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The harmonic Cesàro and Copson operators on the spaces $L^p(\mathbb{R})$, $1 \le p \le 2$

by

Ferenc Móricz (Szeged)

Dedicated to Professor Paul R. Halmos on his 85th birthday

Abstract. The harmonic Cesàro operator $\mathcal{C}$ is defined for a function f in $L^p(\mathbb{R})$ for some $1 \le p < \infty$ by setting $\mathcal{C}(f)(x) := \int_x^\infty (f(u)/u)\,du$ for $x > 0$ and $\mathcal{C}(f)(x) := -\int_{-\infty}^x (f(u)/u)\,du$ for $x < 0$; the harmonic Copson operator $\mathcal{C}^*$ is defined for a function f in $L^1_{\mathrm{loc}}(\mathbb{R})$ by setting $\mathcal{C}^*(f)(x) := (1/x)\int_0^x f(u)\,du$ for $x \ne 0$. The notation indicates that $\mathcal{C}$ and $\mathcal{C}^*$ are adjoint operators in a certain sense.

We present rigorous proofs of the following two commuting relations:

(i) If $f \in L^p(\mathbb{R})$ for some $1 \le p \le 2$, then $(\mathcal{C}(f))^\wedge(t) = \mathcal{C}^*(\widehat{f})(t)$ a.e., where $\widehat{f}$ denotes the Fourier transform of f.

(ii) If $f \in L^p(\mathbb{R})$ for some $1 < p \le 2$, then $(\mathcal{C}^*(f))^\wedge(t) = \mathcal{C}(\widehat{f})(t)$ a.e.

As a by-product of our proofs, we obtain representations of $(\mathcal{C}(f))^\wedge(t)$ and $(\mathcal{C}^*(f))^\wedge(t)$ in terms of Lebesgue integrals in case f belongs to $L^p(\mathbb{R})$ for some $1 < p \le 2$. These representations are valid for almost every t and may be useful in other contexts.

1. Definitions. First, we recall that the *harmonic Cesàro operator* $\mathcal{C}$ is defined for a function f in $L^p(\mathbb{R})$ for some $1 \le p < \infty$ by setting

$$\mathcal{C}(f)(x) := \begin{cases} \int_x^\infty (f(u)/u)\,du & \text{for } x > 0,\\ -\int_{-\infty}^x (f(u)/u)\,du & \text{for } x < 0; \end{cases}$$

the *harmonic Copson operator* $\mathcal{C}^*$ is defined for a function f in $L^1_{\mathrm{loc}}(\mathbb{R})$ by setting

$$\mathcal{C}^*(f)(x) := \frac{1}{x}\int_0^x f(u)\,du \qquad \text{for } x \ne 0.$$

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The notation $\mathcal{C}^*$ (as the adjoint operator of $\mathcal{C}$) is justified by the fact that if $f \in L^p(\mathbb{R})$ for some $1 \le p < \infty$ and $g \in L^{p^*}(\mathbb{R})$, where $1/p + 1/p^* = 1$, then

$$\int_{\mathbb{R}} \mathcal{C}(f)(x) g(x)\, dx = \int_{\mathbb{R}} f(x) \mathcal{C}^*(g)(x)\, dx. \tag{1.1}$$

See, for example, Golubov [3, p. 329] for the case when f and g are defined on $\mathbb{R}_+$.

The integrals on both sides of (1.1) exist in the Lebesgue sense. Indeed, it follows from the well known inequalities of Hardy [4, Theorems 327 and 328] that if $f \in L^p(\mathbb{R})$ for some p, then $\mathcal{C}(f) \in L^p(\mathbb{R})$ in case $1 \le p < \infty$, and $\mathcal{C}^*(f) \in L^p(\mathbb{R})$ in case $1 < p \le \infty$. More exactly, we have

$$\|\mathcal{C}\|_p := \sup_{\|f\|_p \le 1} \|\mathcal{C}(f)\|_p = p \quad \text{and} \quad \|\mathcal{C}^*\|_p = p^*,$$

where

$$\|f\|_p := \Big\{ \int_{\mathbb{R}} |f(x)|^p\, dx \Big\}^{1/p} \qquad \text{for } 1 \le p < \infty,$$
$$\|f\|_\infty := \operatorname{ess\,sup}\{|f(x)| : x \in \mathbb{R}\}.$$

Second, we remind the reader that the Fourier transform of a function f in $L^1(\mathbb{R})$ is defined by

$$\widehat{f}(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-itx}\, dx, \qquad t \in \mathbb{R}. \tag{1.2}$$

It is well known that $\widehat{f}$ is continuous on $\mathbb{R}$ and, by the Riemann–Lebesgue lemma, $\widehat{f}(t)$ vanishes as $|t| \to \infty$. In case $f \in L^p(\mathbb{R})$ for some $1 < p \le 2$, the Fourier transform of f is defined in terms of a limit in the norm of $L^{p^*}(\mathbb{R})$:

$$\widehat{f} := L^{p^*}(\mathbb{R})\text{-}\lim_{a \to \infty} \widehat{f}_a, \quad \text{where} \quad f_a := f\chi_{(-a,a)},\ \widehat{f}_a := (f_a)^\wedge$$

and $\chi_{(-a,a)}$ denotes the characteristic function of the interval $(-a, a)$. (See e.g. [6, Vol. 2, p. 254].) That is, $\widehat{f} \in L^{p^*}(\mathbb{R})$ and

$$\lim_{a \to \infty} \|\widehat{f}_a - \widehat{f}\|_{p^*} = 0. \tag{1.3}$$

We note that if $f \in L^p(\mathbb{R})$ for some $1 < p \le 2$, then the existence of $\widehat{f}(t)$ is guaranteed only at almost every t. In particular, this time $\widehat{f}$ is no longer continuous on $\mathbb{R}$ or vanishes at infinity (unlike the case when $f \in L^1(\mathbb{R})$).

In case $f \in L^p(\mathbb{R})$ for some $2 < p \le \infty$, the Fourier transform of f cannot be defined as an ordinary function in any reasonable way either by making a passage to the limit in the norm of $L^{p^*}(\mathbb{R})$, or by using any linear method of summation. (See e.g. [6, Vol. 2, p. 258].) However, this time $\widehat{f}$ can be defined as a tempered distribution. (See e.g. [5, pp. 19–30].) But we are not concerned with distributions in this paper.

2. Interrelations with Fourier transform. We prove the following two commuting relations.

THEOREM 1. *If $f \in L^p(\mathbb{R})$ for some $1 \le p \le 2$, then*

$$(\mathcal{C}(f))^\wedge(t) = \mathcal{C}^*(\widehat{f})(t) \quad a.e. \tag{2.1}$$

THEOREM 2. *If $f \in L^p(\mathbb{R})$ for some $1 < p \le 2$, then*

$$(\mathcal{C}^*(f))^\wedge(t) = \mathcal{C}(\widehat{f})(t) \quad a.e. \tag{2.2}$$

Theorem 1 justifies the term "harmonic Cesàro operator" since (2.1) can be rewritten in the form

$$(\mathcal{C}(f))^\wedge(t) = \frac{1}{t}\int_0^t \widehat{f}(u)\,du \quad \text{a.e.} \tag{2.3}$$

By the uniqueness theorem for Fourier transforms, we could have defined $\mathcal{C}(f)$ by (2.3), at least for functions f belonging to $L^p(\mathbb{R})$ for some $1 \le p \le 2$. Analogously, for functions f belonging to $L^p(\mathbb{R})$ for some $1 < p \le 2$, we could have defined the harmonic Copson operator $\mathcal{C}^*(f)$ as follows (due to (2.2)):

$$(\mathcal{C}^*(f))^\wedge(t) = \begin{cases} \int_t^\infty (\widehat{f}(u)/u)\,du & \text{for } t > 0, \\ -\int_{-\infty}^t (\widehat{f}(u)/u)\,du & \text{for } t < 0. \end{cases}$$

Theorems 1 and 2 were formulated by Bellman [1] with heuristic motivations. Later Golubov [3, Theorems 3 and 4] presented proofs for them in the case of cosine Fourier transform, without recognizing the forms of the Cesàro and Copson operators. Equality (2.1) for $p = 1$ was independently proved in [2] by Giang and the present author.

Unfortunately, the proofs of [3, Theorems 3 and 4] are not complete. To be specific, there is a deficiency in the proof of [3, Theorem 3] in case $f \in L^p(\mathbb{R})$ for some $1 < p \le 2$. The reason is that this time

$$\int_x^\infty \frac{f(u)}{u}\,du \notin L^1(\mathbb{R}_+) \qquad (\in L^p(\mathbb{R}_+)). \tag{2.4}$$

Therefore, the right-hand side in [3, formula (26)] can be integrated by parts only over a finite interval, say $[x_1, x_2]$ with $0 < x_1 < x_2 < \infty$. As a result, we have

$$\begin{aligned} &\int_{x_1}^{x_2} f(x)\frac{\sin tx}{x}\,dx \\ &\qquad = \Big[-\sin tx \int_x^\infty \frac{f(u)}{u}\,du\Big]_{x=x_1}^{x_2} + t\int_{x_1}^{x_2} \cos tx\,dx \int_x^\infty \frac{f(u)}{u}\,du. \end{aligned} \tag{2.5}$$

In order to obtain [3, formula (27)], we have to let $x_1 \to +0$ and $x_2 \to \infty$ in (2.5). The integrated-out terms converge to 0 at this passage. The problem is that we cannot apply Lebesgue's dominated convergence theorem as $x_1 \to 0$ and $x_2 \to \infty$: for the integral on the left-hand side of (2.5) because $f(x)/x \notin L^1(\mathbb{R}_+)$, and for the integral on the right-hand side because of (2.4).

A similar problem arises in the proof of [3, Theorem 4] when the integral on the right-hand side of the formula preceding [3, formula (34)] is integrated by parts. Namely, due to the fact that $f \in L^p(\mathbb{R})$ for some $1 < p \le 2$, the integral

$$\int_0^\infty \left\{ \frac{1}{x} \int_0^x f(u)\, du \right\} \cos tx\, dx$$

on the right-hand side of [3, formula (34)] does not exist in the Lebesgue sense. Thus, the integration by parts which would yield [3, formula (34)] is not allowed.

Analysing our proof of (2.1), we see that if $f \in L^p(\mathbb{R})$ for some $1 \le p \le 2$, then

$$(\mathcal{C}(f))^\wedge(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) \frac{1 - e^{-itu}}{itu}\, du \qquad \text{a.e.}$$

(see (3.10) below). This can be rewritten in the following form:

$$\begin{aligned} \sqrt{2\pi}\, (\mathcal{C}(f))^\wedge(t) &= \int_{\mathbb{R}} f(u) \left\{ \frac{1}{u} \int_0^u e^{-itx}\, dx \right\} du \\ &= \int_{\mathbb{R}} f(u) \mathcal{C}^*(e^{-it\cdot})(u)\, du \qquad \text{a.e.} \end{aligned} \tag{2.6}$$

Observe that in this way we have actually extended the validity of (1.1) to the case when $f \in L^p(\mathbb{R})$ for some $1 \le p \le 2$ and $g(x) := e^{-itx}$. The left-hand side in (2.6) could be "formally" interpreted as the integral

$$\int_{\mathbb{R}} \mathcal{C}(f)(x) e^{-itx}\, dx.$$

However, this integral exists in the Lebesgue sense only if $p = 1$; while if $1 < p \le 2$, it exists only as the limit of $((\mathcal{C}(f))_a)^\wedge$ as $a \to \infty$ in the norm of $L^{p^*}(\mathbb{R})$:

$$(\mathcal{C}(f))^\wedge(t) := L^{p^*}(\mathbb{R})\text{-}\lim_{a \to \infty} \frac{1}{\sqrt{2\pi}} \int_{|x|<a} \mathcal{C}(f)(x) e^{-itx}\, dx.$$

Analysing our proof of (2.2), we find that if $f \in L^p(\mathbb{R})$ for some $1 < p \le 2$, then

$$\begin{aligned} &\sqrt{2\pi}\, (\mathcal{C}^*(f))^\wedge(t) \\ &\qquad = \int_0^\infty f(x)\, dx \int_x^{\to\infty} \frac{e^{-itu}}{u}\, du - \int_{-\infty}^0 f(x)\, dx \int_{\to -\infty}^x \frac{e^{-itu}}{u}\, du \qquad \text{a.e.} \end{aligned}$$

(see (4.16)). This can be rewritten in the following form:

$$\begin{aligned}\sqrt{2\pi}\,(\mathcal{C}^*(f))^\wedge(t) &= \int_0^\infty f(x)\mathcal{C}(e^{-it\cdot})(x)\,dx + \int_{-\infty}^0 f(x)\mathcal{C}(e^{-it\cdot})(x)\,dx \\ &= \int_{\mathbb{R}} f(x)\mathcal{C}(e^{-it\cdot})(x)\,dx \qquad \text{a.e.,}\end{aligned} \tag{2.7}$$

where $\mathcal{C}(e^{-it\cdot})(x)$ should be defined as an improper integral:

$$\mathcal{C}(e^{-it\cdot})(x) := \begin{cases} \lim\limits_{b\to\infty} \int_x^b \dfrac{e^{-itu}}{u}\,du & \text{for } x > 0, \\ \lim\limits_{b\to-\infty} \int_b^x \dfrac{e^{-itu}}{u}\,du & \text{for } x < 0. \end{cases}$$

The left-hand side in (2.7) could be "formally" interpreted as the integral

$$\int_{\mathbb{R}} \mathcal{C}^*(f)(x)e^{-itx}\,dx.$$

But this integral does not exist generally in the Lebesgue sense; it exists only as the limit of $((\mathcal{C}^*(f))_a)^\wedge$ as $a \to \infty$ in the norm of $L^{p^*}(\mathbb{R})$:

$$(\mathcal{C}^*(f))^\wedge(t) := L^{p^*}(\mathbb{R})\text{-}\lim_{a\to\infty} \int_{|x|<a} \mathcal{C}^*(f)(x)e^{-itx}\,dx.$$

3. Proof of Theorem 1

Case $p = 1$. By definition,

$$\sqrt{2\pi}\,(\mathcal{C}(f))^\wedge(t) := \int_0^\infty e^{-itx}\,dx \int_x^\infty \frac{f(u)}{u}\,du - \int_{-\infty}^0 e^{-itx}\,dx \int_{-\infty}^x \frac{f(u)}{u}\,du,$$

whence by Fubini's theorem,

$$\begin{aligned}\sqrt{2\pi}\,(\mathcal{C}(f))^\wedge(t) &= \int_{\mathbb{R}} \frac{f(u)}{u}\,du \int_0^u e^{-itx}\,dx \\ &= \int_{\mathbb{R}} f(u)\frac{1-e^{-itu}}{itu}\,du, \qquad t \neq 0.\end{aligned} \tag{3.1}$$

The last integral in (3.1) exists in the Lebesgue sense, since

$$\left|\frac{1-e^{-itu}}{itu}\right| = \left|\frac{2\sin(tu/2)}{tu}\right| \qquad \text{for all } t \neq 0 \text{ and } u \neq 0. \tag{3.2}$$

On the other hand,

$$t\sqrt{2\pi}\,\mathcal{C}^*(\widehat{f})(t) := \sqrt{2\pi}\int_0^t \widehat{f}(u)\,du = \int_0^t du \int_{\mathbb{R}} f(x)e^{-iux}\,dx,$$

whence again by Fubini's theorem,

$$\sqrt{2\pi}\,\mathcal{C}^*(\widehat{f})(t) = \frac{1}{t}\int_{\mathbb{R}} f(x)\,dx \int_0^t e^{-iux}\,du = \int_{\mathbb{R}} f(x)\frac{1-e^{-itx}}{itx}\,dx. \tag{3.3}$$

Clearly, the rightmost integrals in (3.1) and (3.3) are identical. Thus, we have proved (2.1) for all $t \neq 0$ provided $f \in L^1(\mathbb{R})$.

We note that for $t = 0$ we clearly have

$$\sqrt{2\pi}\,(\mathcal{C}(f))^\wedge(0) = \int_{\mathbb{R}} f(u)\,du = \sqrt{2\pi}\,\widehat{f}(0),$$

whence

$$(\mathcal{C}(f))^\wedge(0) = \widehat{f}(0).$$

Although $\mathcal{C}^*(f)(x)$ has not been defined for $x = 0$ in general, in the particular case when f is continuous at $x = 0$, it is reasonable to set

$$\mathcal{C}^*(f)(0) := \lim_{x\to 0} \mathcal{C}^*(f)(x) = f(0).$$

Since $\widehat{f}$ is continuous if $f \in L^1(\mathbb{R})$, this supplementary definition applies to $\widehat{f}$ in place of f. So, with this agreement, we have (2.1) for all $t \in \mathbb{R}$ provided $f \in L^1(\mathbb{R})$.

Case $1 < p \leq 2$. Given $a > 0$, let $f_a := f\chi_{(-a,a)}$ and $\widehat{f}_a := (f_a)^\wedge$, where $a > 0$. Since $f_a \in L^1(\mathbb{R})$, we may apply (2.1) to obtain

$$(\mathcal{C}(f_a))^\wedge(t) = \mathcal{C}^*(\widehat{f}_a)(t) = \frac{1}{\sqrt{2\pi}}\int_{|u|<a} \frac{f(u)}{u}\,du \int_0^u e^{-itx}\,dx, \qquad t \in \mathbb{R}. \tag{3.4}$$

First, we claim that

$$\lim_{a\to\infty} \mathcal{C}^*(\widehat{f}_a)(t) = \mathcal{C}^*(\widehat{f})(t) \qquad \text{for all } t \neq 0, \tag{3.5}$$

even uniformly in t provided $|t| \geq t_0$ for some $t_0 > 0$. Indeed, Hölder's inequality and (1.3) yield

$$\begin{aligned} |\mathcal{C}^*(\widehat{f}_a)(t) - \mathcal{C}^*(\widehat{f})(t)| &:= \left|\frac{1}{t}\int_0^t \{\widehat{f}_a(u) - \widehat{f}(u)\}\,du\right| \\ &\leq \frac{1}{|t|}\|\widehat{f}_a - \widehat{f}\|_{p^*}|t|^{1/p} \to 0 \qquad \text{as } a \to \infty. \end{aligned}$$

Second, we claim that

$$L^{p^*}(\mathbb{R})\text{-}\lim_{a\to\infty} (\mathcal{C}(f_a))^\wedge = (\mathcal{C}(f))^\wedge. \tag{3.6}$$

To see this, we start with the definition:

$$\begin{aligned}(\mathcal{C}(f))^\wedge &:= L^{p^*}(\mathbb{R})\text{-}\lim_{a\to\infty}((\mathcal{C}(f))_a)^\wedge \\ &= L^{p^*}(\mathbb{R})\text{-}\lim_{a\to\infty}\frac{1}{\sqrt{2\pi}}\int_{|x|<a}\mathcal{C}(f)(x)e^{-i\cdot x}\,dx.\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}\int_{|x|<a}\mathcal{C}(f)(x)e^{-itx}\,dx &= \int_0^a e^{-itx}dx\int_x^\infty \frac{f(u)}{u}\,du - \int_{-a}^0 e^{-itx}\,dx\int_{-\infty}^x \frac{f(u)}{u}\,du \\ &= \int_0^a \frac{f(u)}{u}\,du\int_0^u e^{-itx}\,dx + \int_a^\infty \frac{f(u)}{u}\,du\int_0^a e^{-itx}\,dx \\ &\quad - \int_{-a}^0 \frac{f(u)}{u}\,du\int_u^0 e^{-itx}\,dx - \int_{-\infty}^{-a} \frac{f(u)}{u}\,du\int_{-a}^0 e^{-itx}\,dx \\ &= \int_{|u|<a}\frac{f(u)}{u}\,du\int_0^u e^{-itx}\,dx \\ &\quad + \int_a^\infty \frac{f(u)}{u}\cdot\frac{1-e^{-iat}}{it}\,du + \int_{-\infty}^{-a}\frac{f(u)}{u}\cdot\frac{1-e^{iat}}{it}\,du,\end{aligned}$$

whence, by (3.4), we conclude that

$$\begin{aligned}(3.7)\qquad (\mathcal{C}(f))^\wedge(t) &= L^{p^*}(\mathbb{R})\text{-}\lim_{a\to\infty}\bigg\{(\mathcal{C}(f_a))^\wedge(t) \\ &\quad + \frac{1}{2\pi}\bigg[\frac{1-e^{-iat}}{it}\int_a^\infty \frac{f(u)}{u}\,du + \frac{1-e^{iat}}{it}\int_{-\infty}^{-a}\frac{f(u)}{u}\,du\bigg]\bigg\}.\end{aligned}$$

Third, we claim that

$$(3.8)\qquad L^{p^*}(\mathbb{R})\text{-}\lim_{a\to\infty}\bigg[\frac{1-e^{-iat}}{it}\int_a^\infty \frac{f(u)}{u}\,du + \frac{1-e^{iat}}{it}\int_{-\infty}^{-a}\frac{f(u)}{u}\,du\bigg] = 0.$$

In fact, by Hölder's inequality we have

$$\begin{aligned}\int_{|u|>a}\left|\frac{f(u)}{u}\right|du &\le \Big\{\int_{|u|>a}|f(u)|^p\,du\Big\}^{1/p}\Big\{\int_{|u|>a}u^{-p^*}\,du\Big\}^{1/p^*} \\ &= o(1)O(a^{-p^*+1})^{1/p^*} = o(a^{-1/p})\quad \text{as } a\to\infty.\end{aligned}$$

On the other hand, by (3.2) and Minkowski's inequality, we find

$$\begin{aligned}
&\left\|\frac{1-e^{-iat}}{it}\int_a^\infty \frac{f(u)}{u}\,du\right\|_{L^{p^*}(dt)}\\
&\qquad \le \int_a^\infty \left|\frac{f(u)}{u}\right| du \left[\Big\{\int_{|t|\le 1/a} a^{p^*}\,dt\Big\}^{1/p^*} + \Big\{\int_{|t|\ge 1/a} \left|\frac{2}{t}\right|^{p^*} dt\Big\}^{1/p^*}\right]\\
&\qquad = o(a^{-1/p})O(a^{1/p}) = o(1) \quad \text{as } a\to\infty.
\end{aligned}$$

The other term in (3.8) can be estimated analogously.

To sum up, (3.7), (3.8) and (3.4) yield

$$(3.9)\qquad (C(f))^\wedge = L^{p^*}(\mathbb{R})\text{-}\lim_{a\to\infty}(\mathcal{C}(f_a))^\wedge = L^{p^*}(\mathbb{R})\text{-}\lim_{a\to\infty}\mathcal{C}^*(\widehat{f}_a).$$

Comparing (3.5) and (3.9) yields (2.1) to be proved. ■

As a by-product of (3.1) and (3.9), we have the following representation: If $f\in L^p(\mathbb{R})$ for some $1\le p\le 2$, then

$$\begin{aligned}
(3.10)\qquad (\mathcal{C}(f))^\wedge(t) &= \lim_{a\to\infty}(\mathcal{C}(f_a))^\wedge(t) = \lim_{a\to\infty}\frac{1}{\sqrt{2\pi}}\int_{|u|<a} f(u)\frac{1-e^{-itu}}{itu}\,du\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(u)\frac{1-e^{-itu}}{itu}\,du \quad \text{a.e.}
\end{aligned}$$

The pointwise limit exists for all $t\ne 0$, since $f\in L^p(\mathbb{R})$ and $(1-e^{-itu})/(itu)$ $\in L^q(\mathbb{R})$ for all $1<q\le\infty$ (cf. (3.2)).

One more remark is appropriate here. Observe that the right-hand side in (2.1) is continuous except possibly at $t=0$ and vanishes at infinity. Consequently, we may change the values of $(\mathcal{C}(f))^\wedge(t)$ (originally defined as a limit in the norm of $L^{p^*}(\mathbb{R})$) on a set of measure zero so that $(\mathcal{C}(f))^\wedge(t)$ becomes continuous except possibly at $t=0$ and vanishes at infinity.

4. Proof of Theorem 2. Consider again the truncated function $f_a := f\chi_{(-a,a)}$, where $a>0$. Then $f_a\in L^1(\mathbb{R})$ and (1.2) applies. For the sake of definiteness, let $0<t<\infty$ be fixed, and let $b>t$. By Fubini's theorem, we may write

$$\sqrt{2\pi}\int_t^b \frac{\widehat{f}_a(u)}{u}\,du = \int_t^b \frac{du}{u}\int_{|x|<a} f(x)e^{-iux}\,dx = \int_{|x|<a} f(x)\,dx\int_t^b \frac{e^{-iux}}{u}\,du.$$

First, letting a tend to ∞ gives

$$(4.1)\qquad \sqrt{2\pi}\int_t^b \frac{\widehat{f}(u)}{u}\,du = \int_{\mathbb{R}} f(x)\,dx\int_t^b \frac{e^{-iux}}{u}\,du.$$

In fact, by (1.3) we have

$$\left|\int_t^b \frac{\widehat{f}_a(u)}{u}\,du - \int_t^b \frac{f_a(u)}{u}\,du\right| \le \|\widehat{f}_a - \widehat{f}\|_{p^*}\left\{\int_t^b \frac{du}{u^p}\right\}^{1/p} \to 0 \qquad \text{as } a\to\infty.$$

On the other hand, introducing the auxiliary function

$$h_{t,b}(x) := \int_t^b \frac{e^{-iux}}{u}\,du, \tag{4.2}$$

from the second mean value theorem for integrals it follows that

$$|h_{t,b}(x)| \le \frac{4}{t|x|} \qquad \text{for all } t>0 \text{ and } x\neq 0. \tag{4.3}$$

Making use of (4.3) and Hölder's inequality, we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}} f(x)h_{t,b}(x)\,dx - \int_{|x|<a} f(x)h_{t,b}(x)\,dx\right| \\ &\qquad \le \frac{4}{t}\int_{|x|>a}\left|\frac{f(x)}{x}\right|dx \le \frac{4}{t}\Big\{\int_{|x|>a}|f(x)|^p\,dx\Big\}^{1/p}\Big\{\int_{|x|>a}\frac{dx}{x^{p^*}}\Big\}^{1/p^*} \to 0\end{aligned}$$

as $a\to\infty$. This completes the proof of (4.1).

In what follows, we need another estimate: if $t|x| \le 1/e$ and $b|x|\ge 1$, then by (4.3) we have

$$|h_{t,b}(x)| \le \int_t^{1/|x|}\frac{1}{u}du + |h_{1/|x|,b}(x)| \le \ln\frac{1}{t|x|} + 4 \le 5\ln\frac{1}{t|x|}. \tag{4.4}$$

Set

$$h_t(x) := \lim_{b\to\infty} h_{t,b}(x) = \int_t^{\to\infty}\frac{e^{-iux}}{u}\,du. \tag{4.5}$$

Clearly, $h_t(x)$ exists as an improper integral. In fact, if $t<b<b_1<\infty$, then by (4.2) and (4.3) we have

$$|h_{t,b}(x) - h_{t,b_1}(x)| = \left|\int_b^{b_1}\frac{e^{-iux}}{u}\,du\right| \le \frac{5}{b|x|} \to 0 \qquad \text{as } b\to\infty.$$

Furthermore, inequalities (4.3) and (4.4) (the latter for $t|x|\le 1/e$) remain valid for h_t in place of $h_{t,b}$.

Second, we claim that letting b tend to ∞ in (4.1) gives

$$\sqrt{2\pi}\,\mathcal{C}(\widehat{f})(t) := \sqrt{2\pi}\int_t^\infty \frac{\widehat{f}(u)}{u}\,du = \int_{\mathbb{R}} f(x)h_t(x)\,dx, \tag{4.6}$$

where $h_t(x)$ is defined in (4.5). To see the validity of (4.6), first we notice that

$$\lim_{b\to\infty}\int_t^b \frac{\widehat{f}(u)}{u}\,du = \int_t^\infty \frac{\widehat{f}(u)}{u}\,du,$$

since $\widehat{f}\in L^{p^*}(\mathbb{R})$ and $1/u\in L^p(t,\infty)$; consequently, $\widehat{f}(u)/u\in L^1(t,\infty)$ for all $t>0$.

On the other hand, by (4.2)–(4.4) and Hölder's inequality we obtain

$$\begin{aligned}(4.7)\quad &\Big|\int_{\mathbb{R}} f(x)h_t(x)\,dx - \int_{\mathbb{R}} f(x)h_{t,b}(x)\,dx\Big| = \Big|\int_{\mathbb{R}} f(x)h_b(x)dx\Big| \\ &\le 5\int_{|x|<1/(be)} |f(x)|\ln\frac{1}{b|x|}\,dx + \frac{4}{b}\int_{|x|>1/(be)}\Big|\frac{f(x)}{x}\Big|\,dx \\ &\le 5\Big\{\int_{|x|<1/(be)} |f(x)|^p dx\Big\}^{1/p}\Big\{\int_{|x|<1/(be)}\Big(\ln\frac{1}{b|x|}\Big)^{p^*}dx\Big\}^{1/p^*} \\ &\quad + \frac{4}{b}\Big\{\int_{|x|>1/(be)} |f(x)|^p dx\Big\}^{1/p}\Big\{\int_{|x|>1/(be)}\frac{dx}{|x|^{p^*}}\Big\}^{1/p^*} \to 0\end{aligned}$$

as $b\to\infty$, since

$$\begin{aligned}\Big\{\int_0^{1/(be)}\Big(\ln\frac{1}{bx}\Big)^{p^*}dx\Big\}^{1/p^*} &\le \Big\{\int_0^{1/(be)}\Big|\ln\frac{1}{b}\Big|^{p^*}dx\Big\}^{1/p^*} + \int_0^{1/be}\Big(\ln\frac{1}{x}\Big)^{p^*}dx\Big\}^{1/p^*} \\ &\le \frac{\ln b}{(be)^{1/p^*}} + \int_{1+\ln b}^\infty t^{p^*}e^{-t}\,dt = O(1)\end{aligned}$$

as $b\to\infty$. This completes the justification of (4.6).

Third, returning to (4.5), we may write

$$h_t(x) := \int_t^{\to\infty}\frac{e^{-iux}}{u}\,du = \int_{tx}^{\to\infty}\frac{e^{-iv}}{v}\,dv = \int_x^{\to\infty}\frac{e^{-itu}}{u}\,du \quad \text{for } x>0$$

(the last integral equals $h_x(t)$), and

$$h_t(x) := -\int_{\to-\infty}^{tx}\frac{e^{-iv}}{v}\,dv = -\int_{\to-\infty}^{x}\frac{e^{-itu}}{u}\,du \quad \text{for } x<0.$$

Substituting these into (4.6), we obtain

$$(4.8)\quad \sqrt{2\pi}\,\mathcal{C}(\widehat{f})(t) = \int_0^\infty f(x)\,dx\int_x^{\to\infty}\frac{e^{-itu}}{u}\,du - \int_{-\infty}^0 f(x)\,dx\int_{\to-\infty}^{x}\frac{e^{-itu}}{u}\,du.$$

Fourth, we claim that

$$\sqrt{2\pi}\,\mathcal{C}(\widehat{f})(t) = \lim_{a\to\infty}\left\{\int_0^a f(x)\,dx\int_x^a \frac{e^{-itu}}{u}\,du - \int_{-a}^0 f(x)\,dx\int_{-a}^x \frac{e^{-itu}}{u}\,du\right\}. \tag{4.9}$$

By (4.8), it is enough to check that

$$\lim_{a\to\infty}\int_0^a f(x)\,dx\int_x^a \frac{e^{-itu}}{u}\,du = \int_0^\infty f(x)\,dx\int_x^{\to\infty} \frac{e^{-itu}}{u}\,du, \tag{4.10}$$

$$\lim_{a\to\infty}\int_{-a}^0 f(x)\,dx\int_{-a}^x \frac{e^{-itu}}{u}\,du = \int_{-\infty}^0 f(x)\,dx\int_{\to-\infty}^x \frac{e^{-itu}}{u}\,du. \tag{4.11}$$

We shall present the proof of (4.10) in detail. It will be done in two steps. By (4.3), (4.5) and Hölder's inequality, we obtain

$$\begin{aligned}\left|\int_0^a f(x)\,dx\int_x^{\to\infty} \frac{e^{-itu}}{u}\,du - \int_0^a f(x)dx\int_x^a \frac{e^{-itu}}{u}\,du\right| \\ = \left|\int_0^a f(x)\,dx\int_a^{\to\infty} \frac{e^{-itu}}{u}\,du\right| \le |h_a(t)|\int_0^a |f(x)|\,dx \\ \le \frac{4}{at}\left\{\int_0^a |f(x)|^p dx\right\}^{1/p} a^{1/p^*} \to 0 \quad \text{as } a\to\infty.\end{aligned} \tag{4.12}$$

Again by (4.3), (4.5) and Hölder's inequality, we find that

$$\begin{aligned}\left|\int_0^a f(x)\,dx\int_x^{\to\infty} \frac{e^{-itu}}{u}\,du - \int_0^\infty f(x)\,dx\int_x^{\to\infty} \frac{e^{-itu}}{u}\,du\right| \\ = \left|\int_a^\infty f(x)\,dx\int_x^{\to\infty} \frac{e^{-itu}}{u}\,du\right| = \left|\int_a^\infty f(x)h_x(t)\,dx\right| \\ \le \left\{\int_a^\infty |f(x)|^p\,dx\right\}^{1/p}\left\{\int_a^\infty |h_x(t)|^{p^*}dx\right\}^{1/p^*} \\ = o(1)\left\{\int_a^\infty \left(\frac{4}{tx}\right)^{p^*} dx\right\}^{1/p^*} \to 0 \quad \text{as } a\to\infty.\end{aligned} \tag{4.13}$$

Clearly, (4.10) follows immediately from (4.12) and (4.13).

The limit relation in (4.11) can be proved in a similar manner.

Fifth, by definition we have

$$(\mathcal{C}^*(f))^\wedge := L^{p^*}(\mathbb{R})\text{-}\lim_{a\to\infty}((\mathcal{C}^*(f))_a)^\wedge. \tag{4.14}$$

Since $(\mathcal{C}^*(f))_a \in L^1(\mathbb{R})$, by (1.2) and Fubini's theorem, we may write

$$\sqrt{2\pi}\,((\mathcal{C}^*(f))_a)^\wedge(t) := \int_{|u|<a} \left\{\frac{1}{u}\int_0^u f(x)\,dx\right\} e^{-itu}\,du \tag{4.15}$$

$$= \int_0^a f(x)\,dx \int_x^a \frac{e^{-itu}}{u}\,du - \int_{-a}^0 f(x)\,dx \int_{-a}^x \frac{e^{-itu}}{u}\,du.$$

A comparison of (4.9), (4.14) and (4.15) completes the proof of (2.2) for $t > 0$. The proof of (2.2) for $t < 0$ can be carried out in an analogous way. ■

As a by-product of (2.2), (4.5) and (4.8), we obtain the following representation: If $f \in L^p(\mathbb{R})$ for some $1 < p \le 2$, then

$$\sqrt{2\pi}\,(\mathcal{C}^*(f))^\wedge(t) \tag{4.16}$$

$$= \int_0^\infty f(x)\,dx \int_x^{\to\infty} \frac{e^{-itu}}{u}\,du - \int_{-\infty}^0 f(x)\,dx \int_{\to-\infty}^x \frac{e^{-itu}}{u}\,du \quad \text{a.e.}$$

Both outer integrals on the right-hand side exist in the Lebesgue sense for all $t \ne 0$, since $f \in L^p(\mathbb{R})$ and

$$\int_x^{\to\infty} \frac{e^{-itu}}{u}\,du \in L^q(\mathbb{R}_+, dx), \qquad \int_{\to-\infty}^x \frac{e^{-itu}}{u}\,du \in L^q(\mathbb{R}_-, dx) \tag{4.17}$$

for all $t \ne 0$ and $1 < q < \infty$. In fact, let $t > 0$, say; then by (4.3)–(4.5) we have

$$\int_0^\infty dx \left| \int_x^{\to\infty} \frac{e^{-itu}}{u}\,du \right|^q \le \left\{ \int_0^{1/(et)} + \int_{1/(et)}^\infty \right\} |h_t(x)|^q\,dx$$

$$\le 5^q \int_0^{1/(et)} \left(\ln\frac{1}{tx}\right)^q dx + \frac{4^q}{t^q}\int_{1/(et)}^\infty \frac{1}{x^q}\,dx < \infty$$

(cf. the computations in (4.7)). The above claim for $t < 0$ as well as the second claim in (4.17) can be proved in a similar way.

We make one more remark. Observe that the right-hand side in (2.2) is continuous except possibly at $t = 0$ and vanishes at infinity. Therefore, we may change the values of $(\mathcal{C}^*(f))^\wedge(t)$ (originally defined as a limit in the norm of $L^{p^*}(\mathbb{R})$) on a set of measure zero so that $(\mathcal{C}^*(f))^\wedge(t)$ becomes continuous except possibly at $t = 0$ and vanishes at infinity.

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Bolyai Institute
University of Szeged
Aradi Vértanúk Tere 1
6720 Szeged, Hungary
E-mail: moricz@math.u-szeged.hu

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