STUDIA MATHEMATICA 153 (1) (2002)

# On the range of the derivative of a real-valued function with bounded support

by

T. Gaspari (Bordeaux)

**Abstract.** We study the set $f'(X) = \{f'(x) : x \in X\}$ when $f : X \to \mathbb{R}$ is a differentiable bump. We first prove that for any $C^2$-smooth bump $f : \mathbb{R}^2 \to \mathbb{R}$ the range of the derivative of $f$ must be the closure of its interior. Next we show that if $X$ is an infinite-dimensional separable Banach space with a $C^p$-smooth bump $b : X \to \mathbb{R}$ such that $\|b^{(p)}\|_\infty$ is finite, then any connected open subset of $X^*$ containing 0 is the range of the derivative of a $C^p$-smooth bump. We also study the finite-dimensional case which is quite different. Finally, we show that in infinite-dimensional separable smooth Banach spaces, every analytic subset of $X^*$ which satisfies a natural linkage condition is the range of the derivative of a $C^1$-smooth bump. We then find an analogue of this condition in the finite-dimensional case.

**1. Introduction.** A bump is a function from a Banach space $X$ to $\mathbb{R}$ with a bounded nonempty support. In this paper we study the set $f'(X) = \{f'(x) : x \in X\}$, which is the range of the derivative of $f$, when $f$ is a Fréchet differentiable bump. More precisely we will try to find necessary or sufficient conditions for a subset $A$ of $X^*$ to be the range of the derivative of a bump.

D. Azagra and M. Jiménez-Sevilla proved in [2] that Rolle's theorem fails in infinite dimensions. As a consequence, they deduce that there is a $C^1$-smooth Lipschitz bump on $l_2$ such that the range of its derivative has an empty interior. However it can be shown by using Ekeland's Variational Principle ([4]) that $0 \in \operatorname{int}(\overline{f'(X)})$ even if $f$ is only Gateaux differentiable. Thus, if $f$ is a $C^1$-smooth bump on $\mathbb{R}^n$, then $f'(\mathbb{R}^n)$ is a compact neighbourhood of 0.

Let us introduce some notations. The symbol $\mathbb{N}$ means the set $\{1, 2, \ldots\}$. We write $B(x, r)$ for the closed ball of centre $x$ and radius $r$, and $S(x, r)$ for the sphere of centre $x$ and radius $r$. Sometimes $B_X$ is used for $B(0, 1)$. For a function $f : X \to \mathbb{R}$, the *support* of $f$ is $\operatorname{supp}(f) = \{x \in X : f(x) \neq 0\}$. As said before, $f$ is called a *bump* if its support is nonempty and bounded. Recall that a function $f : X \to \mathbb{R}$ is said to be *Fréchet differentiable* at

2000 *Mathematics Subject Classification*: 46G05, 26B05, 46B20, 46T20.

$x_0 \in X$ if there exists $f'(x_0)$ in $X^*$ such that

$$\lim_{y\to 0} \frac{f(x_0+y)-f(x_0)-f'(x_0)(y)}{\|y\|} = 0.$$

$f'(x_0)$ is then called the *derivative* of $f$ at $x_0$. The set $f'(X) = \{f'(x) : x \in X\}$ is the range of the derivative of $f$. We will be concerned only with Fréchet differentiability.

Let us recall some notations for multiindices. The symbol $\mathbb{N}^{<\mathbb{N}}$ stands for the set of finite sequences of natural numbers. If $\sigma = (q_1,\ldots,q_k) \in \mathbb{N}^{<\mathbb{N}}$, then $k$ is called the *length* of $\sigma$ and we write $k = |\sigma|$. If $k \geq 2$ we define $\sigma_- = (q_1,\ldots,q_{k-1})$. For $j \in \{1,\ldots,k\}$, $\sigma(j) = q_j$ and $\sigma|j = (\sigma(1),\ldots,\sigma(j))$. For $\tau = (r_1,\ldots,r_m) \in \mathbb{N}^{<\mathbb{N}}$, $\sigma\hat{\ }\tau = (q_1,\ldots,q_k,r_1,\ldots,r_m)$. The symbol $\mathbb{N}^{\mathbb{N}}$ denotes the set of infinite sequences of natural numbers. For $\sigma = (q_j)_{j\geq 1} \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$, $\sigma(j) = q_j$ and $\sigma|j = (\sigma(1),\ldots,\sigma(j))$.

Now we describe our main results and the organization of the paper.

The goal in Section 2 is to try to answer the following question of [3]: If $f : \mathbb{R}^n \to \mathbb{R}$ is a $C^1$-smooth bump, is $f'(\mathbb{R}^n)$ equal to the closure of its interior? We give a partial answer when $n = 2$ and $f$ is $C^2$-smooth in Theorem 2.1. Notice that in infinite dimensions, $f'(X)$ has no reason to be closed and $\operatorname{int}(f'(X))$ can be empty (see [5]).

Section 3 is devoted to finding sufficient conditions for a connected open set to be the range of the derivative of a bump. We recall that $f'(X)$ is connected if $f$ is a Fréchet differentiable bump. This extension of Darboux's theorem is proved by J. Malý in [7]. However $f'(X)$ is not always simply connected (see [3]). In finite dimensions we prove that any connected open subset of $\mathbb{R}^n$ containing 0 is the range of the derivative of a Fréchet differentiable bump (Theorem 3.1). We then extend this result to the case when $X$ is an infinite-dimensional separable Banach space with a $C^p$-smooth bump $b : X \to \mathbb{R}$ such that $\|b^{(p)}\|_\infty$ is finite (Theorem 3.6).

In Section 4, we find a sufficient condition for an analytic subset of $X^*$ to be the range of the derivative of a $C^1$-smooth bump when $X^*$ is separable (Proposition 4.2). We then exhibit analytic sets, neither closed nor open, which are the range of the derivative of a $C^1$-smooth bump (Theorem 4.4). We obtain an analogue of Proposition 4.2 in finite dimensions in Theorem 4.6. Finally, we study the relationship between Theorem 4.6 and a result of [3].

**2. The range of the derivative of a $C^n$-bump.** In this section we focus on the case $X = \mathbb{R}^n$ with $n \geq 2$. Our main result is

THEOREM 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$-smooth bump. Then $f'(\mathbb{R}^2)$ is equal to the closure of its interior.*

Before proceeding with the proof of this result we recall that the range of the derivative of a $C^1$-smooth bump on $\mathbb{R}^n$ is a connected compact neighbourhood of the origin. We now show other properties which, applied to the case $n = 2$, will allow us to prove Theorem 2.1.

PROPOSITION 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^n$-smooth function. If $f' = 0$ on a compact connected set $K$, then $f$ is constant on $K$.*

*Proof.* If $\mathcal{C}$ is the set of critical points of $f$, Sard's Theorem shows that $f(\mathcal{C})$ is of Lebesgue measure 0. Since $K$ is a compact connected subset of $\mathcal{C}$, $f(K)$ is a compact interval of $\mathbb{R}$ of measure 0, and hence a single point. ∎

We need a result on connectedness.

LEMMA 2.3. *Let $C$ be a connected compact subset of $\mathbb{R}^n$ and $G$ the unbounded connected component of $\mathbb{R}^n \setminus C$. Then $\partial G$, the boundary of $G$, is connected.*

This follows from [6, §52.III.6 and §52.I.9].

PROPOSITION 2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^n$-smooth bump and $z \in \partial(f'(\mathbb{R}^n))$. Then $\mathbb{R}^n \setminus {f'}^{-1}(z)$ is connected.*

*Proof.* Assume that $\mathbb{R}^n \setminus {f'}^{-1}(z)$ is not connected. Since $z \neq 0$, ${f'}^{-1}(z)$ is bounded and thus $\mathbb{R}^n \setminus {f'}^{-1}(z)$ has a bounded nonempty connected component, which we call $B$. If we denote by $G$ the unbounded connected component of $\mathbb{R}^n \setminus \bar{B}$, Lemma 2.3 asserts that $\partial G$ is connected. We put $g(x) = f(x) - \langle z, x\rangle$ for $x \in \mathbb{R}^n$. Since $\partial G \subset \partial B$ (see [6, §44.III.3]), $g'(x) = 0$ for all $x$ in $\partial G$. Proposition 2.2 implies that $g$ is constant, equal to some $C$ on $\partial G$. We define $h(x) = 0$ if $x \in G$ and $h(x) = g(x) - C$ if $x \notin G$. Then $\operatorname{supp} h$ is bounded and nonempty, since $h'(x) = f'(x) - z \neq 0$ if $x \in B$. Clearly $h$ is $C^1$, so $h$ is a $C^1$-smooth bump, and hence $0 \in \operatorname{int}(h'(\mathbb{R}^n))$. But $h'(\mathbb{R}^n) \subset f'(\mathbb{R}^n) - z$, so $z \in \operatorname{int}(f'(\mathbb{R}^n))$. This contradicts the fact that $z \in \partial(f'(\mathbb{R}^n))$. ∎

PROPOSITION 2.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^n$-smooth bump. Then $f'(\mathbb{R}^n)$ cannot be the union of compact sets $A$ and $B$ such that $0 \notin B \not\subset A$ and $A \cap B$ is a totally disconnected subset of $\partial(f'(\mathbb{R}^n))$.*

*Proof.* We suppose that $f'(\mathbb{R}^n) = A \cup B$ with $A$ and $B$ as in the statement. Let $K = {f'}^{-1}(B)$. Then $K$ is compact, since $B$ is closed and $0 \notin B$. Let $x_0 \in K$ be so that $f'(x_0) \notin A \cap B$. We denote by $C$ the connected component of $x_0$ in $K$ and by $G$ the unbounded connected component of $\mathbb{R}^n \setminus C$. Then $\partial G \subset \partial C \subset \partial K$ ([6, §44.III.3]) and $\partial G$ is connected (Lemma 2.3). Thus $f'(\partial G)$ is a connected subset of $A \cap B$ and hence $f'(\partial G)$ is a single point, called $y$. Proposition 2.4 asserts that $\mathbb{R}^n \setminus {f'}^{-1}(y)$ is connected. Recall that $0 \notin B$, hence $y \neq 0$ and $\mathbb{R}^n \setminus {f'}^{-1}(y)$ is unbounded. Since $f'(x_0) \notin A \cap B$, $x_0 \in \mathbb{R}^n \setminus {f'}^{-1}(y)$. So it is possible to join $x_0$ to infinity with a continuous path staying in $\mathbb{R}^n \setminus f'^{-1}(y)$. This is absurd, because such a path must cross $\partial G$ which is included in $f'^{-1}(y)$. ∎

Corollary 2.6. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be a $C^2$-smooth bump. Let $y\in f'(\mathbb{R}^2)$. Then there is $\alpha>0$ such that for all $0<\varepsilon<\alpha$, the set $f'(\mathbb{R}^2)\cap S(y,\varepsilon)$ contains a nontrivial arc of a circle.*

*Proof.* Let $y\in f'(\mathbb{R}^2)$. If $y=0$ the conclusion is obvious. If $y\neq 0$, let $\varepsilon\in\,]0,\|y\|/2[$. If $S(y,\varepsilon)\cap\operatorname{int}(f'(\mathbb{R}^2))\neq\emptyset$ the result follows. Otherwise, $S(y,\varepsilon)\cap f'(\mathbb{R}^2)\subset\partial(f'(\mathbb{R}^2))$. We define $A=f'(\mathbb{R}^2)\cap\{z:\|z-y\|\geq\varepsilon\}$ and $B=f'(\mathbb{R}^2)\cap\{z:\|z-y\|\leq\varepsilon\}$. The sets $A$ and $B$ are both compact, $0\notin B$ and $y\in B\setminus A$. By Proposition 2.5, $f'(\mathbb{R}^2)\cap S(y,\varepsilon)=A\cap B$ cannot be a totally disconnected subset of $\partial(f'(\mathbb{R}^2))$. So $f'(\mathbb{R}^2)\cap S(y,\varepsilon)$ has a nontrivial connected component. It is easy to see that a closed connected subset of $S(y,\varepsilon)$ is an arc. ∎

*Proof of Theorem 2.1.* We set $K=f'(\mathbb{R}^2)$. As $K$ is closed, $\overline{\operatorname{int}K}\subset K$. To show the other inclusion, let $y\in K$. For our $f$ and $y$ we find $\alpha>0$ by Corollary 2.6. We fix $0<\beta<\alpha$. For $q\in\mathbb{N}$ and $k\in\{1,\dots,2q\}$ we define

$$U_k(q)=\{y+t(\cos\theta,\sin\theta):t\in[0,\beta],\ \theta\in[(k-1)\pi/q,k\pi/q]\},$$
$$F_{q,k}=\{\varepsilon\in[0,\beta]:U_k(q)\cap S(y,\varepsilon)\subset K\}.$$

Thanks to Corollary 2.6,

$$[0,\beta]=\bigcup_{q\in\mathbb{N}}\bigcup_{k=1}^{2q}F_{q,k}.$$

Furthermore each $F_{q,k}$ is closed. Indeed, let $(\varepsilon_j)_j$ be a sequence in $F_{q,k}$ which has a limit $\varepsilon$. Then $\varepsilon\in[0,\beta]$. Let $z\in U_k(q)\cap S(y,\varepsilon)$ and $\theta\in[(k-1)\pi/q,k\pi/q]$ so that $z=y+\varepsilon(\cos\theta,\sin\theta)$. Then $z_j=y+\varepsilon_j(\cos\theta,\sin\theta)$ is a sequence in $K$ which converges to $z$. Thus $z\in K$ and $U_k(q)\cap S(y,\varepsilon)\subset K$. So $\varepsilon\in F_{q,k}$ and $F_{q,k}$ is closed.

By Baire's theorem, there are $q_0\in\mathbb{N}$ and $k_0\in\{1,\dots,2q_0\}$ such that $F_{q_0,k_0}$ has a nonempty interior. Thus

$$U_{k_0}(q_0)\cap\{y+t(\cos\theta,\sin\theta):t\in\operatorname{int}F_{q_0,k_0},\ \theta\in[0,2\pi]\}$$

is an open subset of $K\cap B(y,\beta)$. Since $\beta$ can be taken arbitrarily small, $y\in\overline{\operatorname{int}K}$. ∎

**3. Connected open subsets of $X^*$ and ranges of derivative.** First we study the finite-dimensional case. Our main result is

Theorem 3.1. *Let $U$ be a connected open subset of $\mathbb{R}^n$ containing $0$. Then there is a differentiable bump $f:\mathbb{R}^n\to\mathbb{R}$ such that $f'(\mathbb{R}^n)=U$.*

We first recall some tools introduced in [3].

DEFINITION 3.2. Let $(y,a) \in (\mathbb{R}^n)^2$ and $0 < \varepsilon < \|y\|$. We define

$$D_\varepsilon(y) = \{(1-t)u + \sqrt{t}\, y : t \in [0,1],\ \|u\| \le \varepsilon\}.$$

The set $T(a,y,\varepsilon) = a + D_\varepsilon(y-a)$ is called the *drop* with centre $a$, vertex $y$, and thickness $\varepsilon$.

We also introduce the notion of stationary images.

DEFINITION 3.3. Let $g : X \to Y$ be a mapping and $y \in Y$. We call $y$ a *stationary image* of $g$ if there is a nonempty open subset $\Omega$ of $X$ such that $g(\Omega) = \{y\}$.

The following lemma is proved in [3].

LEMMA 3.4. *For every $y \in \mathbb{R}^n \setminus \{0\}$ and every $0 < \varepsilon < \|y\|$ there exists a $C^1$-smooth bump $g : \mathbb{R}^n \to \mathbb{R}$ such that $g'(\mathbb{R}^n) = D_\varepsilon(y)$ and $y$ is a stationary image of $g'$.*

LEMMA 3.5. *Let $q \in \mathbb{N}$ and $T_1, \ldots, T_q$ be drops with $T_i = T(a_i, y_i, \varepsilon_i)$, $a_{i+1} = y_i$ for all $i$ in $\{1, \ldots, q-1\}$ and $a_1 = 0$. Then there exists a $C^1$-smooth bump $g : \mathbb{R}^n \to \mathbb{R}$ such that*

$$g'(\mathbb{R}^n) = T_1 \cup \ldots \cup T_q.$$

*Proof.* The proof is a simple induction. We want to show that the following holds for every $q \in \mathbb{N}$: "For every $T_1, \ldots, T_q$ as in the lemma there is a $C^1$-smooth bump $g$ such that $g'(\mathbb{R}^n) = T_1 \cup \ldots \cup T_q$ and $y_q$ is a stationary image of $g'$ ".

If $q = 1$ this is Lemma 3.4. Suppose that the property is true for some $q \ge 1$. Consider a finite set $T_1, \ldots, T_{q+1}$ of drops with $T_i = T(a_i, y_i, \varepsilon_i)$, $a_1 = 0$, $a_{i+1} = y_i$ for $1 \le i \le q$. There are a $C^1$-smooth bump $g : \mathbb{R}^n \to \mathbb{R}$, $x_0 \in X$ and $r > 0$ such that $g'(\mathbb{R}^n) = \bigcup_{1 \le i \le q} T_i$ and $g'(x) = y_q$ for all $x$ in $B(x_0, r)$. We apply Lemma 3.4 with the drop $T_{q+1} - a_{q+1} = T(0, y_{q+1} - y_q, \varepsilon_{q+1})$. It gives a $C^1$-smooth bump $h$ so that $h'(\mathbb{R}^n) = T_{q+1} - y_q$ and $y_{q+1} - y_q$ is a stationary image of $h'$. Let $M$ be large enough to ensure that $\operatorname{supp}(h) \subset B(0, M)$. Define $b(x) = g(x) + (2M)^{-1} r h(2Mr^{-1}(x - x_0))$ for $x \in \mathbb{R}^n$. The function $b$ is a $C^1$-smooth bump, $y_{q+1}$ is a stationary image of $b'$, and

$$b'(\mathbb{R}^n) = g'(\mathbb{R}^n) \cup (y_q + h'(\mathbb{R}^n)) = \bigcup_{1 \le i \le q+1} T_i. \ \blacksquare$$

Now we can prove Theorem 3.1. The idea is the following: Lemma 3.5 allows us to write any finite union of drops as the range of the derivative of a smooth bump. We cover $U$ by a countable sequence of such sets. We show that the bumps can be taken in such a way that the series is convergent, differentiable, and that the range of its derivative is $U$.

*Proof of Theorem 3.1.*

STEP 1: *$U$ is covered by a countable sequence of good finite unions of drops.*

Consider the following set:

$$\begin{aligned} W = \{y \in U : &\text{ there are } q \in \mathbb{N} \text{ and } q \text{ drops} \\ & T_1 = T(a_1, y_1, \varepsilon_1), \dots, T_q = T(a_q, y_q, \varepsilon_q) \text{ in } U \text{ such that} \\ & a_1 = 0,\ y_q = y \text{ and } a_{i+1} = y_i \text{ for all } 1 \le i \le q-1\}. \end{aligned}$$

We are going to show that $W = U$. Since $U$ is connected, it is sufficient to prove that $W$ is a closed open nonempty subset of $U$. Of course $0 \in W$, so $W \neq \emptyset$. Let $y \in W$ and $\varepsilon > 0$ with $B(y, \varepsilon) \subset U$. If $z \in B(y, \varepsilon/2)$, then $T(y, z, \|z - y\|/10) \subset U$, so $z \in W$ and $W$ is open. We take a sequence $(z_k)_k$ in $W$ which has a limit $z$ in $U$. There is $\varepsilon > 0$ with $B(z, 2\varepsilon) \subset U$. Find $k > 0$ so that $z_k \in B(z, \varepsilon)$. Then $T(z_k, z, \|z - z_k\|/10) \subset U$, thus $z \in W$. Therefore $W$ is a closed subset of $U$. Hence $W = U$.

If $y \in U = W$, there exist $q$ drops $T_1 = T(a_1, y_1, \varepsilon_1), \dots, T_q = T(a_q, y_q, \varepsilon_q)$ in $U$ such that $a_1 = 0$, $y_q = y$ and $a_{i+1} = y_i$ for all $1 \le i \le q-1$. We take $\varepsilon_y > 0$ such that $B(y, 2\varepsilon_y) \subset U$ and $w_y$ in $B(y, \varepsilon_y)$. We define $P_y = T_1 \cup \dots T_q \cup T(y, w_y, \|w_y - y\|/10)$. Then

$$U = \bigcup_{y \in U} \operatorname{int} P_y.$$

By Lindelöf's theorem ([8]), there exists a countable sequence $(y_k)_{k \in \mathbb{N}}$ in $U$ such that

$$U = \bigcup_{k \ge 1} \operatorname{int} P_{y_k}.$$

STEP 2: *There is a differentiable bump $f$ such that each $P_{y_k}$ is in $f'(\mathbb{R}^n)$.*

According to Lemma 3.5, for all $k \in \mathbb{N}$, there is a $C^1$-smooth bump $f_k$ with $f_k'(\mathbb{R}^n) = P_{y_k}$. After a possible homothety we can suppose that $\|f_k\|_\infty \le 1$. Let $M_k \ge 1$ be such that $\operatorname{supp}(f_k) \subset B(0, M_k)$. We define

$$x_k = (2^{-1} + \dots + 2^{-k}, 0, \dots, 0), \qquad b_k(x) = 8^{-k} {M_k}^{-1} f_k(8^k M_k (x - x_k)).$$

Then $b_k'(\mathbb{R}^n) = P_{y_k}$ and $\operatorname{supp}(b_k) \subset B(x_k, 8^{-k}) = S_k$. If $k \neq j$, then $S_k \cap S_j = \emptyset$ and $\bigcup_{k \in \mathbb{N}} S_k \subset B(0, 2)$. We denote by $x_\infty$ the point $(1, 0, \dots, 0)$. The function

$$f = \sum_{k \ge 1} b_k$$

is obviously $C^1$ on $\mathbb{R}^n \setminus \{x_\infty\}$. Let $x \in \mathbb{R}^n$ and $k \ge 1$. If $x \notin S_k$, then $b_k(x) = 0$. If $x \in S_k$, then $|b_k(x)| \le 8^{-k} {M_k}^{-1} \|f_k\|_\infty \le 8^{-k}$ and $\|x - x_\infty\| \ge$

$1-((2^{-1}+\ldots+2^{-k})+8^{-k})\geq 2^{-k-1}$. Thus $|b_k(x)|\leq 4\|x-x_\infty\|^2$ and

$$\frac{|f(x)-f(x_\infty)|}{\|x-x_\infty\|}\leq\frac{\sup_k|b_k(x)|}{\|x-x_\infty\|}\leq 4\|x-x_\infty\|,$$

so $f$ is differentiable at $x_\infty$ and $f'(x_\infty)=0$. Therefore $f$ is a differentiable bump on $\mathbb{R}^n$ and

$$f'(\mathbb{R}^n)=\bigcup_{k\in\mathbb{N}}P_{y_k}=U.\ \blacksquare$$

We remark that $f$ is not $C^1$-smooth because if it were, $U$ would be closed. $f$ is nevertheless $C^1$-smooth on $\mathbb{R}^n\setminus\{x_\infty\}$.

We now obtain similar results in infinite dimensions. Our main result is

THEOREM 3.6. *Let $X$ be an infinite-dimensional Banach space with a separable dual. Let $p\in\mathbb{N}$ be such that there exists a $C^p$-smooth bump $b:X\to\mathbb{R}$ with $\|b^{(p)}\|_\infty$ finite. Let $U$ be a connected open subset of $X^*$ containing* 0*. Then there is a $C^p$-smooth bump $f:X\to\mathbb{R}$ such that $f'(X)=U$.*

Until the end of this section, $X$ is as in Theorem 3.6. Notice that the separability of $X^*$ implies that there exists indeed $p\geq 1$ and a $C^p$-smooth bump $b:X\to\mathbb{R}$ such that $\|b^{(p)}\|_\infty$ is finite ([4, p. 58]). We remark that the mean value theorem implies that $\|b^{(j)}\|_\infty$ is finite for all $j$ in $\{0,\ldots,p\}$. In [1], it was proved that there is a $C^1$-smooth bump such that the range of its derivative is equal to $X^*$. Theorem 3.6 is an improvement of this result. We now establish results which will be used to prove Theorem 3.6.

LEMMA 3.7. *There is a $C^p$-smooth bump $F:X\to\mathbb{R}$ such that $B_{X^*}\subset F'(X)$ and $\|F^{(p)}\|_\infty$ is finite.*

*Proof.*

STEP 1: *There is a $C^p$-smooth bump $f$ so that $f(x)=1$ for all $x\in 2B_X$ and $\|f^{(p)}\|_\infty$ is finite.*

After maybe a translation and multiplication by $-1$, we can suppose $b(0)>0$. We take a $C^\infty$-smooth bump $\varphi:\mathbb{R}\to\mathbb{R}$ such that $0\leq\varphi\leq 1$, $\varphi(t)=1$ if $t\in[2^{-1}b(0),2^{-1}3b(0)]$, and $\varphi(0)<1$. By the continuity of $b$ there is $\delta>0$ such that $b(x)\in[2^{-1}b(0),2^{-1}3b(0)]$ if $x\in\delta B_X$. We put $f(x)=(1-\varphi(0))^{-1}(\varphi(b(\delta x/2))-\varphi(0))$ and the result follows.

STEP 2: *There is a $C^p$-smooth bump $f_0$ such that the stationary images of $f_0'$ are dense in $B_{X^*}$ and $\|f_0^{(p)}\|_\infty$ is finite.*

Since $X^*$ is separable, there is a dense sequence $(y_k^*)_{k\geq 1}$ in $B_{X^*}$. Let $M>1$ be so large that $\operatorname{supp}(f)\subset MB_X$ and $\|f^{(j)}\|_\infty<M$ for all $j$ in $\{0,\ldots,p\}$. Fix now a sequence $(x_k)_{k\geq 1}$ in $X$ so that $\|x_k-x_q\|\geq 2M+1>3$

if $k \neq q$ and $\|x_k\| < 4M + 3$. We define

$$f_0(x) = \sum_{k \geq 1} \langle y_k^*, x \rangle f(x - x_k),$$

which is a sum of $C^p$-smooth functions with separated supports. Thus $f_0$ is $C^p$-smooth, $\operatorname{supp}(f_0) \subset (5M+3)B_X$ and $f_0'(x) = y_k^*$ if $x \in B(x_k, 1)$. If $x \in \operatorname{supp}(f_0)$, then

$$\begin{aligned} \|f_0^{(p)}(x)\| &\leq \sup_{k \geq 1} \{\|y_k^*\| \cdot \|x\| \cdot \|f^{(p)}(x - x_k)\| + p\|y_k^*\| \cdot \|f^{(p-1)}(x - x_k)\|\} \\ &\leq (5M+3)M + pM = (5M+3+p)M. \end{aligned}$$

Step 3: *We construct a sequence $(f_j)_{j \geq 1}$ of $C^p$-smooth bump functions.*

We set $L = 5M+3$. Then $L \geq 8$, $\operatorname{supp}(f_0) \subset LB_X$ and $\|x_k\| < L-1$ for all $k \geq 1$. For $j \geq 0$ we define

$$f_{j+1}(x) = \sum_{k \geq 1} L^{-p-1} f_j(L(x - x_k)).$$

For $\sigma = (k_1, \ldots, k_j) \in \mathbb{N}^{<\mathbb{N}}$ we put

$$S(\sigma) = B(x_{k_1} + L^{-1}x_{k_2} + \ldots + L^{-j+1}x_{k_j}, L^{-j+1})$$

and we prove that

$$\begin{cases} S(\sigma\hat{\ }k) \subset S(\sigma) \textit{ for all } \sigma \in \mathbb{N}^{<\mathbb{N}} \textit{ and } k \in \mathbb{N}. \\ \textit{For all } \sigma, \tau \textit{ in } \mathbb{N}^{<\mathbb{N}},\ |\sigma| = |\tau| \textit{ and } \sigma \neq \tau \Rightarrow S(\sigma) \cap S(\tau) = \emptyset. \end{cases}$$

For $j \geq 1$ we denote by $\mathcal{P}(j)$ the following statement:

$$\begin{cases} \operatorname{supp}(f_j) \subset \bigcup_{\sigma \in \mathbb{N}^j} S(\sigma) \textit{ and } f_j \textit{ is } C^p\textit{-smooth.} \\ \textit{For all } \sigma \in \mathbb{N}^j \textit{ and } k \in \mathbb{N},\ x \in S(\sigma\hat{\ }k) \Rightarrow f_j'(x) = L^{-jp} y_k^*. \end{cases}$$

We have $\operatorname{supp}(f_1) \subset \bigcup_{\sigma \in \mathbb{N}} S(\sigma)$. Let $x \in \operatorname{supp}(f_1)$ and $\sigma \in \mathbb{N}$ so that $x \in S(\sigma)$. If $z$ is in a small neighbourhood of $x$, then $f_1(z) = L^{-p-1} f_0(L(z - x_\sigma))$. Therefore $f_1$ is $C^p$-smooth. Let $k \in \mathbb{N}$ and $x \in S(\sigma\hat{\ }k)$. We have $S(\sigma\hat{\ }k) \subset S(\sigma)$ so $f_1(z) = L^{-p-1} f_0(L(z - x_\sigma))$ in a neighbourhood of $x$. Thus $f_1'(x) = L^{-p} f_0'(L(x - x_\sigma)) = L^{-p} y_k^*$, since $L(x - x_\sigma) \in B(x_k, 1)$. Consequently, $\mathcal{P}(1)$ holds.

Let $j \geq 1$ and suppose that $\mathcal{P}(j)$ holds. Then

$$\begin{aligned} \operatorname{supp}(f_{j+1}) &\subset \bigcup_{k \geq 1} \operatorname{supp}(x \mapsto f_j(L(x - x_k))) \subset \bigcup_{k \geq 1} (x_k + L^{-1} \operatorname{supp}(f_j)) \\ &\subset \bigcup_{k \geq 1} \bigcup_{\sigma \in \mathbb{N}^j} S(k\hat{\ }\sigma) \subset \bigcup_{\sigma \in \mathbb{N}^{j+1}} S(\sigma). \end{aligned}$$

Let $x \in \operatorname{supp}(f_{j+1})$ and $\sigma \in \mathbb{N}^{j+1}$ be such that $x \in S(\sigma)$. Clearly $f_{j+1}(z) = L^{-p-1} f_j(L(z - x_{\sigma(1)}))$ in a neighbourhood of $x$, so $f_{j+1}$ is $C^p$-smooth. Let $\sigma \in \mathbb{N}^{j+1}$, $k \in \mathbb{N}$ and $x \in S(\sigma\hat{\ }k)$. In a neighbourhood of $x$, $f_{j+1}(z) =$

$L^{-p-1}f_j(L(z-x_{\sigma(1)}))$. Thus $f'_{j+1}(x)=L^{-p}f'_j(L(x-x_{\sigma(1)}))=L^{-(j+1)p}y^*_k$, since $L(x-x_{\sigma(1)})\in S(\sigma(2),\dots,\sigma(j+1),k)$. Finally, $\mathcal{P}(j+1)$ holds.

STEP 4: $F=\sum_{j\ge 0}f_j$ *is a* $C^p$*-smooth function and* $\|F^{(p)}\|_\infty$ *is finite.*

For all $j\ge 0$, $\|f_{j+1}\|_\infty\le L^{-p-1}\|f_j\|_\infty$. Thus the series of the $\|f_j\|_\infty$ is convergent. This proves the existence of $F$ and its continuity. For $j\ge 1$ and $\sigma\in\mathbb{N}^j$, $S(\sigma)\subset S(\sigma(1))\subset LB_X$. Thus $\operatorname{supp}(f_j)\subset LB_x$ for all $j\ge 0$ and hence $F$ has a bounded support. If $m\in\{0,\dots,p\}$, then $\|f^{(m)}_{j+1}\|_\infty\le L^{m-p-1}\|f^{(m)}_j\|_\infty\le L^{-1}\|f^{(m)}_j\|_\infty$, so $\sum_{j\ge0}\|f^{(m)}_j\|_\infty<\infty$. Therefore $F$ is a $C^p$-smooth function and $\|F^{(m)}\|_\infty$ is finite for all $0\le m\le p$.

STEP 5: *Any point in* $B_{X^*}$ *is in the range of the derivative of* $F$.

Fix $z^*$ in $B_{X^*}$. There exists $k_1\ge 1$ such that $\|z^*-y^*_{k_1}\|\le L^{-p}$. Then $L^p(z^*-y^*_{k_1})$ is in $B_{X^*}$, so there is $k_2\ge 1$ such that $\|L^p(z^*-y^*_{k_1})-y^*_{k_2}\|\le L^{-p}$. Thus $\|z^*-(y^*_{k_1}+L^{-p}y^*_{k_2})\|\le L^{-2p}$. We construct inductively a sequence $\sigma=(k_j)_{j\ge1}\in\mathbb{N}^{\mathbb{N}}$ such that $\|z^*-(y^*_{\sigma(1)}+L^{-p}y^*_{\sigma(2)}+\dots+L^{-(j-1)p}y^*_{\sigma(j)})\|\le L^{-jp}$ for all $j\ge 1$. Then

$$z^*=\sum_{j\ge 0}L^{-jp}y^*_{\sigma(j+1)}.$$

For $q\ge 1$ we define $z^*_q=\sum_{j=0}^{q-1}L^{-jp}y^*_{\sigma(j+1)}$ and $F_q=\sum_{j=0}^{q-1}f_j$. Let $w=\sum_{j\ge0}L^{-j}x_{\sigma(j+1)}$ and $w_q=\sum_{j=0}^{q-1}L^{-j}x_{\sigma(j+1)}$. For all $j\in\{0,\dots,q-1\}$, $w_q\in S(\sigma|j+1)$ so $f'_j(w_q)=L^{-jp}y^*_{\sigma(j+1)}$. Thus $F'_q(w_q)=z^*_q$. The sequence $(F'_q)_q$ is uniformly convergent, $(w_q)_q$ converges to $w$ and $(z^*_q)_q$ converges to $z^*$, so $F'(w)=z^*$. ∎

The next result provides the existence of plateau functions.

LEMMA 3.8. *There is a* $C^p$*-smooth bump* $b:X\to\mathbb{R}$ *such that*

$$b(X)\subset[0,1],\qquad b(x)=1\quad\textit{if }\|x\|\le 2\quad\textit{and}\quad\|b'\|_\infty\le 1.$$

*Proof.* Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a $C^\infty$-smooth function so that $\varphi(t)=0$ if $t\le 0$, $0\le\varphi\le 1$, $\varphi(t)=1$ if $t\ge 2$, and $|\varphi'(t)|\le 1$ for all $t\in\mathbb{R}$. Let $b_0:X\to\mathbb{R}$ be a $C^p$-smooth bump with $b_0(0)>2$ and $\|b_0^{(p)}\|_\infty<\infty$. We define $b(x)=b_0(rx)$ with $r>0$ small enough to have $b(x)\ge 2$ if $\|x\|\le 2$, and $\|b'\|_\infty\le 1$. Then the function given by $F(x)=\varphi(b(x))$ satisfies the conditions of the lemma. ∎

LEMMA 3.9. *There is a constant* $K$ *such that for all* $y^*$ *in* $X^*$, *there are a* $C^p$*-smooth bump* $f:X\to\mathbb{R}$ *and a real number* $a>0$ *such that*

$$y^*+aB_{X^*}\subset f'(X)\subset K\|y^*\|B_{X^*}\quad\textit{and}\quad f'(x)=y^*\quad\textit{if }\|x\|\le 1.$$

*Proof.* Let $b$ be the $C^p$-smooth bump given by Lemma 3.8 and $G$ the $C^p$-smooth bump given by Lemma 3.7. There is an $A > 1$ such that $B_{X^*} \subset G'(X) \subset AB_{X^*}$, $\operatorname{supp}(G) \subset AB_X$ and $\operatorname{supp}(b) \subset AB_X$. We put $F(x) = A^{-2}\|y^*\|G(Ax)$. Then $A^{-1}\|y^*\|B_{X^*} \subset F'(X) \subset \|y^*\|B_{X^*}$ and $\operatorname{supp}(F) \subset B_X$. We now fix a point $x_0 \in X$ with $\|x_0\| = 3/2$ and we define

$$f(x) = 2y^*(x/2 - x_0)b(x/2 - x_0) + 2F(x/2 - x_0).$$

Then $\operatorname{supp}(f) \subset (2A+3)B_X$. We set $K = 2A + 8$ and $a = A^{-1}\|y^*\|$. We remark that $K$ is independent of $y^*$. It is clear that $K$ and $f$ satisfy the conditions of the lemma. ∎

In what follows, $K$ is the constant given by Lemma 3.9.

LEMMA 3.10. *Let $U$ be a connected open subset of $X^*$. Let $y^* \in U$ be such that there are $q \geq 1$ and a sequence $y_0^*, \ldots, y_q^*$ of points of $U$ with $y_0^* = 0$, $y_q^* = y^*$ and $B(y_i^*, K\|y_{i+1}^* - y_i^*\|) \subset U$ for all $i \in \{0, \ldots, q-1\}$. Then there exist a $C^p$-smooth bump $f : X \to \mathbb{R}$ and $\delta > 0$ such that*

$$y^* \in \operatorname{int}(f'(X)), \quad f'(X) \subset U \quad \textit{and} \quad f'(x) = y^* \quad \textit{if } \|x\| \leq \delta.$$

*Proof* (by induction). The case $q = 1$ is immediate from Lemma 3.9. We fix $q \geq 2$ and suppose that the property is true for $q-1$. Let $y_0^*, \ldots, y_q^*$ satisfy the hypotheses. By the induction hypothesis we have a $C^p$-smooth bump $g$ and $\alpha > 0$ such that $y_{q-1}^* \in \operatorname{int}(g'(X))$, $g'(X) \subset U$ and $g'(x) = y_{q-1}^*$ for all $x \in \alpha B_X$. Furthermore Lemma 3.9 gives a $C^p$-smooth bump $h$ such that $y_q^* - y_{q-1}^* \in \operatorname{int}(h'(X))$, $h'(X) \subset K\|y_q^* - y_{q-1}^*\|B_{X^*}$ and $h'(x) = y_q^* - y_{q-1}^*$ for all $x \in B_X$. We take $L \geq 1$ large enough to have $\operatorname{supp}(h) \subset LB_X$ and we define

$$f(x) = g(x) + L^{-1}\alpha h(\alpha^{-1}Lx).$$

Then $y_q^* \in \operatorname{int}(f'(X))$, $f'(X) \subset g'(X) \cup (y_{q-1}^* + h'(X)) \subset U$ and $f'(x) = y_q^*$ if $\|x\| \leq L^{-1}\alpha$. ∎

We are now able to prove Theorem 3.6.

*Proof of Theorem 3.6.*

STEP 1: *Each point $y^*$ in $U$ satisfies the condition of Lemma* 3.10.

Define

$$\mathcal{A} = \{y^* \in U : \exists q \in \mathbb{N},\ \exists (y_0^* = 0, y_1^*, \ldots, y_q^* = y^*) \in U^{q+1} \text{ so that } B(y_i^*, K\|y_{i+1}^* - y_i^*\|) \subset U \text{ for all } i \in \{0, \ldots, q-1\}\}.$$

We are going to prove that $\mathcal{A} = U$. Since $0 \in \mathcal{A}$, $\mathcal{A}$ is not empty. Clearly $\mathcal{A}$ is an open subset of $U$. Let $(y_k^*)_k$ be a sequence in $\mathcal{A}$ which has a limit $y^*$ in $U$. There is $\alpha > 0$ such that $B(y^*, 2\alpha) \subset U$. If $k_0$ is large enough, then $y_{k_0}^* \in B(y^*, K^{-1}\alpha)$. Thus $B(y_{k_0}^*, K\|y^* - y_{k_0}^*\|) \subset U$ and hence $y^* \in \mathcal{A}$. Therefore $\mathcal{A}$ is a closed subset of $U$. Since $U$ is connected, $\mathcal{A} = U$.

STEP 2: *There is a sequence* $(f_k)_{k\ge 1}$ *of* $C^p$*-smooth bumps with* $U = \bigcup_{k\ge 1} f_k'(X)$.

If $y^* \in U$, then $y^* \in \mathcal{A}$ so Lemma 3.10 can be applied. We let $f_{y^*}$ be the function given by Lemma 3.10. We have

$$U = \bigcup_{y^* \in U} \operatorname{int}(f_{y^*}'(X)).$$

As $X^*$ is separable, we can apply Lindelöf's theorem ([8]): There is a countable sequence $(y_k^*)_k$ in $U$ such that

$$U = \bigcup_{k\ge 1} \operatorname{int}(f_{y_k^*}'(X)) \quad \text{and therefore} \quad U = \bigcup_{k\ge 1} f_{y_k^*}'(X).$$

We put $f_k = f_{y_k^*}$.

STEP 3: *There is a* $C^p$*-smooth bump* $f$ *such that* $U = f'(X)$.

After possible homotheties we can suppose that $\operatorname{supp}(f_k) \subset B_X$ for all $k \ge 1$. Since $X$ is infinite-dimensional, there exists a sequence $(x_k)_{k\ge 1}$ in $X$ such that $\|x_k\| < 7$ for every $k \ge 1$ and $\|x_k - x_q\| > 3$ if $q \ne k$. We define

$$f(x) = \sum_{k\ge 1} f_k(x - x_k).$$

If $\|x - x_k\| > 3/2$ for all $k$, then $f$ is zero and so is $C^p$-smooth in a neighbourhood of $x$. If there is $k$ so that $\|x - x_k\| \le 3/2$, then $\|x - x_q\| > 3/2$ for all $q \ne k$, so $f(z) = f_k(z)$ and $f'(z) = f_k'(z)$ when $z$ is in a neighbourhood of $x$. Thus $f$ is a $C^p$-smooth function and $f'(X) = \bigcup_{k\ge 1} f_k'(X) = U$. ∎

We give a stronger version of Theorem 3.6 which will be needed in what follows.

PROPOSITION 3.11. *Let* $X$ *be as in Theorem* 3.6. *Let* $U$ *be a connected open subset of* $X^*$ *containing* 0. *Let* $(z_k^*)_{k\ge 1}$ *be a sequence of points of* $U$. *There is a* $C^p$*-smooth bump* $f : X \to \mathbb{R}$ *such that* $f'(X) = U$ *and each* $z_k^*$ *is a stationary image of* $f'$.

*Proof.* In the proof of Theorem 3.6, when we use Lindelöf's theorem to extract the sequence $(y_k^*)_k$, we can add to this family some elements in such a way that $\{z_q^* : q \in \mathbb{N}\} \subset \{y_k^* : k \in \mathbb{N}\}$. The function $f$ which is then constructed satisfies the following statement: For all $k$, there is $\delta_k > 0$ so that $f'(x) = y_k^*$ if $\|x - x_k\| < \delta_k$. So every $z_k^*$ is a stationary image of $f'$. ∎

**4. Well-linked sets and ranges of derivative.** In finite dimensions the range of the derivative of a $C^1$-smooth bump is compact. If $X$ is an infinite-dimensional separable Banach space we see, by the definition, that the range of the derivative of a $C^1$-smooth bump is an analytic set. Moreover, if $f$ is a $C^1$-smooth bump and $f'$ is Lipschitzian, there exists $M > 0$ such that each point of $f'(X)$ can be joined to 0 by an $M$-Lipschitzian path contained in $f'(X)$. It is sufficient to consider the path $\gamma(t) = f'((1-t)x_0+tx)$ with $x_0$ so that $f'(x_0) = 0$. Furthermore we have seen in Section 2 that it makes sense to assume $f'(X) = \overline{\operatorname{int}(f'(X))}$. Consequently, Proposition 4.2 and Theorem 4.6 are partial converses of these necessary conditions. In the first result of this section (Proposition 4.2), we give a sufficient condition for an analytic subset of $X^*$ to be the range of the derivative of a $C^1$-smooth bump when $X^*$ is separable. Let us introduce this condition.

DEFINITION 4.1. Let $F$ be a subset of $X^*$. We say that $F$ satisfies *condition* $(\mathcal{A}_\infty)$ if there are a mapping $\varphi : \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}} \to X^*$ and a summable sequence $(\delta_k)_{k\geq 1}$ of positive numbers such that

$$\begin{cases} \varphi(\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}) = F. \\ \sigma \in \mathbb{N}^{<\mathbb{N}} \text{ and } |\sigma| = 1 \Rightarrow [0, \varphi(\sigma)] \subset \operatorname{int} F \text{ and } \|\varphi(\sigma)\| < \delta_1. \\ \sigma \in \mathbb{N}^{<\mathbb{N}} \text{ and } |\sigma| \geq 2 \Rightarrow [\varphi(\sigma_-), \varphi(\sigma)] \subset \operatorname{int} F \text{ and } \|\varphi(\sigma) - \varphi(\sigma_-)\| < \delta_{|\sigma|}. \\ \sigma \in \mathbb{N}^{\mathbb{N}} \Rightarrow \varphi(\sigma) = \lim_k \varphi(\sigma|k). \end{cases}$$

PROPOSITION 4.2. *Let $X$ be an infinite-dimensional Banach space with a separable dual. Let $F$ be a subset of $X^*$. If $F$ satisfies $(\mathcal{A}_\infty)$, then there is a $C^1$-smooth bump $f : X \to \mathbb{R}$ such that $f'(X) = F$.*

*Proof.* Since $X^*$ is separable, Theorem 3.6 and Proposition 3.11 can be applied with $p = 1$. Since $X$ is infinite-dimensional, for a given $x \in X$, there is a sequence $(w_k)_{k\in\mathbb{N}}$ in $B(x, \beta/2)$ such that $\|w_k - w_q\| > \beta/5$ if $k \neq q$. We write $w_k = w_k(x, \beta)$. We will proceed by induction on $k := |\sigma|$. In the following, if $|\sigma| = 1$, we put $\varphi(\sigma_-) = 0$, $\alpha_{\sigma_-} = 1$, $x_{\sigma_-} = 0$.

For $k \in \mathbb{N}$, denote by $\mathcal{P}(k)$ the following statement: "For all $\sigma \in \mathbb{N}^{<\mathbb{N}}$ with $|\sigma| = k$, there are $x_\sigma \in B_X$, $\alpha_\sigma \in ]0, 2^{-k}[$, $\varepsilon_\sigma \in ]0, \min(2^{-k}, \delta_k)[$ and a $C^1$-smooth bump $h_\sigma : X \to \mathbb{R}$ such that

(i) $\varphi(\sigma_-) + h'_\sigma(X) = [\varphi(\sigma_-), \varphi(\sigma)] + \varepsilon_\sigma \operatorname{int} B_{X^*} \subset \operatorname{int} F$.
(ii) $h'_\sigma(x) = \varphi(\sigma) - \varphi(\sigma_-)$ for all $x \in B(x_\sigma, \alpha_\sigma)$.
(iii) $\operatorname{supp}(h_\sigma) \subset B(x_{\sigma_-}, \alpha_{\sigma_-}) \subset B_X$.
(iv) If $|\tau| = |\sigma|$ and $\tau \neq \sigma$, then $\operatorname{supp}(h_\sigma) \cap \operatorname{supp}(h_\tau) = \emptyset$."

STEP 1: $\mathcal{P}(1)$ *holds.*

Let $\sigma \in \mathbb{N}^{<\mathbb{N}}$ with $|\sigma| = 1$. Since $[0, \varphi(\sigma)] \subset \operatorname{int} F$, there is $0 < \varepsilon_\sigma < \delta_1$ with $[0, \varphi(\sigma)] + \varepsilon_\sigma B_{X^*} \subset \operatorname{int} F$. We apply Proposition 3.11 to obtain a $C^1$-smooth bump $g_\sigma$ such that $g'_\sigma(X) = [0, \varphi(\sigma)] + \varepsilon_\sigma \operatorname{int} B_{X^*}$ and $\varphi(\sigma)$ is a stationary image of $g'_\sigma$. We can suppose that $\operatorname{supp}(g_\sigma) \subset B_X$. Define

$$h_\sigma(x) = 12^{-1} g_\sigma(12(x - w_{\sigma(1)}(0, 1))).$$

Then $\operatorname{supp}(h_\sigma) \subset B(w_{\sigma(1)}(0, 1), 12^{-1}) \subset B_X$. Moreover there are $x_\sigma$ in $B_X$ and $0 < \alpha_\sigma < 1$ such that $h'_\sigma(x) = \varphi(\sigma)$ for all $x$ in $B(x_\sigma, \alpha_\sigma)$.

Finally, if $|\sigma| = |\tau| = 1$ and $\sigma \neq \tau$, then $\mathrm{supp}(h_\sigma) \cap \mathrm{supp}(h_\tau) = \emptyset$, because $\|w_{\sigma(1)}(0,1) - w_{\tau(1)}(0,1)\| > 5^{-1}$.

STEP 2: *$\mathcal{P}(k)$ holds for all $k \geq 1$.*

Take $k \geq 1$ and suppose that $\mathcal{P}(k)$ holds. Let $\sigma \in \mathbb{N}^{<\mathbb{N}}$ with $|\sigma| = k+1$. There is $0 < \varepsilon_\sigma < \delta_{k+1}$ such that $[\varphi(\sigma_-), \varphi(\sigma)] + \varepsilon_\sigma B_{X^*} \subset \mathrm{int}\, F$. Proposition 3.11 gives a $C^1$-smooth bump $g_\sigma$ such that $g'_\sigma(X) = [0, \varphi(\sigma) - \varphi(\sigma_-)] + \varepsilon_\sigma \,\mathrm{int}\, B_{X^*}$, $\varphi(\sigma) - \varphi(\sigma_-)$ is a stationary image of $g'_\sigma$ and $\mathrm{supp}(g_\sigma) \subset B_X$. We put

$$h_\sigma(x) = 12^{-1}\alpha_{\sigma_-} g_\sigma(12\alpha_{\sigma_-}^{-1}(x - w_{\sigma(k+1)}(x_{\sigma_-}, \alpha_{\sigma_-}))).$$

We have $\mathrm{supp}(h_\sigma) \subset B(w_{\sigma(k+1)}(x_{\sigma_-}, \alpha_{\sigma_-}), 12^{-1}\alpha_{\sigma_-}) \subset B(x_{\sigma_-}, \alpha_{\sigma_-}) \subset B_X$. If $|\sigma| = |\tau| = k+1$ and $\sigma \neq \tau$, we can easily check that

$$B(w_{\sigma(k+1)}(x_{\sigma_-}, \alpha_{\sigma_-}), 12^{-1}\alpha_{\sigma_-}) \cap B(w_{\tau(k+1)}(x_{\tau_-}, \alpha_{\tau_-}), 12^{-1}\alpha_{\tau_-}) = \emptyset,$$

so $\mathrm{supp}(h_\sigma) \cap \mathrm{supp}(h_\tau) = \emptyset$. Moreover $\varphi(\sigma) - \varphi(\sigma_-)$ is clearly a stationary image of $h'_\sigma$. So there are $x_\sigma \in B_X$ and $\alpha_\sigma \in \,]0, 2^{-k}[$ such that $h'_\sigma(x) = \varphi(\sigma) - \varphi(\sigma_-)$ for all $x \in B(x_\sigma, \alpha_\sigma)$. Finally, $\mathcal{P}(k+1)$ holds.

STEP 3: *The function $f = \sum_{k\geq 1}\sum_{|\sigma|=k} h_\sigma$ is a $C^1$-smooth bump.*

For $k \geq 1$ we define $G_k(x) = \sum_{|\sigma|=k} h_\sigma(x)$. Since this is a sum of $C^1$-smooth functions with disjoint supports, it is $C^1$-smooth. We recall that for all $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $h'_\sigma(X) = g'_\sigma(X) = [0, \varphi(\sigma) - \varphi(\sigma_-)] + \varepsilon_\sigma \,\mathrm{int}\, B_{X^*}$. For all $x \in X$,

$$\begin{aligned} \|G'_k(x)\| &\leq \sup\{\|h'_\sigma(x)\| : |\sigma| = k\} \\ &\leq \sup\{\|\varphi(\sigma) - \varphi(\sigma_-)\| + \varepsilon_\sigma : |\sigma| = k\} \leq 2\delta_k. \end{aligned}$$

By the mean value theorem we get $|G_k(x)| \leq 2\delta_k$ since $\mathrm{supp}(G_k) \subset B_X$. Therefore $f$ is a $C^1$-smooth bump.

STEP 4: *$f'(X)$ is equal to $F$.*

Let $f_k(x) = \sum_{1\leq j\leq k} G_j(x)$. For all $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $B(x_\sigma, \alpha_\sigma) \subset B(x_{\sigma_-}, \alpha_{\sigma_-})$. Thus, if $k \geq 1$ and $|\sigma| = k$, then $G'_j(x_\sigma) = \varphi(\sigma|j) - \varphi(\sigma|j-1)$ for all $1 \leq j \leq k$ and hence $f'_k(x_\sigma) = \varphi(\sigma)$.

Let $x \in X$. Three cases can arise:

Case 1: For all $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $x \notin B(x_\sigma, \alpha_\sigma)$. Then $f'(x) = 0$.

Case 2: There is $\sigma \in \mathbb{N}^{\mathbb{N}}$ so that $x \in B(x_{\sigma|k}, \alpha_{\sigma|k})$ for all $k \geq 1$. Thus $(x_{\sigma|k})_k$ converges to $x$ and since $(f'_k)_k$ is uniformly convergent, we have $f'(x) = \lim_k f'_k(x_{\sigma|k}) = \lim_k \varphi(\sigma|k) = \varphi(\sigma)$.

Case 3: There is $\sigma \in \mathbb{N}^{<\mathbb{N}}$ such that $x \in B(x_\sigma, \alpha_\sigma)$ and $x \notin \bigcup_{j\in\mathbb{N}} B(x_{\sigma\hat{}j}, \alpha_{\sigma\hat{}j})$. Then $f'(x) = f'_k(x) = \varphi(\sigma)$.

It is therefore clear that $f'(X) = \varphi(\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}) = F$. ∎

For closed sets we can rewrite condition $(\mathcal{A}_\infty)$ using sequences. Indeed, it is not hard to prove that *a closed subset $F$ of $X^*$ satisfies $(\mathcal{A}_\infty)$ if and only if there are a summable sequence $(\delta_k)_{k\geq 1}$ of positive numbers and a sequence $(y_k^*)_{k\geq 1}$ of points in $\operatorname{int} F$ with $y_1^* = 0$ such that for all $y^*$ in $F$, there is a nondecreasing function $\psi : \mathbb{N} \to \mathbb{N}$ so that $\lim_{k\to\infty} y^*_{\psi(k)} = y^*$, $\psi(1) = 1$ and for all $k \geq 1$,*

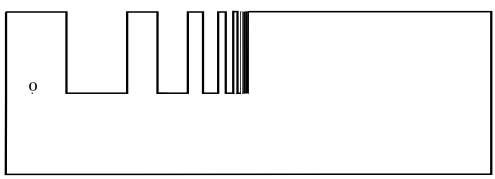
$$[y^*_{\psi(k)}, y^*_{\psi(k+1)}] \subset \operatorname{int} F \quad \textit{and} \quad \|y^*_{\psi(k+1)} - y^*_{\psi(k)}\| < \delta_k.$$

Proposition 4.2 is false in finite dimensions. Indeed, we can construct a compact subset $P$ of $\mathbb{R}^2$ which satisfies condition $(\mathcal{A}_\infty)$ but which is not the range of the derivative of a $C^1$-smooth bump. Because of its form, we call this set a *comb*. We define

$$P_1 = ([-1,2] \times [-1,0]) \cup ([1,2] \times [-1,1]),$$
$$P_2 = \Big(\bigcup_{q\geq 1} [2^{-1} + \ldots + 2^{-q} - 8^{-q}, 2^{-1} + \ldots + 2^{-q} + 8^{-q}]\Big) \times [0,1]$$

(comb's teeth) and

$$P = (-3/2, 0) + (P_1 \cup P_2).$$




The comb in $\mathbb{R}^2$

If $n \geq 2$, then $P \times B_{\mathbb{R}^{n-2}}$ is not the range of the derivative of a $C^1$-smooth bump, because of the following lemma:

LEMMA 4.3. *For $x$ and $y$ in $F$ define $r(x,y) = \inf\{\operatorname{diam}(\gamma([0,1])) : \gamma : [0,1] \to F$ is continuous, $\gamma(0) = x$ and $\gamma(1) = y\}$. If $F = b'(\mathbb{R}^n)$ with $b : \mathbb{R}^n \to \mathbb{R}$ a $C^1$-smooth bump, then for all $\varepsilon > 0$ there exists a finite $\varepsilon$-net in $F$ for the metric $r$.*

The proof of this lemma is clear: Since $b'$ is uniformly continuous on $\operatorname{supp}(b)$, we find $\delta > 0$ such that $\|b'(x) - b'(y)\| < \varepsilon$ if $\|x - y\| < \delta$. Take a finite $\delta$-net in $\operatorname{supp}(b)$ for the norm; then its range under $b'$ is a finite $\varepsilon$-net in $F$ for the metric $r$. Notice that if $H$ is an infinite-dimensional separable Hilbert space, then $P \times B_H$ is a subset of $\mathbb{R}^2 \times H$ which satisfies condition $(\mathcal{A}_\infty)$, hence is the range of the derivative of a $C^1$-smooth bump on $\mathbb{R}^2 \times H$.

We now give examples of subsets of $X^*$, neither closed nor open, which satisfy $(\mathcal{A}_\infty)$.

THEOREM 4.4. *Let $X$ be an infinite-dimensional Banach space with a separable dual. Let $U$ be a bounded open convex subset of $X^*$ containing 0 and let $U \subset A \subset \bar{U}$ be any analytic set. Then there exists a $C^1$-smooth bump $f : X \to \mathbb{R}$ such that $f'(X) = A$.*

*Proof.* Let $U$ and $A$ be as in the theorem. We put $\alpha_k = 2^{-k}$, $k \in \mathbb{N}$.

STEP 1: *We construct a mapping $\psi : \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}} \to X^*$ such that*

$$\begin{cases} \psi(\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}) = A = \psi(\mathbb{N}^{\mathbb{N}}). \\ \sigma \in \mathbb{N}^{<\mathbb{N}} \text{ and } |\sigma| \geq 2 \Rightarrow \|\psi(\sigma) - \psi(\sigma_-)\| < \alpha_{|\sigma|}. \\ \sigma \in \mathbb{N}^{\mathbb{N}} \Rightarrow \psi(\sigma) = \lim_k \psi(\sigma|k). \end{cases}$$

Let $g$ be a bijection from $\mathbb{N}$ onto $\mathbb{N}^{<\mathbb{N}}$. Since $A$ is analytic, there is a continuous mapping $\chi_0$ on $\mathbb{N}^{\mathbb{N}}$ such that $\chi_0(\mathbb{N}^{\mathbb{N}}) = A$. We define the map $\chi$ on $\mathbb{N}^{\mathbb{N}} \cup \mathbb{N}^{<\mathbb{N}}$ by $\chi(\sigma) = \chi_0(\sigma)$ if $\sigma \in \mathbb{N}^{\mathbb{N}}$, and $\chi(\sigma) \in \{\chi_0(\tau) : \tau \in \mathbb{N}^{\mathbb{N}}$ and $\sigma < \tau\}$ if $\sigma \in \mathbb{N}^{<\mathbb{N}}$. Then $\chi(\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}) = A$ and for all $\sigma \in \mathbb{N}^{\mathbb{N}}$, $(\chi(\sigma|k))_k$ converges and $\chi(\sigma) = \lim_k \chi(\sigma|k)$.

We will define $h : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by induction over $k := |\sigma|$. If $|\sigma| = 1$, then $h(\sigma) = g(\sigma(1))$. If $|\sigma| = k \geq 2$, we put

$$h(\sigma) = \begin{cases} h(\sigma_-)\hat{\ }g(\sigma(k)) & \text{if } \|\chi(h(\sigma_-)\hat{\ }g(\sigma(k))) - \chi(h(\sigma_-))\| < \alpha_k, \\ h(\sigma_-) & \text{otherwise.} \end{cases}$$

So, if $\sigma \in \mathbb{N}^{<\mathbb{N}}$, there is a unique $u(\sigma) \in \mathbb{N}^{<\mathbb{N}} \cup \{\emptyset\}$ such that $h(\sigma) = h(\sigma_-)\hat{\ }u(\sigma)$. Let $\sigma \in \mathbb{N}^{\mathbb{N}}$. There is a unique sequence $(u(\sigma|k))_{k\geq 1}$ in $\mathbb{N}^{<\mathbb{N}} \cup \{\emptyset\}$ such that $h(\sigma|k) = u(\sigma|1)\hat{\ }\ldots\hat{\ }u(\sigma|k)$ for all $k$. We then define

$$h(\sigma) = u(\sigma|1)\hat{\ }\ldots\hat{\ }u(\sigma|k)\hat{\ }\ldots$$

The mapping $h$ is a surjection from $\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}$ onto itself. Indeed, for each $\sigma \in \mathbb{N}^{<\mathbb{N}}$, there is $p \in \mathbb{N}$ with $g(p) = \sigma$, thus $h(p) = g(p) = \sigma$. Now let $\sigma \in \mathbb{N}^{\mathbb{N}}$. There exists a stricly increasing sequence $(q_j)_{j\geq 1}$ of positive integers such that

$$\forall j \geq 1,\ \forall k, p \geq q_j, \quad \|\chi(\sigma|k) - \chi(\sigma|p)\| < \alpha_{j+1}.$$

We take $q_0 = 0$. For all $k \geq 1$, there is a unique $m_k \in \mathbb{N}$ so that $g(m_k) = (\sigma(q_{k-1} + 1), \ldots, \sigma(q_k))$. We set $\tau = (m_k)_{k\geq 1}$. For all $k \geq 2$ and all $j \in \{2, \ldots, k\}$,

$$\begin{aligned} &\|\chi(g(m_1)\hat{\ }\ldots\hat{\ }g(m_j)) - \chi(g(m_1)\hat{\ }\ldots\hat{\ }g(m_{j-1}))\| \\ &\qquad = \|\chi(\sigma|q_j) - \chi(\sigma|q_{j-1})\| < \alpha_j \end{aligned}$$

so $h(\tau|k) = g(m_1)\hat{\ }\ldots\hat{\ }g(m_k) = \sigma|q_k$ and hence $h(\tau) = \sigma$.

We define $\psi$ on $\mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}$ by $\psi(\sigma) = \chi(h(\sigma))$. The range of $\psi$ is clearly included in $A$. Let $a \in A$ and $\sigma \in \mathbb{N}^{\mathbb{N}}$ with $a = \chi(\sigma)$. We have proved that there exists $\tau \in \mathbb{N}^{\mathbb{N}}$ such that $h(\tau) = \sigma$. Then $\psi(\tau) = \chi(h(\tau)) = \chi(\sigma) = a$, so $A \subset \psi(\mathbb{N}^{\mathbb{N}})$. It is clear that if $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $|\sigma| \geq 2$, then

$$\|\psi(\sigma) - \psi(\sigma_{\text{-}})\| \leq \|\chi(h(\sigma)) - \chi(h(\sigma_{\text{-}}))\| < \alpha_{|\sigma|}.$$

Finally, if $\sigma \in \mathbb{N}^{\mathbb{N}}$, then $(\psi(\sigma|k))_k$ converges and

$$\lim_k \psi(\sigma|k) = \lim_k \chi(h(\sigma|k)) = \lim_k \chi((h(\sigma))|n_k) = \chi(h(\sigma)) = \psi(\sigma).$$

Step 2: *$A$ satisfies* $(\mathcal{A}_\infty)$.

We define $\varphi : \mathbb{N}^{\mathbb{N}} \cup \mathbb{N}^{<\mathbb{N}} \to X^*$ by $\varphi(\sigma) = (1 - \alpha_{|\sigma|})\psi(\sigma)$ if $\sigma \in \mathbb{N}^{<\mathbb{N}}$, and $\varphi(\sigma) = \psi(\sigma)$ if $\sigma \in \mathbb{N}^{\mathbb{N}}$. We can easily verify that $(\mathcal{A}_\infty)$ holds with $\delta_k = 2\alpha_k$. Then Proposition 4.2 completes the proof. ∎

We now introduce a sufficient condition in finite dimensions which is not far from condition $(\mathcal{C}_\infty)$.

Definition 4.5. Let $F$ be a subset of $\mathbb{R}^n$. We say that $F$ satisfies *condition* $(\mathcal{C})$ if $F$ is closed, there are a summable sequence $(\delta_k)_{k\geq 2}$, a sequence $(q_k)_{k\geq 1}$ of positive integers with $q_1 = 1$ and a mapping $\varphi : D \cup \bigcup_{k\geq 1} D_k \to F$ (where $D = \prod_{j\geq 1} \{1, \ldots, q_j\}$ and $D_k = \prod_{1\leq j\leq k} \{1, \ldots, q_j\}$) such that

$$\begin{cases} \varphi(D \cup \bigcup_{k\geq 1} D_k) = F,\ \varphi(1) = 0 \text{ and, for all } k \geq 2, \\ \sigma \in D_k \Rightarrow [\varphi(\sigma_{\text{-}}), \varphi(\sigma)] \subset \operatorname{int} F \text{ and } \|\varphi(\sigma) - \varphi(\sigma_{\text{-}})\| < \delta_k. \\ \sigma \in D \Rightarrow \varphi(\sigma) = \lim_k \varphi(\sigma|k). \end{cases}$$

Again, we can rewrite this condition in terms of sequences: $F$ satisfies condition $(\mathcal{C})$ if and only if $F$ is closed, there are a sequence $(y_k^*)_{k\geq 1}$ of points in $\operatorname{int} F$ with $y_1^* = 0$, a nondecreasing sequence $(I_k)_{k\geq 1}$ of finite subsets of $\mathbb{N}$ with $I_1 = \{1\}$ and a summable sequence $(\delta_k)_{k\geq 1}$ of positive numbers such that for all $y^*$ in $F$, there is a function $\psi : \mathbb{N} \to \mathbb{N}$ so that $\lim_k y^*_{\psi(k)} = y^*$ and for all $k \geq 1$, $\psi(k) \in I_k$, $[y^*_{\psi(k)}, y^*_{\psi(k+1)}] \subset \operatorname{int} F$ and $\|y^*_{\psi(k+1)} - y^*_{\psi(k)}\| < \delta_k$.

Using the same ideas as in the proof of Proposition 4.2, we get

Theorem 4.6. *Let $n \geq 1$ and $F$ be a subset of $\mathbb{R}^n$. If $F$ satisfies condition $(\mathcal{C})$, then there is a $C^1$-smooth bump $f : \mathbb{R}^n \to \mathbb{R}$ such that $f'(\mathbb{R}^n) = F$.*

Let us now recall the condition introduced in [3]:

A subset $F$ of $X^*$ is said to satisfy *condition* $(*)$ if there are a summable sequence $(\delta_k)_{k\geq 1}$ of positive numbers and a sequence $(C_k)_{k\geq 1}$ of bounded closed subsets of $X^*$ such that $F = \overline{\bigcup_{k\geq 1} C_k}$, $C_1$ is convex and contains 0, for all $k \geq 1$, $C_k \subset \operatorname{int} C_{k+1}$ and for all $y$ in $C_{k+1} \setminus \operatorname{int} C_k$, there is $z$ in $C_k$ such that $[z, y] \subset C_{k+1}$ and $\|y - z\| < \delta_k$.

The authors of [3] prove that any subset of $\mathbb{R}^n$ satisfying $(*)$ is the range of the derivative of a $C^1$-smooth bump. We are going to show that condition $(*)$ is equivalent to condition $(\mathcal{C})$. Consequently, Theorem 4.6 is nothing but Theorem 12 of [3]. Later we will explain the advantages of condition $(\mathcal{C})$.

Proposition 4.7. *If $X = \mathbb{R}^n$, then condition $(*)$ is equivalent to $(\mathcal{C})$.*

*Proof.* Let $F$ be a subset of $\mathbb{R}^n$.

STEP 1: *Condition* $(*) \Rightarrow$ *Condition* $(\mathcal{C})$.

We suppose that $F$ satisfies $(*)$. We put $S_1 = \{0\}$. For $k \geq 1$ we define $\varepsilon_k = 2^{-1}\min(\delta_k, \operatorname{dist}(C_{k+1}, \partial F))$,

$$S_{k+1} = S_k \cup \{\text{a finite } \varepsilon_k\text{-net of } C_k\},$$

$q_k = \operatorname{Card} S_k$, $D = \prod_{j\geq 1}\{1,\ldots,q_j\}$ and $D_k = \prod_{1\leq j\leq k}\{1,\ldots,q_j\}$. We define $\varphi : \bigcup_{k\geq 1} D_k \to \mathbb{R}^n$ by induction. First we set $\delta_0 = \operatorname{diam}(C_1)$ and $\varphi(1) = 0$. We fix $k \geq 1$ and assume that $\varphi$ is defined on $D_k$, $\varphi(D_k) = S_k$ and for all $\sigma \in D_k$, $[\varphi(\sigma_-), \varphi(\sigma)] \subset \operatorname{int} F$ and $\|\varphi(\sigma) - \varphi(\sigma_-)\| < 2\delta_{k-1}$. We remark that if $y \in C_k$, then there is $z \in S_k$ such that $\|y - z\| < 2\delta_{k-1}$ and $[z, y] \subset \operatorname{int} F$. If $\sigma \in D_k$ we set $T_\sigma = \{y \in S_{k+1} : \|y - \varphi(\sigma)\| < 2\delta_{k-1}$ and $[\varphi(\sigma), y] \subset \operatorname{int} F\}$. We can write $T_\sigma = \{z_1, \ldots, z_r\}$ with $r \leq q_{k+1}$. We define $\varphi(\sigma\hat{\ }j) = z_j$ if $1 \leq j \leq r$ and $\varphi(\sigma\hat{\ }j) = \varphi(\sigma)$ if $r < j \leq q_{k+1}$. Then $\varphi$ is defined on $D_{k+1}$ and has all the required properties. The fact that $\varphi(D_{k+1}) = S_{k+1}$ follows from the remark. In this way we define $\varphi$ on $\bigcup_{k\geq 1} D_k$. If $\sigma \in D$, then the sequence $(\varphi(\sigma|k))_k$ is convergent and we define $\varphi(\sigma) = \lim_k \varphi(\sigma|k)$.

Let $y \in F$. There is a sequence $(z_k)_{k\geq 1}$ such that $\lim_k z_k = y$ and $z_k \in C_k$ for all $k \geq 1$. For all $k \geq 1$, there is $\sigma_k \in D_{k+1}$ with $\|z_k - \varphi(\sigma_k)\| < \delta_k$ and $[\varphi(\sigma_k), z_k] \subset \operatorname{int} F$. The sequence $(\sigma_k(1))_k$ takes a finite number of values in $\{1, \ldots, q_1\}$. Thus there is $r_1$ in $\{1, \ldots, q_1\}$ so that $\{k : \sigma_k(1) = r_1\}$ is infinite. By induction we build a sequence $(r_j)_{j\geq 1}$ in $D$ such that for all $j$, $\{k : \sigma_k(i) = r_i \text{ for } 1 \leq i \leq j\}$ is infinite. Then $\tau = (r_1, \ldots, r_j, \ldots)$ is in $D$ and $\varphi(\tau) = y$. Therefore $(*)$ implies $(\mathcal{C})$.

STEP 2: *Condition* $(\mathcal{C}) \Rightarrow$ *Condition* $(*)$.

We assume that $F$ satisfies $(\mathcal{C})$. There is $\varepsilon_1 > 0$ such that $B(0, \varepsilon_1) \subset \operatorname{int} F$. We define $C_1 = B(0, \varepsilon_1)$. For $k \geq 1$, if $\sigma \in D_k$, then there is $0 < \varepsilon_\sigma < \delta_k$ with $[\varphi(\sigma_-), \varphi(\sigma)] + B(0, \varepsilon_\sigma) \subset \operatorname{int} F$. The set

$$B_k = C_k \cup \Big(\bigcup_{\sigma\in D_k} [\varphi(\sigma_-), \varphi(\sigma)] + B(0, \varepsilon_\sigma)\Big)$$

is compact and is in $\operatorname{int} F$. So $\alpha_k = \frac{1}{2}\min(\delta_k, \operatorname{dist}(B_k, \partial F)) > 0$. Finally, we define $C_{k+1} = B_k + B(0, \alpha_k)$. The sequence $(C_k)_{k\geq 1}$ satisfies all the required conditions, thus $F$ satisfies condition $(*)$. ∎

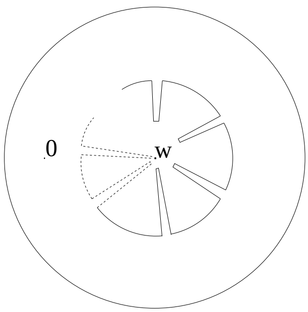
We have proved that condition $(\mathcal{C})$ can be extended to infinite dimensions. Indeed, Proposition 4.2 shows that $(\mathcal{A}_\infty)$ is a sufficient condition in smooth infinite-dimensional Banach spaces and $(\mathcal{A}_\infty)$ can be considered as an extension of $(\mathcal{C})$. The situation is different with condition $(*)$. In fact, if $X$ is an infinite-dimensional Banach space, we can construct a subset $R$ of $X^*$

which satisfies condition $(*)$ but which is not the range of the derivative of a $C^1$-smooth bump. Let us describe $R$. Since $X$ is infinite-dimensional, there is $\varepsilon > 0$ and a $3\varepsilon$-separated sequence $(e_k)_{k\geq 1}$ in $S_{X^*}$. We fix a point $w$ in $X^*$ with $\|w\| = 3/2$. We define

$$D_k = \{tx : x \in S_{X^*} \cap B(e_k, \varepsilon),\ 1/k \leq t \leq 1\}, \qquad D = \bigcup_{k\geq 1} D_k,$$

$$R = \overline{w + (\{x \in X^* : 1 \leq \|x\| \leq 2\} \cup D)}$$
$$= (w + (\{x \in X^* : 1 \leq \|x\| \leq 2\} \cup D)) \cup \{w\}.$$

We remark that the construction of $R$ is only possible in an infinite-dimensional Banach space. Here is a 2-dimensional representation of $R$:




The wheel with broken spokes

Because of its form, $R$ is called the "wheel with broken spokes". In fact, in infinite dimensions, we can imagine that each spoke is in a new direction and comes closer to $w$, the centre of the wheel. Then $R$ satisfies condition $(*)$ but $R$ is not the range of the derivative of a $C^1$-smooth bump, because $w$ cannot be joined to 0 by a continuous path in $R$. Thus condition $(*)$ is not sufficient in infinite dimensions.

**Acknowledgments.** I wish to thank Robert Deville for his valuable help and the referee for his valuable suggestions and comments.

### References

[1] D. Azagra and R. Deville, *James' theorem fails for starlike bodies*, J. Funct. Anal. 180 (2001), 328–346.

[2] D. Azagra and M. Jiménez-Sevilla, *The failure of Rolle's theorem in infinite-dimensional Banach spaces*, ibid. 182 (2001), 207–226.

[3] J. M. Borwein, M. Fabian, I. Kortezov and P. D. Loewen, *The range of the gradient of a continuously differentiable bump*, J. Nonlinear Convex Anal. 2 (2001), 1–19.

[4] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monogr. Surveys Pure Appl. Math. 64, Longman, 1993.

[5] M. Jiménez-Sevilla and J. P. Moreno, *A note on norm attaining functionals*, Proc. Amer. Math. Soc. 126 (1998), 1989–1997.
[6] C. Kuratowski, *Topologie II*, PWN, Warszawa, 1961.
[7] J. Malý, *The Darboux property for gradients*, Real Anal. Exchange 22 (1996/1997), 167–173.
[8] G. F. Simmons, *Introduction to Topology and Modern Analysis*, McGraw-Hill, 1963.

Mathématiques Pures de Bordeaux (MPB), UMR 5467 CNRS
Université Bordeaux 1
351, cours de la Libération
33400 Talence, France
E-mail: gaspari@math.u-bordeaux.fr

*Received July 5, 2001*
*Revised version April 15, 2002* (4771)