - x_k),
\]

which is a sum of \( C^p \)-smooth functions with separated supports. Thus \( f_0 \) is \( C^p \)-smooth, \( \text{supp}(f_0) \subset (5M + 3)B_X \) and \( f_0^{(j)}(x) = y_k^* \) if \( x \in B(x_k, 1) \). If \( x \in \text{supp}(f_0) \), then

\[
  \| f_0^{(p)}(x) \| \leq \sup_{k \geq 1} \{ \| y_k^* \| \cdot \| x \| \cdot \| f^{(p)}(x - x_k) \| + p \| y_k^* \| \cdot \| f^{(p-1)}(x - x_k) \| \} \\
  \leq (5M + 3)M + pM = (5M + 3 + p)M.
\]

**Step 3:** We construct a sequence \( (f_j)_{j \geq 1} \) of \( C^p \)-smooth bump functions.

We set \( L = 5M + 3 \). Then \( L \geq 8 \), \( \text{supp}(f_0) \subset LB_X \) and \( \| x_k \| < L - 1 \) for all \( k \geq 1 \). For \( j \geq 0 \) we define

\[
  f_{j+1}(x) = \sum_{k \geq 1} L^{-p-1} f_j(L(x - x_k)).
\]

For \( \sigma = (k_1, \ldots, k_j) \in \mathbb{N}^< \mathbb{N} \) we put

\[
  S(\sigma) = B(x_{k_1} + L^{-1}x_{k_2} + \ldots + L^{-j+1}x_{k_j}, L^{-j+1})
\]

and we prove that

\[
  \begin{cases}
    S(\sigma \kappa) \subset S(\sigma) \text{ for all } \sigma \in \mathbb{N}^< \mathbb{N} \text{ and } k \in \mathbb{N}.
    \\
    \text{For all } \sigma, \tau \text{ in } \mathbb{N}^< \mathbb{N}, |\sigma| = |\tau| \text{ and } \sigma \neq \tau \Rightarrow S(\sigma) \cap S(\tau) = \emptyset.
  \end{cases}
\]

For \( j \geq 1 \) we denote by \( \mathcal{P}(j) \) the following statement:

\[
  \begin{cases}
    \text{supp}(f_j) \subset \bigcup_{\sigma \in \mathbb{N}^j} S(\sigma) \text{ and } f_j \text{ is } C^p \text{-smooth.}
    \\
    \text{For all } \sigma \in \mathbb{N}^j \text{ and } k \in \mathbb{N}, x \in S(\sigma \kappa) \Rightarrow f'_j(x) = L^{-j} p y_k^*.
  \end{cases}
\]

We have \( \text{supp}(f_1) \subset \bigcup_{\sigma \in \mathbb{N}} S(\sigma) \). Let \( x \in \text{supp}(f_1) \) and \( \sigma \in \mathbb{N} \) so that \( x \in S(\sigma) \). If \( z \) is in a small neighbourhood of \( x \), then \( f_1(z) = L^{-p-1} f_0(L(z - x_\sigma)) \). Therefore \( f_1 \) is \( C^p \)-smooth. Let \( k \in \mathbb{N} \) and \( x \in S(\sigma \kappa) \). We have \( S(\sigma \kappa) \subset S(\sigma) \) so \( f_1(z) = L^{-p-1} f_0(L(z - x_\sigma)) \) in a neighbourhood of \( x \). Thus \( f'_j(x) = L^{-p} f'_0(L(x - x_\sigma)) = L^{-p} y_k^* \), since \( L(x - x_\sigma) \in B(x_k, 1) \). Consequently, \( \mathcal{P}(1) \) holds.

Let \( j \geq 1 \) and suppose that \( \mathcal{P}(j) \) holds. Then

\[
  \text{supp}(f_{j+1}) \subset \bigcup_{k \geq 1} \text{supp}(x \mapsto f_j(L(x - x_k))) \subset \bigcup_{k \geq 1} (x_k + L^{-1} \text{supp}(f_j))
\]

\[
  \subset \bigcup_{k \geq 1} \bigcup_{\sigma \in \mathbb{N}^j} S(k^\sigma) \subset \bigcup_{\sigma \in \mathbb{N}^{j+1}} S(\sigma).
\]

Let \( x \in \text{supp}(f_{j+1}) \) and \( \sigma \in \mathbb{N}^{j+1} \) be such that \( x \in S(\sigma) \). Clearly \( f_{j+1}(z) = L^{-p-1} f_j(L(z - x_{\sigma(1)})) \) in a neighbourhood of \( x \), so \( f_{j+1} \) is \( C^p \)-smooth. Let \( \sigma \in \mathbb{N}^{j+1} \), \( k \in \mathbb{N} \) and \( x \in S(\sigma \kappa) \). In a neighbourhood of \( x \), \( f_{j+1}(z) = \)
$L^{-p-1} f_j(L(z - x_{\sigma(1)}))$. Thus $f_{j+1}'(x) = L^{-p} f_j'(L(x - x_{\sigma(1)})) = L^{-(j+1)p} y_k^*$, since $L(x - x_{\sigma(1)}) \in S(\sigma(2), \ldots, \sigma(j+1), k)$. Finally, $P(j+1)$ holds.

**STEP 4:** $F = \sum_{j \geq 0} f_j$ is a $C^p$-smooth function and $\|F^{(p)}\|_\infty$ is finite.

For all $j \geq 0$, $\|f_{j+1}\|_\infty \leq L^{-p-1} \|f_j\|_\infty$. Thus the series of the $\|f_j\|_\infty$ is convergent. This proves the existence of $F$ and its continuity. For $j \geq 1$ and $\sigma \in \mathbb{N}$, $S(\sigma) \subseteq S(\sigma(1)) \subseteq LB_X$. Thus $\text{supp}(f_j) \subseteq LB_x$ for all $j \geq 0$ and hence $F$ has a bounded support. If $m \in \{0, \ldots, p\}$, then $\|f_j^{(m)}\|_\infty \leq L^{m-p-1} \|f_j^{(m)}\|_\infty \leq L^{-1} \|f_j^{(m)}\|_\infty$, so $\sum_{j \geq 0} \|f_j^{(m)}\|_\infty < \infty$. Therefore $F$ is a $C^p$-smooth function and $\|F^{(m)}\|_\infty$ is finite for all $0 \leq m \leq p$.

**STEP 5:** Any point in $B_{X^*}$ is in the range of the derivative of $F$.

Fix $z^*$ in $B_{X^*}$. There exists $k_1 \geq 1$ such that $\|z^* - y_{k_1}^*\|_2 \leq L^{-p}$. Then $L^p(z^* - y_{k_1}^*)$ is in $B_{X^*}$, so there is $k_2 \geq 1$ such that $\|L^p(z^* - y_{k_1}^* - y_{k_2}^*)\|_2 \leq L^{-p}$. Thus $\|z^* - (y_{k_1}^* + L^{-p} y_{k_2}^*)\|_2 \leq L^{-2p}$. We construct inductively a sequence $\sigma = (k_j)_{j \geq 1} \in \mathbb{N}^\mathbb{N}$ such that $\|z^* - (y_{\sigma(1)}^* + L^{-p} y_{\sigma(2)}^* + \ldots + L^{-j-1} p y_{\sigma(j)}^*)\|_2 \leq L^{-jp}$ for all $j \geq 1$. Then

$$z^* = \sum_{j \geq 0} L^{-jp} y_{\sigma(j+1)}^*.$$ 

For $q \geq 1$ we define $z_q^* = \sum_{j=0}^{q-1} L^{-jp} y_{\sigma(j+1)}^*$ and $F_q = \sum_{j=0}^{q-1} f_j$. Let $w = \sum_{j \geq 0} L^{-j} x_{\sigma(j+1)}$ and $w_q = \sum_{j=0}^{q-1} L^{-j} x_{\sigma(j+1)}$. For all $j \in \{0, \ldots, q-1\}$, $w_q \in S(\sigma, j+1)$ so $f_j(w_q) = L^{-jp} y_{\sigma(j+1)}$. Thus $F_q'(w_q) = z_q^*$. The sequence $(F_q')_q$ is uniformly convergent, $(w_q)_q$ converges to $w$ and $(z_q^*)_q$ converges to $z^*$, so $F'(w) = z^*$.

The next result provides the existence of plateau functions.

**LEMMA 3.8.** There is a $C^p$-smooth bump $b : X \to \mathbb{R}$ such that $b(X) \subseteq [0, 1]$, $b(x) = 1$ if $\|x\| \leq 2$ and $\|b'\|_\infty \leq 1$.

**Proof.** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-smooth function so that $\varphi(t) = 0$ if $t \leq 0$, $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ if $t \geq 2$, and $|\varphi'(t)| \leq 1$ for all $t \in \mathbb{R}$. Let $b_0 : X \to \mathbb{R}$ be a $C^p$-smooth bump with $b_0(0) > 2$ and $\|b_0^{(p)}\|_\infty < \infty$. We define $b(x) = b_0(rx)$ with $r > 0$ small enough to have $b(x) \geq 2$ if $\|x\| \leq 2$, and $\|b'\|_\infty \leq 1$. Then the function given by $F(x) = \varphi(b(x))$ satisfies the conditions of the lemma.

**LEMMA 3.9.** There is a constant $K$ such that for all $x^*$ in $X^*$, there are a $C^p$-smooth bump $f : X \to \mathbb{R}$ and a real number $a > 0$ such that $y^* + aB_{X^*} \subset f'(X) \subset K\|y^*\|B_{X^*}$ and $f'(x) = y^*$ if $\|x\| \leq 1$.
Proof. Let $b$ be the $C^p$-smooth bump given by Lemma 3.8 and $G$ the $C^p$-smooth bump given by Lemma 3.7. There is an $A > 1$ such that $B_{X*} \subset G'(X) \subset AB_{X*}$, supp$(G) \subset AB_X$ and supp$(b) \subset AB_X$. We put $F(x) = A^{-2}||y^*||G(Ax)$. Then $A^{-1}||y^*||B_{X*} \subset F'(X) \subset ||y^*||B_{X*}$ and supp$(F) \subset B_X$. We now fix a point $x_0 \in X$ with $|x_0| = 3/2$ and we define

$$f(x) = 2y^*(x/2 - x_0)b(x/2 - x_0) + 2F(x/2 - x_0).$$

Then supp$(f) \subset (2A + 3)B_X$. We set $K = 2A + 8$ and $a = A^{-1}||y^*||$. We remark that $K$ is independent of $y^*$. It is clear that $K$ and $f$ satisfy the conditions of the lemma. ■

In what follows, $K$ is the constant given by Lemma 3.9.

Lemmas 3.10. Let $U$ be a connected open subset of $X^*$. Let $y^* \in U$ be such that there are $q \geq 1$ and a sequence $y_0^*, \ldots, y_q^*$ of points of $U$ with $y_0^* = 0$, $y_1^* = y^*$ and $B(y_i^*, K||y_{i+1}^* - y_i^*||) \subset U$ for all $i \in \{0, \ldots, q - 1\}$. Then there exist a $C^p$-smooth bump $f : X \to \mathbb{R}$ and $\delta > 0$ such that

$$y^* \in \text{int}(f'(X)), \quad f'(X) \subset U \quad \text{and} \quad f'(x) = y^* \quad \text{if} \quad |x| \leq \delta.$$

Proof (by induction). The case $q = 1$ is immediate from Lemma 3.9. We fix $q \geq 2$ and suppose that the property is true for $q - 1$. Let $y_0^*, \ldots, y_q^*$ satisfy the hypotheses. By the induction hypothesis we have a $C^p$-smooth bump $g$ and $\alpha > 0$ such that $y_{q-1}^* \in \text{int}(g'(X))$, $g'(X) \subset U$ and $g'(x) = y_{q-1}^*$ for all $x \in \alpha B_X$. Furthermore Lemma 3.9 gives a $C^p$-smooth bump $h$ such that $y_q^* - y_{q-1}^* \in \text{int}(h'(X))$, $h'(X) \subset K||y_q^* - y_{q-1}^*||B_{X*}$ and $h'(x) = y_q^* - y_{q-1}^*$ for all $x \in B_X$. We take $L \geq 1$ large enough to have supp$(h) \subset LB_X$ and we define

$$f(x) = g(x) + L^{-1}\alpha h(\alpha^{-1}Lx).$$

Then $y_q^* \in \text{int}(f'(X))$, $f'(X) \subset g'(X) \cup (y_{q-1}^* + h'(X)) \subset U$ and $f'(x) = y_q^*$ if $|x| \leq L^{-1}\alpha$. ■

We are now able to prove Theorem 3.6.

Proof of Theorem 3.6.

Step 1: Each point $y^*$ in $U$ satisfies the condition of Lemma 3.10.

Define

$$\mathcal{A} = \{y^* \in U : \exists q \in \mathbb{N}, \exists (y_0^* = 0, y_1^*, \ldots, y_q^* = y^*) \in U^{q+1} \text{ so that }$$

$$B(y_i^*, K||y_{i+1}^* - y_i^*||) \subset U \text{ for all } i \in \{0, \ldots, q - 1\}\}.$$

We are going to prove that $\mathcal{A} = U$. Since $0 \in \mathcal{A}$, $\mathcal{A}$ is not empty. Clearly $\mathcal{A}$ is an open subset of $U$. Let $(y_k^*)_k$ be a sequence in $\mathcal{A}$ which has a limit $y^*$ in $U$. There is $\alpha > 0$ such that $B(y^*, 2\alpha) \subset U$. If $k_0$ is large enough, then $y_{k_0}^* \in B(y^*, K^{-1}\alpha)$. Thus $B(y_{k_0}^*, K||y^* - y_{k_0}^*||) \subset U$ and hence $y^* \in \mathcal{A}$. Therefore $\mathcal{A}$ is a closed subset of $U$. Since $U$ is connected, $\mathcal{A} = U$.  

STEP 2: There is a sequence \((f_k)_{k \geq 1}\) of \(C^p\)-smooth bumps with \(U = \bigcup_{k \geq 1} f'_k(X)\).

If \(y^* \in U\), then \(y^* \in \mathcal{A}\) so Lemma 3.10 can be applied. We let \(f_{y^*}\) be the function given by Lemma 3.10. We have

\[
U = \bigcup_{y^* \in U} \text{int}(f'_{y^*}(X)).
\]

As \(X^*\) is separable, we can apply Lindelöf’s theorem ([8]): There is a countable sequence \((y^*_k)_{k \geq 1}\) in \(U\) such that

\[
U = \bigcup_{k \geq 1} \text{int}(f'_{y^*_k}(X)) \quad \text{and therefore} \quad U = \bigcup_{k \geq 1} f'_{y^*_k}(X).
\]

We put \(f_k = f_{y^*_k}\).

STEP 3: There is a \(C^p\)-smooth bump \(f\) such that \(U = f'(X)\).

After possible homotheties we can suppose that supp\((f_k) \subset B_X\) for all \(k \geq 1\). Since \(X\) is infinite-dimensional, there exists a sequence \((x_k)_{k \geq 1}\) in \(X\) such that \(\|x_k\| < 7\) for every \(k \geq 1\) and \(\|x_k - x_q\| > 3\) if \(q \neq k\). We define

\[
f(x) = \sum_{k \geq 1} f_k(x - x_k).
\]

If \(\|x - x_k\| > 3/2\) for all \(k\), then \(f\) is zero and so is \(C^p\)-smooth in a neighbourhood of \(x\). If there is \(k\) so that \(\|x - x_k\| \leq 3/2\), then \(\|x - x_q\| > 3/2\) for all \(q \neq k\), so \(f(z) = f_k(z)\) and \(f'(z) = f'_k(z)\) when \(z\) is in a neighbourhood of \(x\). Thus \(f\) is a \(C^p\)-smooth function and \(f'(X) = \bigcup_{k \geq 1} f'_k(X) = U\).

We give a stronger version of Theorem 3.6 which will be needed in what follows.

PROPOSITION 3.11. Let \(X\) be as in Theorem 3.6. Let \(U\) be a connected open subset of \(X^*\) containing 0. Let \((z^*_k)_{k \geq 1}\) be a sequence of points of \(U\). There is a \(C^p\)-smooth bump \(f : X \to \mathbb{R}\) such that \(f'(X) = U\) and each \(z^*_k\) is a stationary image of \(f'\).

Proof. In the proof of Theorem 3.6, when we use Lindelöf’s theorem to extract the sequence \((y^*_k)_{k}\), we can add to this family some elements in such a way that \(\{z^*_q : q \in \mathbb{N}\} \subset \{y^*_k : k \in \mathbb{N}\}\). The function \(f\) which is then constructed satisfies the following statement: For all \(k\), there is \(\delta_k > 0\) so that \(f'(x) = y^*_k\) if \(\|x - x_k\| < \delta_k\). So every \(z^*_k\) is a stationary image of \(f'\).

4. Well-linked sets and ranges of derivative. In finite dimensions the range of the derivative of a \(C^1\)-smooth bump is compact. If \(X\) is an infinite-dimensional separable Banach space we see, by the definition, that the range of the derivative of a \(C^1\)-smooth bump is an analytic set. Moreover, if \(f\) is a \(C^1\)-smooth bump and \(f'\) is Lipschitzian, there exists \(M > 0\) such that
each point of $f'(X)$ can be joined to 0 by an $M$-Lipschitzian path contained in $f'(X)$. It is sufficient to consider the path $\gamma(t) = f'((1-t)x_0 + tx)$ with $x_0$ so that $f'(x_0) = 0$. Furthermore we have seen in Section 2 that it makes sense to assume $f'(X) = \text{int}(f'(X))$. Consequently, Proposition 4.2 and Theorem 4.6 are partial converses of these necessary conditions. In the first result of this section (Proposition 4.2), we give a sufficient condition for an analytic subset of $X^*$ to be the range of the derivative of a $C^1$-smooth bump when $X^*$ is separable. Let us introduce this condition.

**Definition 4.1.** Let $F$ be a subset of $X^*$. We say that $F$ satisfies condition $(A_{\infty})$ if there are a mapping $\varphi: \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^\mathbb{N} \to X^*$ and a summable sequence $(\delta_k)_{k \geq 1}$ of positive numbers such that

\[
\begin{align*}
\varphi(\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^\mathbb{N}) &= F, \\
\sigma \in \mathbb{N}^{<\mathbb{N}} \text{ and } |\sigma| = 1 &\Rightarrow [0, \varphi(\sigma)] \subset \text{int} F \text{ and } \|\varphi(\sigma)\| < \delta_1, \\
\sigma \in \mathbb{N}^{<\mathbb{N}} \text{ and } |\sigma| \geq 2 &\Rightarrow [\varphi(\sigma), \varphi(\sigma)] \subset \text{int} F \text{ and } \|\varphi(\sigma) - \varphi(\sigma_\cdot)\| < \delta_{|\sigma|}, \\
\sigma \in \mathbb{N}^\mathbb{N} &\Rightarrow \varphi(\sigma) = \lim_k \varphi(\sigma|k).
\end{align*}
\]

**Proposition 4.2.** Let $X$ be an infinite-dimensional Banach space with a separable dual. Let $F$ be a subset of $X^*$. If $F$ satisfies $(A_{\infty})$, then there is a $C^1$-smooth bump $f: X \to \mathbb{R}$ such that $f'(X) = F$.

**Proof.** Since $X^*$ is separable, Theorem 3.6 and Proposition 3.11 can be applied with $p = 1$. Since $X$ is infinite-dimensional, for a given $x \in X$, there is a sequence $(w_k)_{k \in \mathbb{N}}$ in $B(x, \beta/2)$ such that $\|w_k - w_q\| > \beta/5$ if $k \neq q$. We write $w_k = w_k(x, \beta)$. We will proceed by induction on $k := |\sigma|$. In the following, if $|\sigma| = 1$, we put $\varphi(\sigma) = 0$, $\alpha_{\sigma_\cdot} = 1$, $x_{\sigma_\cdot} = 0$.

For $k \in \mathbb{N}$, denote by $P(k)$ the following statement: “For all $\sigma \in \mathbb{N}^{<\mathbb{N}}$ with $|\sigma| = k$, there are $x_{\sigma} \in B_X$, $\alpha_{\sigma} \in ]0, 2^{-k}[, \varepsilon_{\sigma} \in ]0, \min(2^{-k}, \delta_k)]$ and a $C^1$-smooth bump $h_{\sigma}: X \to \mathbb{R}$ such that

\[
\begin{align*}
(i) &\quad \varphi(\sigma_\cdot) + h'_{\sigma}(x) = [\varphi(\sigma_\cdot), \varphi(\sigma)] + \varepsilon_{\sigma} \text{ int } B_X \subset \text{int } F, \\
(ii) &\quad h'_{\sigma}(x) = \varphi(\sigma) - \varphi(\sigma_\cdot) \text{ for all } x \in B(x_{\sigma}, \alpha_{\sigma}), \\
(iii) &\quad \text{supp}(h_{\sigma}) \subset B(x_{\sigma_\cdot}, \alpha_{\sigma_\cdot}) \subset B_X, \\
(iv) &\quad \text{If } |\tau| = |\sigma| \text{ and } \tau \neq \sigma, \text{ then supp}(h_{\sigma}) \cap \text{supp}(h_{\tau}) = \emptyset.
\end{align*}
\]

**Step 1:** $P(1)$ holds.

Let $\sigma \in \mathbb{N}^{<\mathbb{N}}$ with $|\sigma| = 1$. Since $[0, \varphi(\sigma)] \subset \text{int } F$, there is $0 < \varepsilon_{\sigma} < \delta_1$ with $[0, \varphi(\sigma)] + \varepsilon_{\sigma} B_{X^*} \subset \text{int } F$. We apply Proposition 3.11 to obtain a $C^1$-smooth bump $g_{\sigma}$ such that $g'_{\sigma}(X) = [0, \varphi(\sigma)] + \varepsilon_{\sigma} \text{ int } B_{X^*}$ and $\varphi(\sigma)$ is a stationary image of $g'_{\sigma}$. We can suppose that supp($g_{\sigma}$) $\subset B_X$. Define

\[
h_{\sigma}(x) = 12^{-1}g_{\sigma}(12(x - w_{\sigma(1)}(0, 1))).
\]

Then supp($h_{\sigma}$) $\subset B(w_{\sigma(1)}(0, 1), 12^{-1}) \subset B_X$. Moreover there are $x_{\sigma}$ in $B_X$ and $0 < \alpha_{\sigma} < 1$ such that $h'_{\sigma}(x) = \varphi(\sigma)$ for all $x$ in $B(x_{\sigma}, \alpha_{\sigma})$. 


Finally, if $|\sigma| = |\tau| = 1$ and $\sigma \neq \tau$, then supp$(h_\sigma) \cap$ supp$(h_\tau) = \emptyset$, because $\|w_{\sigma(1)}(0,1) - w_{\tau(1)}(0,1)\| > 5^{-1}$.

**Step 2:** $\mathcal{P}(k)$ holds for all $k \geq 1$.

Take $k \geq 1$ and suppose that $\mathcal{P}(k)$ holds. Let $\sigma \in \mathbb{N}^{<\mathbb{N}}$ with $|\sigma| = k + 1$. There is $0 < \varepsilon_\sigma < \delta_{k+1}$ such that $[\varphi(\sigma_-), \varphi(\sigma)] + \varepsilon_\sigma B_{X^*} \subset \text{int } F$. Proposition 3.11 gives a $C^1$-smooth bump $g_\sigma$ such that $g'_\sigma(X) = [0, \varphi(\sigma) - \varphi(\sigma_-)] + \varepsilon_\sigma \text{int } B_{X^*}$, $\varphi(\sigma) - \varphi(\sigma_-)$ is a stationary image of $g'_\sigma$ and supp$(g_\sigma) \subset B_X$. We put

$$h_\sigma(x) = 12^{-1} \alpha_\sigma g_\sigma(12\alpha_\sigma^{-1}(x - w_{\sigma(k+1)}(x_{\sigma_-}, \alpha_{\sigma_-}))).$$

We have supp$(h_\sigma) \subset B(w_{\sigma(k+1)}(x_{\sigma_-}, \alpha_{\sigma_-}), 12^{-1}\alpha_{\sigma_-}) \subset B(x_{\sigma_-}, \alpha_{\sigma_-}) \subset B_X$. If $|\sigma| = |\tau| = k + 1$ and $\sigma \neq \tau$, we can easily check that

$$B(w_{\sigma(k+1)}(x_{\sigma_-}, \alpha_{\sigma_-}), 12^{-1}\alpha_{\sigma_-}) \cap B(w_{\tau(k+1)}(x_{\tau_-}, \alpha_{\tau_-}), 12^{-1}\alpha_{\tau_-}) = \emptyset,$$

so supp$(h_\sigma)$ \cap supp$(h_\tau) = \emptyset$. Moreover $\varphi(\sigma) - \varphi(\sigma_-)$ clearly a stationary image of $h'_\sigma$. So there are $x_\sigma \in B_X$ and $\alpha_\sigma \in [0, 2^{-k}]$ such that $h'_\sigma(x) = \varphi(\sigma) - \varphi(\sigma_-)$ for all $x \in B(x_\sigma, \alpha_\sigma)$. Finally, $\mathcal{P}(k + 1)$ holds.

**Step 3:** The function $f = \sum_{k \geq 1} \sum_{|\sigma| = k} h_\sigma$ is a $C^1$-smooth bump.

For $k \geq 1$ we define $G_k(x) = \sum_{|\sigma| = k} h_\sigma(x)$. Since this is a sum of $C^1$-smooth functions with disjoint supports, it is $C^1$-smooth. We recall that for all $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $h'_\sigma(X) = g'_\sigma(X) = [0, \varphi(\sigma) - \varphi(\sigma_-)] + \varepsilon_\sigma \text{int } B_{X^*}$. For all $x \in X$,

$$\|G'_k(x)\| \leq \sup\{\|h'_\sigma(x)\| : |\sigma| = k\} \leq \sup\{\|\varphi(\sigma) - \varphi(\sigma_-)\| + \varepsilon_\sigma : |\sigma| = k\} \leq 2\delta_k.$$

By the mean value theorem we get $|G_k(x)| \leq 2\delta_k$ since supp$(G_k) \subset B_X$. Therefore $f$ is a $C^1$-smooth bump.

**Step 4:** $f'(X)$ is equal to $F$.

Let $f_k(x) = \sum_{1 \leq j \leq k} G_j(x)$. For all $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $B(x_\sigma, \alpha_\sigma) \subset B(x_{\sigma_-}, \alpha_{\sigma_-})$. Thus, if $k \geq 1$ and $|\sigma| = k$, then $G'_j(x_\sigma) = \varphi(\sigma|j) - \varphi(\sigma|j-1)$ for all $1 \leq j \leq k$ and hence $f'_k(x_\sigma) = \varphi(\sigma)$.

Let $x \in X$. Three cases can arise:

**Case 1:** For all $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $x \notin B(x_\sigma, \alpha_\sigma)$. Then $f'(x) = 0$.

**Case 2:** There is $\sigma \in \mathbb{N}^\mathbb{N}$ so that $x \in B(x_{\sigma|k}, \alpha_{\sigma|k})$ for all $k \geq 1$. Thus $(x_{\sigma|k})_k$ converges to $x$ and since $(f'_k)_k$ is uniformly convergent, we have $f'(x) = \lim_k f'_k(x_{\sigma|k}) = \lim_k \varphi(\sigma|k) = \varphi(\sigma)$.

**Case 3:** There is $\sigma \in \mathbb{N}^{<\mathbb{N}}$ such that $x \in B(x_\sigma, \alpha_\sigma)$ and $x \notin \bigcup_{j \in \mathbb{N}} B(x_{\sigma^-j}, \alpha_{\sigma^-j})$. Then $f'(x) = f'_k(x) = \varphi(\sigma)$.

It is therefore clear that $f'(X) = \varphi(\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^\mathbb{N}) = F$. ■
For closed sets we can rewrite condition \((A_\infty)\) using sequences. Indeed, it is not hard to prove that a closed subset \(F\) of \(X^*\) satisfies \((A_\infty)\) if and only if there are a summable sequence \((\delta_k)_{k \geq 1}\) of positive numbers and a sequence \((y^*_k)_{k \geq 1}\) of points in \(\text{int } F\) with \(y^*_1 = 0\) such that for all \(y^*\) in \(F\), there is a nondecreasing function \(\psi : \mathbb{N} \to \mathbb{N}\) so that \(\lim_{k \to \infty} y^*_\psi(k) = y^*\), \(\psi(1) = 1\) and for all \(k \geq 1\),

\[
[y^*_\psi(k), y^*_\psi(k+1)] \subset \text{int } F \quad \text{and} \quad \|y^*_\psi(k+1) - y^*_\psi(k)\| < \delta_k.
\]

Proposition 4.2 is false in finite dimensions. Indeed, we can construct a compact subset \(P\) of \(\mathbb{R}^2\) which satisfies condition \((A_\infty)\) but which is not the range of the derivative of a \(C^1\)-smooth bump. Because of its form, we call this set a *comb*. We define

\[
P_1 = ([1,2] \times [-1,0]) \cup ([1,2] \times [-1,1]),
\]

\[
P_2 = \left( \bigcup_{q \geq 1} [2^{-1} + \ldots + 2^{-q} - 8^{-q}, 2^{-1} + \ldots + 2^{-q} + 8^{-q}] \right) \times [0,1]
\]

(comb’s teeth) and

\[
P = (-3/2, 0) + (P_1 \cup P_2).
\]

The comb in \(\mathbb{R}^2\)

If \(n \geq 2\), then \(P \times B_{\mathbb{R}^{n-2}}\) is not the range of the derivative of a \(C^1\)-smooth bump, because of the following lemma:

**Lemma 4.3.** For \(x\) and \(y\) in \(F\) define \(r(x, y) = \inf\{\text{diam}(\gamma([0,1])) : \gamma : [0,1] \to F\text{ is continuous, } \gamma(0) = x \text{ and } \gamma(1) = y\}\). If \(F = b'(\mathbb{R}^n)\) with \(b : \mathbb{R}^n \to \mathbb{R}\) a \(C^1\)-smooth bump, then for all \(\varepsilon > 0\) there exists a finite \(\varepsilon\)-net in \(F\) for the metric \(r\).

The proof of this lemma is clear: Since \(b'\) is uniformly continuous on \(\text{supp}(b)\), we find \(\delta > 0\) such that \(\|b'(x) - b'(y)\| < \varepsilon\) if \(\|x - y\| < \delta\). Take a finite \(\delta\)-net in \(\text{supp}(b)\) for the norm; then its range under \(b'\) is a finite \(\varepsilon\)-net in \(F\) for the metric \(r\). Notice that if \(H\) is an infinite-dimensional separable Hilbert space, then \(P \times B_H\) is a subset of \(\mathbb{R}^2 \times H\) which satisfies condition \((A_\infty)\), hence is the range of the derivative of a \(C^1\)-smooth bump on \(\mathbb{R}^2 \times H\).

We now give examples of subsets of \(X^*\), neither closed nor open, which satisfy \((A_\infty)\).
Theorem 4.4. Let $X$ be an infinite-dimensional Banach space with a separable dual. Let $U$ be a bounded open convex subset of $X^*$ containing 0 and let $U \subset A \subset \overline{U}$ be any analytic set. Then there exists a $C^1$-smooth bump $f : X \to \mathbb{R}$ such that $f'(X) = A$.

Proof. Let $U$ and $A$ be as in the theorem. We put $\alpha_k = 2^{-k}, k \in \mathbb{N}$.

Step 1: We construct a mapping $\psi : \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^\mathbb{N} \to X^*$ such that

\[
\begin{align*}
\psi(\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^\mathbb{N}) &= A = \psi(\mathbb{N}^\mathbb{N}) , \\
\sigma \in \mathbb{N}^{<\mathbb{N}} \text{ and } |\sigma| &\geq 2 \Rightarrow \|\psi(\sigma) - \psi(\sigma^-)\| < \alpha_{|\sigma|} , \\
\sigma \in \mathbb{N}^\mathbb{N} \Rightarrow \psi(\sigma) &= \lim_k \psi(\sigma|k) .
\end{align*}
\]

Let $g$ be a bijection from $\mathbb{N}$ onto $\mathbb{N}^{<\mathbb{N}}$. Since $A$ is analytic, there is a continuous mapping $\chi_0$ on $\mathbb{N}^\mathbb{N}$ such that $\chi_0(\mathbb{N}^\mathbb{N}) = A$. We define the map $\chi$ on $\mathbb{N}^\mathbb{N} \cup \mathbb{N}^{<\mathbb{N}}$ by $\chi(\sigma) = \chi_0(\sigma)$ if $\sigma \in \mathbb{N}^{<\mathbb{N}}$, and $\chi(\sigma) \in \{\chi_0(\tau) : \tau \in \mathbb{N}^\mathbb{N} \text{ and } \sigma < \tau\}$ if $\sigma \in \mathbb{N}^{<\mathbb{N}}$. Then $\chi(\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^\mathbb{N}) = A$ and for all $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $(\chi(\sigma|k))_k$ converges and $\chi(\sigma) = \lim_k \chi(\sigma|k)$.

We will define $h : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by induction over $k := |\sigma|$. If $|\sigma| = 1$, then $h(\sigma) = g(\sigma(1))$. If $|\sigma| = k \geq 2$, we put

\[
h(\sigma) = \begin{cases} 
   h(\sigma^-)g(\sigma(k)) & \text{if } \|\chi(h(\sigma^-)g(\sigma(k))) - \chi(h(\sigma^-))\| < \alpha_k , \\
   h(\sigma^-) & \text{otherwise}.
\end{cases}
\]

So, if $\sigma \in \mathbb{N}^{<\mathbb{N}}$, there is a unique $u(\sigma) \in \mathbb{N}^{<\mathbb{N}} \cup \{\emptyset\}$ such that $h(\sigma) = h(\sigma^-)u(\sigma)$. Let $\sigma \in \mathbb{N}^\mathbb{N}$. There is a unique sequence $(u(\sigma|k))_{k \geq 1}$ in $\mathbb{N}^{<\mathbb{N}} \cup \{\emptyset\}$ such that $h(\sigma|k) = u(\sigma|1) \ldots u(\sigma|k)$ for all $k$. We then define

\[
h(\sigma) = u(\sigma|1) \ldots u(\sigma|k) \ldots
\]

The mapping $h$ is a surjection from $\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^\mathbb{N}$ onto itself. Indeed, for each $\sigma \in \mathbb{N}^{<\mathbb{N}}$, there is $p \in \mathbb{N}$ with $g(p) = \sigma$, thus $h(p) = g(p) = \sigma$. Now let $\sigma \in \mathbb{N}^\mathbb{N}$. There exists a strictly increasing sequence $(q_j)_{j \geq 1}$ of positive integers such that

\[
\forall j \geq 1, \forall k, p \geq q_j, \quad \|\chi(\sigma|k) - \chi(\sigma|p)\| < \alpha_{j+1}.
\]

We take $q_0 = 0$. For all $k \geq 1$, there is a unique $m_k \in \mathbb{N}$ so that $g(m_k) = (\sigma(q_{k-1} + 1), \ldots, \sigma(q_k))$. We set $\tau = (m_k)_{k \geq 1}$. For all $k \geq 2$ and all $j \in \{2, \ldots, k\}$,

\[
\|\chi(g(m_1) \ldots g(m_j)) - \chi(g(m_1) \ldots g(m_{j-1}))\| = \|\chi(\sigma|q_j) - \chi(\sigma|q_{j-1})\| < \alpha_j
\]

so $h(\tau|k) = g(m_1) \ldots g(m_k) = \sigma|q_k$ and hence $h(\tau) = \sigma$.

We define $\psi$ on $\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^\mathbb{N}$ by $\psi(\sigma) = \chi(h(\sigma))$. The range of $\psi$ is clearly included in $A$. Let $a \in A$ and $\sigma \in \mathbb{N}^\mathbb{N}$ with $a = \chi(\sigma)$. We have proved that there exists $\tau \in \mathbb{N}^\mathbb{N}$ such that $h(\tau) = \sigma$. Then $\psi(\tau) = \chi(h(\tau)) = \chi(\sigma) = a$. 

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so \( A \subset \psi(\mathbb{N}^N) \). It is clear that if \( \sigma \in \mathbb{N}^{<N} \) and \( |\sigma| \geq 2 \), then
\[
\|\psi(\sigma) - \psi(\sigma_-)\| \leq \|\chi(h(\sigma)) - \chi(h(\sigma_-))\| < \alpha_{|\sigma|}.
\]
Finally, if \( \sigma \in \mathbb{N}^N \), then \( (\psi(\sigma|k))_k \) converges and
\[
\lim_k \psi(\sigma|k) = \lim_k \chi(h(\sigma|k)) = \lim_k \chi((h(\sigma)|n_k) = \chi(h(\sigma)) = \psi(\sigma).
\]

**Step 2:** \( A \) satisfies \( (A_\infty) \).

We define \( \varphi : \mathbb{N}^N \cup \mathbb{N}^{<N} \to X^* \) by \( \varphi(\sigma) = (1 - \alpha_{|\sigma|}) \psi(\sigma) \) if \( \sigma \in \mathbb{N}^{<N} \), and \( \varphi(\sigma) = \psi(\sigma) \) if \( \sigma \in \mathbb{N}^N \). We can easily verify that \( (A_\infty) \) holds with \( \delta_k = 2\alpha_k \).

Then Proposition 4.2 completes the proof.

We now introduce a sufficient condition in finite dimensions which is not far from condition \( (C_\infty) \).

**Definition 4.5.** Let \( F \) be a subset of \( \mathbb{R}^n \). We say that \( F \) satisfies condition \( (C) \) if \( F \) is closed, there are a summable sequence \((\delta_k)_{k \geq 2}\), a sequence \((q_k)_{k \geq 1}\) of positive integers with \( q_1 = 1 \) and a mapping \( \varphi : D \cup \bigcup_{k \geq 1} D_k \to \mathbb{R}^n \)

(\( D = \prod_{j \geq 1} \{1, \ldots, q_j \} \) and \( D_k = \prod_{1 \leq j \leq k} \{1, \ldots, q_j \} \)) such that
\[
\begin{align*}
\varphi(D \cup \bigcup_{k \geq 1} D_k) &= F, \quad \varphi(1) = 0 \quad \text{and, for all } k \geq 2, \\
\sigma \in D_k &\Rightarrow [\varphi(\sigma_-), \varphi(\sigma)] \subset \text{int } F \quad \text{and } \|\varphi(\sigma) - \varphi(\sigma_-)\| < \delta_k. \\
\sigma \in D &\Rightarrow \varphi(\sigma) = \lim_k \varphi(\sigma|k).
\end{align*}
\]

Again, we can rewrite this condition in terms of sequences: \( F \) satisfies condition \( (C) \) if and only if \( F \) is closed, there is a sequence \((y_k^*)_{k \geq 1}\) of points in \( \text{int } F \) with \( y_1^* = 0 \), a nondecreasing sequence \((I_k)_{k \geq 1}\) of finite subsets of \( \mathbb{N} \)

with \( I_1 = \{1\} \) and a summable sequence \((\delta_k)_{k \geq 1}\) of positive numbers such that for all \( y^* \) in \( F \), there is a function \( \psi : \mathbb{N} \to \mathbb{N} \) so that \( \lim_k y_{\psi(k)}^* = y^* \) and

for all \( k \geq 1 \), \( \psi(k) \in I_k \), \([y_{\psi(k)}^*, y_{\psi(k+1)}^*] \subset \text{int } F \) and \( \|y_{\psi(k+1)} - y_{\psi(k)}\| < \delta_k \).

Using the same ideas as in the proof of Proposition 4.2, we get

**Theorem 4.6.** Let \( n \geq 1 \) and \( F \) be a subset of \( \mathbb{R}^n \). If \( F \) satisfies condition \( (C) \), then there is a \( C^1 \) -smooth bump \( f : \mathbb{R}^n \to \mathbb{R} \) such that \( f'(\mathbb{R}^n) = F \).

Let us now recall the condition introduced in [3]:

A subset \( F \) of \( X^* \) is said to satisfy condition \( *(*) \) if there are a summable sequence \((\delta_k)_{k \geq 1}\) of positive numbers and a sequence \((C_k)_{k \geq 1}\) of bounded closed subsets of \( X^* \) such that \( F = \bigcup_{k \geq 1} C_k \), \( C_1 \) is convex and contains 0,

for all \( k \geq 1 \), \( C_k \subset \text{int } C_{k+1} \) and for all \( y \) in \( C_{k+1} \setminus \text{int } C_k \), there is \( z \) in \( C_k \)

such that \([z, y] \subset C_{k+1} \) and \( \|y - z\| < \delta_k \).

The authors of [3] prove that any subset of \( \mathbb{R}^n \) satisfying \( (*) \) is the range of the derivative of a \( C^1 \) -smooth bump. We are going to show that condition \( (*) \) is equivalent to condition \( (C) \). Consequently, Theorem 4.6 is nothing but

**Proposition 4.7.** If \( X = \mathbb{R}^n \), then condition \( (*) \) is equivalent to \( (C) \).
Proof. Let $F$ be a subset of $\mathbb{R}^n$.

**Step 1:** Condition (*) $\Rightarrow$ Condition (C).

We suppose that $F$ satisfies (*). We put $S_1 = \{0\}$. For $k \geq 1$ we define $\varepsilon_k = 2^{-1} \min(\delta_k, \text{dist}(C_{k+1}, \partial F))$,

$$S_{k+1} = S_k \cup \{\text{a finite } \varepsilon_k\text{-net of } C_k\},$$

$q_k = \text{Card } S_k$, $D = \prod_{j \geq 1} \{1, \ldots, q_j\}$ and $D_k = \prod_{1 \leq j \leq k} \{1, \ldots, q_j\}$. We define $\varphi : \bigcup_{k \geq 1} D_k \to \mathbb{R}^n$ by induction. First we set $\delta_0 = \text{diam}(C_1)$ and $\varphi(1) = 0$. We fix $k \geq 1$ and assume that $\varphi$ is defined on $D_k$, $\varphi(D_k) = S_k$ and for all $\sigma \in D_k$, $[\varphi(\sigma_-), \varphi(\sigma)] \subset \text{int } F$ and $\|\varphi(\sigma) - \varphi(\sigma_-)\| < 2\delta_{k-1}$. We remark that if $y \in C_k$, then there is $z \in S_k$ such that $\|y - z\| < 2\delta_{k-1}$ and $[z, y] \subset \text{int } F$. If $\sigma \in D_k$ we set $T_\sigma = \{y \in S_{k+1} : \|y - \varphi(\sigma)\| < 2\delta_{k-1}$ and $[\varphi(\sigma), y] \subset \text{int } F\}$. We can write $T_\sigma = \{z_1, \ldots, z_r\}$ with $r \leq q_{k+1}$. We define $\varphi(\sigma^* j) = z_j$ if $1 \leq j \leq r$ and $\varphi(\sigma^- j) = \varphi(\sigma)$ if $r < j \leq q_{k+1}$. Then $\varphi$ is defined on $D_{k+1}$ and has all the required properties. The fact that $\varphi(D_{k+1}) = S_{k+1}$ follows from the remark. In this way we define $\varphi$ on $\bigcup_{k \geq 1} D_k$. If $\sigma \in D$, then the sequence $(\varphi(\sigma|k))_k$ is convergent and we define $\varphi(\sigma) = \lim_k \varphi(\sigma|k)$.

Let $y \in F$. There is a sequence $(z_k)_{k \geq 1}$ such that $\lim_k z_k = y$ and $z_k \in C_k$ for all $k \geq 1$. For all $k \geq 1$, there is $\sigma_k \in D_{k+1}$ with $\|z_k - \varphi(\sigma_k)\| < \delta_k$ and $[\varphi(\sigma_k), z_k] \subset \text{int } F$. The sequence $(\sigma_k(1))_k$ takes a finite number of values in $\{1, \ldots, q_1\}$. Thus there is $r_1$ in $\{1, \ldots, q_1\}$ so that $\{k : \sigma_k(1) = r_1\}$ is infinite. By induction we build a sequence $(r_j)_{j \geq 1}$ in $D$ such that for all $j$, $\{k : \sigma_k(i) = r_i \text{ for } 1 \leq i \leq j\}$ is infinite. Then $\tau = (r_1, \ldots, r_j, \ldots)$ is in $D$ and $\varphi(\tau) = y$. Therefore (*) implies (C).

**Step 2:** Condition (C) $\Rightarrow$ Condition (*).

We assume that $F$ satisfies (C). There is $\varepsilon_1 > 0$ such that $B(0, \varepsilon_1) \subset \text{int } F$. We define $C_1 = B(0, \varepsilon_1)$. For $k \geq 1$, if $\sigma \in D_k$, then there is $0 < \varepsilon_\sigma < \delta_k$ with $[\varphi(\sigma_-), \varphi(\sigma)] + B(0, \varepsilon_\sigma) \subset \text{int } F$. The set

$$B_k = C_k \cup \left( \bigcup_{\sigma \in D_k} [\varphi(\sigma_-), \varphi(\sigma)] + B(0, \varepsilon_\sigma) \right)$$

is compact and is in $\text{int } F$. So $\alpha_k = \frac{1}{2} \min(\delta_k, \text{dist}(B_k, \partial F)) > 0$. Finally, we define $C_{k+1} = B_k + B(0, \alpha_k)$. The sequence $(C_k)_{k \geq 1}$ satisfies all the required conditions, thus $F$ satisfies condition (*). ■

We have proved that condition (C) can be extended to infinite dimensions. Indeed, Proposition 4.2 shows that $(A_\infty)$ is a sufficient condition in smooth infinite-dimensional Banach spaces and $(A_\infty)$ can be considered as an extension of (C). The situation is different with condition (*). In fact, if $X$ is an infinite-dimensional Banach space, we can construct a subset $R$ of $X^*$
which satisfies condition (\( \ast \)) but which is not the range of the derivative of a \( C^1 \)-smooth bump. Let us describe \( R \). Since \( X \) is infinite-dimensional, there is \( \varepsilon > 0 \) and a \( 3\varepsilon \)-separated sequence \((e_k)_{k \geq 1}\) in \( S_{X^*} \). We fix a point \( w \) in \( X^* \) with \( \|w\| = 3/2 \). We define

\[
D_k = \{tx : x \in S_{X^*} \cap B(e_k, \varepsilon), 1/k \leq t \leq 1\}, \quad D = \bigcup_{k \geq 1} D_k,
\]

\[
R = w + (\{x \in X^* : 1 \leq \|x\| \leq 2\} \cup D)
\]

\[
= (w + (\{x \in X^* : 1 \leq \|x\| \leq 2\} \cup D)) \cup \{w\}.
\]

We remark that the construction of \( R \) is only possible in an infinite-dimensional Banach space. Here is a 2-dimensional representation of \( R \):

\begin{center}
\includegraphics[width=0.3\textwidth]{wheel.png}
\end{center}

The wheel with broken spokes

Because of its form, \( R \) is called the “wheel with broken spokes”. In fact, in infinite dimensions, we can imagine that each spoke is in a new direction and comes closer to \( w \), the centre of the wheel. Then \( R \) satisfies condition (\( \ast \)) but \( R \) is not the range of the derivative of a \( C^1 \)-smooth bump, because \( w \) cannot be joined to 0 by a continuous path in \( R \). Thus condition (\( \ast \)) is not sufficient in infinite dimensions.

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References

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