The harmonic Cesàro and Copson operators
on the spaces $L^p(\mathbb{R}), \ 1 \leq p \leq 2$

by

FERENC MÓRICZ (Szeged)

Dedicated to Professor Paul R. Halmos on his 85th birthday

Abstract. The harmonic Cesàro operator $C$ is defined for a function $f$ in $L^p(\mathbb{R})$ for some $1 \leq p < \infty$ by setting

$$C(f)(x) := \begin{cases} \int_{x}^{\infty} (f(u)/u) \, du & \text{for } x > 0, \\ -\int_{-\infty}^{x} (f(u)/u) \, du & \text{for } x < 0, \end{cases}$$

the harmonic Copson operator $C^*$ is defined for a function $f$ in $L^1_{\text{loc}}(\mathbb{R})$ by setting

$$C^*(f)(x) := \frac{1}{x} \int_{0}^{x} f(u) \, du$$

for $x \neq 0$.

We present rigorous proofs of the following two commuting relations:

(i) If $f \in L^p(\mathbb{R})$ for some $1 \leq p \leq 2$, then $(C(f))^\wedge(t) = C^*(\hat{f})(t)$ a.e., where $\hat{f}$ denotes the Fourier transform of $f$.

(ii) If $f \in L^p(\mathbb{R})$ for some $1 < p \leq 2$, then $(C^*(f))^\wedge(t) = C(\hat{f})(t)$ a.e.

As a by-product of our proofs, we obtain representations of $(C(f))^\wedge(t)$ and $(C^*(f))^\wedge(t)$ in terms of Lebesgue integrals in case $f$ belongs to $L^p(\mathbb{R})$ for some $1 < p \leq 2$. These representations are valid for almost every $t$ and may be useful in other contexts.

1. Definitions. First, we recall that the harmonic Cesàro operator $C$ is defined for a function $f$ in $L^p(\mathbb{R})$ for some $1 \leq p < \infty$ by setting

$$C(f)(x) := \begin{cases} \int_{x}^{\infty} (f(u)/u) \, du & \text{for } x > 0, \\ -\int_{-\infty}^{x} (f(u)/u) \, du & \text{for } x < 0; \end{cases}$$

the harmonic Copson operator $C^*$ is defined for a function $f$ in $L^1_{\text{loc}}(\mathbb{R})$ by setting

$$C^*(f)(x) := \frac{1}{x} \int_{0}^{x} f(u) \, du$$

for $x \neq 0$.

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The notation $C^*$ (as the adjoint operator of $C$) is justified by the fact that if $f \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$ and $g \in L^{p^*}(\mathbb{R})$, where $1/p + 1/p^* = 1$, then
\begin{equation}
\int_{\mathbb{R}} C(f)(x)g(x) \, dx = \int_{\mathbb{R}} f(x)C^*(g)(x) \, dx.
\end{equation}

See, for example, Golubov [3, p. 329] for the case when $f$ and $g$ are defined on $\mathbb{R}_+$.

The integrals on both sides of (1.1) exist in the Lebesgue sense. Indeed, it follows from the well known inequalities of Hardy [4, Theorems 327 and 328] that if $f \in L^p(\mathbb{R})$ for some $p$, then $C(f) \in L^p(\mathbb{R})$ in case $1 < p < \infty$, and $C^*(f) \in L^p(\mathbb{R})$ in case $1 < p < \infty$. More exactly, we have

$$
\|C\|_p := \sup_{\|f\|_p \leq 1} \|C(f)\|_p = p \quad \text{and} \quad \|C^*\|_p = p^*;
$$

where

$$
\|f\|_p := \left\{ \int_{\mathbb{R}} |f(x)|^p \, dx \right\}^{1/p} \quad \text{for } 1 \leq p < \infty,
$$

$$
\|f\|_\infty := \text{ess sup}\{|f(x)| : x \in \mathbb{R}\}.
$$

Second, we remind the reader that the Fourier transform of a function $f$ in $L^1(\mathbb{R})$ is defined by
\begin{equation}
\hat{f}(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-itx} \, dx, \quad t \in \mathbb{R}.
\end{equation}

It is well known that $\hat{f}$ is continuous on $\mathbb{R}$ and, by the Riemann–Lebesgue lemma, $\hat{f}(t)$ vanishes as $|t| \to \infty$. In case $f \in L^p(\mathbb{R})$ for some $1 < p \leq 2$, the Fourier transform of $f$ is defined in terms of a limit in the norm of $L^{p^*}(\mathbb{R})$:
\begin{equation}
\hat{f} := L^{p^*}(\mathbb{R})- \lim_{a \to \infty} \hat{f}_a, \quad \text{where} \quad f_a := f \chi_{(-a,a)}, \quad \hat{f}_a := \left( f_a \right)^{\wedge}
\end{equation}

and $\chi_{(-a,a)}$ denotes the characteristic function of the interval $(-a,a)$. (See e.g. [6, Vol. 2, p. 254].) That is, $\hat{f} \in L^{p^*}(\mathbb{R})$ and
\begin{equation}
\lim_{a \to \infty} \|\hat{f}_a - \hat{f}\|_{p^*} = 0.
\end{equation}

We note that if $f \in L^p(\mathbb{R})$ for some $1 < p \leq 2$, then the existence of $\hat{f}(t)$ is guaranteed only at almost every $t$. In particular, this time $\hat{f}$ is no longer continuous on $\mathbb{R}$ or vanishes at infinity (unlike the case when $f \in L^1(\mathbb{R})$).

In case $f \in L^p(\mathbb{R})$ for some $2 < p \leq \infty$, the Fourier transform of $f$ cannot be defined as an ordinary function in any reasonable way either by making a passage to the limit in the norm of $L^{p^*}(\mathbb{R})$, or by using any linear method of summation. (See e.g. [6, Vol. 2, p. 258].) However, this time $\hat{f}$ can be defined as a tempered distribution. (See e.g. [5, pp. 19–30].) But we are not concerned with distributions in this paper.
2. Interrelations with Fourier transform. We prove the following two commuting relations.

**Theorem 1.** If \( f \in L^p(\mathbb{R}) \) for some \( 1 \leq p \leq 2 \), then

\[
(C(f))^\wedge(t) = C^*(\hat{f})(t) \quad \text{a.e.}
\]

**Theorem 2.** If \( f \in L^p(\mathbb{R}) \) for some \( 1 < p \leq 2 \), then

\[
(C^*(f))^\wedge(t) = C(\hat{f})(t) \quad \text{a.e.}
\]

Theorem 1 justifies the term “harmonic Cesàro operator” since (2.1) can be rewritten in the form

\[
(C(f))^\wedge(t) = \frac{1}{t} \int_0^t \hat{f}(u) \, du \quad \text{a.e.}
\]

By the uniqueness theorem for Fourier transforms, we could have defined \( C(f) \) by (2.3), at least for functions \( f \) belonging to \( L^p(\mathbb{R}) \) for some \( 1 \leq p \leq 2 \). Analogously, for functions \( f \) belonging to \( L^p(\mathbb{R}) \) for some \( 1 < p \leq 2 \), we could have defined the harmonic Copson operator \( C^*(f) \) as follows (due to (2.2)):

\[
(C^*(f))^\wedge(t) = \begin{cases} 
\int_t^{\infty} (\hat{f}(u)/u) \, du & \text{for } t > 0, \\
\int_{-\infty}^t (\hat{f}(u)/u) \, du & \text{for } t < 0.
\end{cases}
\]

Theorems 1 and 2 were formulated by Bellman [1] with heuristic motivations. Later Golubov [3, Theorems 3 and 4] presented proofs for them in the case of cosine Fourier transform, without recognizing the forms of the Cesàro and Copson operators. Equality (2.1) for \( p = 1 \) was independently proved in [2] by Giang and the present author.

Unfortunately, the proofs of [3, Theorems 3 and 4] are not complete. To be specific, there is a deficiency in the proof of [3, Theorem 3] in case \( f \in L^p(\mathbb{R}) \) for some \( 1 < p \leq 2 \). The reason is that this time

\[
\int_x^{\infty} \frac{f(u)}{u} \, du \notin L^1(\mathbb{R}+) \quad (\in L^p(\mathbb{R}^+)).
\]

Therefore, the right-hand side in [3, formula (26)] can be integrated by parts only over a finite interval, say \([x_1, x_2]\) with \( 0 < x_1 < x_2 < \infty \). As a result, we have

\[
\int_{x_1}^{x_2} f(x) \frac{\sin tx}{x} \, dx = \left[ -\sin tx \int_x^{\infty} \frac{f(u)}{u} \, du \right]_{x=x_1}^{x_2} + t \int_{x_1}^{x_2} \cos tx \, dx \int_x^{\infty} \frac{f(u)}{u} \, du.
\]

In order to obtain [3, formula (27)], we have to let \( x_1 \to +0 \) and \( x_2 \to \infty \) in (2.5). The integrated-out terms converge to 0 at this passage. The problem is
that we cannot apply Lebesgue’s dominated convergence theorem as $x_1 \to 0$ and $x_2 \to \infty$: for the integral on the left-hand side of (2.5) because $f(x)/x \notin L^1(\mathbb{R}_+)$, and for the integral on the right-hand side because of (2.4).

A similar problem arises in the proof of [3, Theorem 4] when the integral on the right-hand side of the formula preceding [3, formula (34)] is integrated by parts. Namely, due to the fact that $f \in L^p(\mathbb{R})$ for some $1 < p \leq 2$, the integral

$$\int_0^\infty \left\{ \frac{x}{x} f(u) du \right\} \cos tx \, dx$$

on the right-hand side of [3, formula (34)] does not exist in the Lebesgue sense. Thus, the integration by parts which would yield [3, formula (34)] is not allowed.

Analysing our proof of (2.1), we see that if $f \in L^p(\mathbb{R})$ for some $1 \leq p \leq 2$, then

$$(C(f))^\wedge(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) \frac{1 - e^{-itu}}{itu} \, du \quad \text{a.e.}$$

(see (3.10) below). This can be rewritten in the following form:

$$\sqrt{2\pi} (C(f))^\wedge(t) = \int_{\mathbb{R}} f(u) \left\{ \frac{1}{u} \int_0^u e^{-itx} \, dx \right\} du$$

$$= \int_{\mathbb{R}} f(u) C^*(e^{-it\cdot})(u) \, du \quad \text{a.e.}$$

Observe that in this way we have actually extended the validity of (1.1) to the case when $f \in L^p(\mathbb{R})$ for some $1 \leq p \leq 2$ and $g(x) := e^{-itx}$. The left-hand side in (2.6) could be “formally” interpreted as the integral

$$\int_{\mathbb{R}} C(f)(x)e^{-itx} \, dx.$$

However, this integral exists in the Lebesgue sense only if $p = 1$; while if $1 < p \leq 2$, it exists only as the limit of $((C(f))_a)^\wedge$ as $a \to \infty$ in the norm of $L^p^*(\mathbb{R})$:

$$(C(f))^\wedge(t) := L^p^*(\mathbb{R})- \lim_{a \to \infty} \frac{1}{\sqrt{2\pi}} \int_{|x|<a} C(f)(x)e^{-itx} \, dx.$$  

Analysing our proof of (2.2), we find that if $f \in L^p(\mathbb{R})$ for some $1 < p \leq 2$, then

$$\sqrt{2\pi} (C^*(f))^\wedge(t)$$

$$= \int_0^\infty f(x) \, dx \int_0^\infty \frac{e^{-itu}}{u} \, du - \int_{-\infty}^0 f(x) \, dx \int_{-\infty}^0 \frac{e^{-itu}}{u} \, du \quad \text{a.e.}$$
(see (4.16)). This can be rewritten in the following form:

\[ \sqrt{2\pi} (\mathcal{C}^* (f))^\wedge (t) = \int_0^\infty f(x) \mathcal{C}(e^{-it\cdot})(x) dx + \int_{-\infty}^0 f(x) \mathcal{C}(e^{-it\cdot})(x) dx \]

\[ = \int_{\mathbb{R}} f(x) \mathcal{C}(e^{-it\cdot})(x) dx \text{ a.e.,} \]

where \( \mathcal{C}(e^{-it\cdot})(x) \) should be defined as an improper integral:

\[ C(e^{-it\cdot})(x) := \begin{cases} 
\lim_{b \to -\infty} \int_x^b \frac{e^{-itu}}{u} du & \text{for } x > 0, \\
\lim_{b \to \infty} \int_b^x \frac{e^{-itu}}{u} du & \text{for } x < 0.
\end{cases} \]

The left-hand side in (2.7) could be “formally” interpreted as the integral

\[ \int_{\mathbb{R}} \mathcal{C}^*(f)(x)e^{-it\cdot} dx. \]

But this integral does not exist generally in the Lebesgue sense; it exists only as the limit of \( (\mathcal{C}^*(f))_a^\wedge \) as \( a \to \infty \) in the norm of \( L^p(\mathbb{R}) \):

\[ (\mathcal{C}^*(f))^\wedge (t) := L^p(\mathbb{R})- \lim_{a \to \infty} \int_{|x|<a} \mathcal{C}^*(f)(x)e^{-it\cdot} dx. \]

### 3. Proof of Theorem 1

**Case** \( p = 1 \). By definition,

\[ \sqrt{2\pi} (\mathcal{C}(f))^\wedge (t) := \int_0^\infty e^{-it\cdot} dx \int_{\mathbb{R}} \frac{f(u)}{u} du - \int_{-\infty}^0 e^{-it\cdot} dx \int_{-\infty}^u \frac{f(u)}{u} du, \]

whence by Fubini’s theorem,

\[ 2\pi (\mathcal{C}^*(f))^\wedge (t) = \int_{\mathbb{R}} \frac{f(u)}{u} du \int_0^u e^{-itu} du = \int_{\mathbb{R}} f(u) \frac{1 - e^{-itu}}{itu} du, \quad t \neq 0. \]

The last integral in (3.1) exists in the Lebesgue sense, since

\[ \left| \frac{1 - e^{-itu}}{itu} \right| = \left| \frac{2 \sin(tu/2)}{tu} \right| \quad \text{for all } t \neq 0 \text{ and } u \neq 0. \]
On the other hand,
\[ t\sqrt{2\pi} C^* (\hat{f})(t) := \sqrt{2\pi} \int_0^t \hat{f}(u) \, du = \int_0^t f(x) e^{-iu x} \, dx, \]
whence again by Fubini’s theorem,
\[ \sqrt{2\pi} C^* (\hat{f})(t) = \frac{1}{t} \int_0^t f(x) \, dx \int_0^t e^{-iu x} \, du = \int_0^t f(x) \frac{1 - e^{-it x}}{it x} \, dx. \]

Clearly, the rightmost integrals in (3.1) and (3.3) are identical. Thus, we have proved (2.1) for all \( t \neq 0 \) provided \( f \in L^1(\mathbb{R}) \).

We note that for \( t = 0 \) we clearly have
\[ \sqrt{2\pi} (C(f))^\wedge (0) = \int_{\mathbb{R}} f(u) \, du = \sqrt{2\pi} \hat{f}(0), \]
whence
\[ (C(f))^\wedge (0) = \hat{f}(0). \]

Although \( C^*(f)(x) \) has not been defined for \( x = 0 \) in general, in the particular case when \( f \) is continuous at \( x = 0 \), it is reasonable to set
\[ C^*(f)(0) := \lim_{x \to 0} C^*(f)(x) = f(0). \]

Since \( \hat{f} \) is continuous if \( f \in L^1(\mathbb{R}) \), this supplementary definition applies to \( \hat{f} \) in place of \( f \). So, with this agreement, we have (2.1) for all \( t \in \mathbb{R} \) provided \( f \in L^1(\mathbb{R}) \).

Case \( 1 < p \leq 2 \). Given \( a > 0 \), let \( f_a := f_{\chi_{(-a,a)}} \) and \( \hat{f}_a := (f_a)^\wedge \), where \( a > 0 \). Since \( f_a \in L^1(\mathbb{R}) \), we may apply (2.1) to obtain
\[ (C(f_a))^\wedge (t) = C^*(\hat{f}_a)(t) = \frac{1}{\sqrt{2\pi}} \int_{|u| < a} \frac{f(u)}{u} \, du \int_0^u e^{-it x} \, dx, \quad t \in \mathbb{R}. \]

First, we claim that
\[ \lim_{a \to \infty} C^*(\hat{f}_a)(t) = C^*(\hat{f})(t) \quad \text{for all} \ t \neq 0, \]
even uniformly in \( t \) provided \( |t| \geq t_0 \) for some \( t_0 > 0 \). Indeed, Hölder’s inequality and (1.3) yield
\[ |C^*(\hat{f}_a)(t) - C^*(\hat{f})(t)| = \left| \frac{1}{t} \int_0^t \{\hat{f}_a(u) - \hat{f}(u)\} \, du \right| \]
\[ \leq \frac{1}{|t|} \|\hat{f}_a - \hat{f}\|_p |t|^{1/p} \to 0 \quad \text{as} \ a \to \infty. \]

Second, we claim that
\[ L^p(\mathbb{R})- \lim_{a \to \infty} (C(f_a))^\wedge = (C(f))^\wedge. \]
To see this, we start with the definition:
\[
(C(f))^\wedge := L^p^*(\mathbb{R})\lim_{a \to \infty} ((C(f))_a)^\wedge \\
= L^p^*(\mathbb{R})\lim_{a \to \infty} \frac{1}{\sqrt{2\pi}} \int_{|x| < a} C(f)(x)e^{-ix} \, dx.
\]

By Fubini’s theorem,
\[
\int_{|x| < a} C(f)(x)e^{-itx} \, dx = \int_{u < a} e^{-itx} \int_{x}^{\infty} f(u) du - \int_{-a}^{0} e^{-itx} \int_{-\infty}^{x} f(u) du \\
= \int_{0}^{a} \frac{f(u)}{u} du \int_{0}^{u} e^{-itx} dx + \int_{a}^{\infty} \frac{f(u)}{u} du \int_{0}^{a} e^{-itx} dx \\
- \int_{-a}^{0} \frac{f(u)}{u} du \int_{-\infty}^{0} e^{-itx} dx - \int_{-\infty}^{-a} \frac{f(u)}{u} du \int_{-a}^{0} e^{-itx} dx \\
= \int_{|u| < a} \frac{f(u)}{u} du \int_{0}^{u} e^{-itx} dx \\
+ \int_{a}^{\infty} \frac{f(u)}{u} \cdot \frac{1 - e^{-iat}}{it} du - \int_{-\infty}^{a} \frac{f(u)}{u} \cdot \frac{1 - e^{iat}}{it} du,
\]
whence, by (3.4), we conclude that
\[
(C(f))^\wedge(t) = L^p^*(\mathbb{R})\lim_{a \to \infty} \left\{ (C(f)_a)^\wedge(t) + \frac{1}{2\pi} \left[ \frac{1 - e^{-iat}}{it} \int_{a}^{\infty} \frac{f(u)}{u} du + \frac{1 - e^{iat}}{it} \int_{-\infty}^{-a} \frac{f(u)}{u} du \right] \right\}.
\]

Third, we claim that
\[
L^p^*(\mathbb{R})\lim_{a \to \infty} \left[ \frac{1 - e^{-iat}}{it} \int_{a}^{\infty} \frac{f(u)}{u} du + \frac{1 - e^{iat}}{it} \int_{-\infty}^{-a} \frac{f(u)}{u} du \right] = 0.
\]

In fact, by Hölder’s inequality we have
\[
\int_{|u| > a} \left| \frac{f(u)}{u} \right| \, du \leq \left\{ \int_{|u| > a} |f(u)|^p \, du \right\}^{1/p} \left\{ \int_{|u| > a} u^{-p^*} \, du \right\}^{1/p^*} \\
= o(1)O(a^{-p^*+1})^{1/p^*} = o(a^{-1/p}) \quad \text{as} \quad a \to \infty.
\]
On the other hand, by (3.2) and Minkowski’s inequality, we find
\[
\left\| \frac{1 - e^{-iat}}{it} \int_a^\infty \frac{f(u)}{u} \, du \right\|_{L^{p^*}(dt)} \\
\leq \int_a^\infty \left\| \frac{f(u)}{u} \right\|_{|t| \leq 1/a} \, du \left\{ \int_{|t| \geq 1/a} \frac{ap^*}{t} \, dt \right\}^{1/p^*} + \left\{ \int_{|t| \leq 1/a} \frac{ap^*}{t} \, dt \right\}^{1/p^*} \\
= o(a^{-1/p})O(a^{1/p}) = o(1) \quad \text{as } a \to \infty.
\]
The other term in (3.8) can be estimated analogously.
To sum up, (3.7), (3.8) and (3.4) yield
\[
(3.9) \quad (C(f))^\wedge = L^{p^*}(\mathbb{R}) - \lim_{a \to \infty} (C(f_a))^\wedge = L^{p^*}(\mathbb{R}) - \lim_{a \to \infty} C^*(\hat{f}_a).
\]
Comparing (3.5) and (3.9) yields (2.1) to be proved. ■

As a by-product of (3.1) and (3.9), we have the following representation:
If \( f \in L^p(\mathbb{R}) \) for some \( 1 \leq p \leq 2 \), then
\[
(3.10) \quad (C(f))^\wedge(t) = \lim_{a \to \infty} (C(f_a))^\wedge(t) = \lim_{a \to \infty} \frac{1}{\sqrt{2\pi}} \int_{|u| < a} f(u) \frac{1 - e^{-itu}}{itu} \, du \\
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) \frac{1 - e^{-itu}}{itu} \, du \quad \text{a.e.}
\]
The pointwise limit exists for all \( t \neq 0 \), since \( f \in L^p(\mathbb{R}) \) and \((1 - e^{-itu})/(itu) \in L^q(\mathbb{R})\) for all \( 1 < q \leq \infty \) (cf. (3.2)).

One more remark is appropriate here. Observe that the right-hand side in (2.1) is continuous except possibly at \( t = 0 \) and vanishes at infinity. Consequently, we may change the values of \((C(f))^\wedge(t)\) (originally defined as a limit in the norm of \( L^{p^*}(\mathbb{R}) \)) on a set of measure zero so that \((C(f))^\wedge(t)\) becomes continuous except possibly at \( t = 0 \) and vanishes at infinity.

4. Proof of Theorem 2. Consider again the truncated function \( f_a := f \chi_{(-a,a)} \), where \( a > 0 \). Then \( f_a \in L^1(\mathbb{R}) \) and (1.2) applies. For the sake of definiteness, let \( 0 < t < \infty \) be fixed, and let \( b > t \). By Fubini’s theorem, we may write
\[
\sqrt{2\pi} \int_t^b \frac{\hat{f}(u)}{u} \, du = \int_t^b \frac{du}{u} \int_{|x| < a} f(x) e^{-ixu} \, dx = \int_{|x| < a} f(x) dx \int_t^b \frac{e^{-ixu}}{u} \, du.
\]

First, letting \( a \) tend to \( \infty \) gives
\[
(4.1) \quad \sqrt{2\pi} \int_t^b \frac{\hat{f}(u)}{u} \, du = \int_{\mathbb{R}} f(x) dx \int_t^b \frac{e^{-ixu}}{u} \, du.
\]
In fact, by (1.3) we have
\[
\left| \int_t^b \frac{\hat{f}_a(u)}{u} \, du - \int_t^b \frac{f_a(u)}{u} \, du \right| \leq \|\hat{f}_a - \hat{f}\|_p \left\{ \int_t^b \frac{du}{u^p} \right\}^{1/p} \to 0 \quad \text{as } a \to \infty.
\]

On the other hand, introducing the auxiliary function
\[
(4.2) \quad h_{t,b}(x) := \int_t^b \frac{e^{-iux}}{u} \, du,
\]
from the second mean value theorem for integrals it follows that
\[
(4.3) \quad |h_{t,b}(x)| \leq \frac{4}{t|x|} \quad \text{for all } t > 0 \text{ and } x \neq 0.
\]

Making use of (4.3) and Hölder’s inequality, we obtain
\[
\left| \int_{\mathbb{R}} f(x)h_{t,b}(x) \, dx - \int_{|x|<a} f(x)h_{t,b}(x) \, dx \right|
\leq \frac{4}{t} \int_{|x|>a} \frac{|f(x)|}{x} \, dx \leq \frac{4}{t} \left\{ \int_{|x|>a} |f(x)|^p \, dx \right\}^{1/p} \left\{ \int_{|x|>a} \frac{dx}{|x|^{p'}} \right\}^{1/p'} \to 0
\]
as \(a \to \infty\). This completes the proof of (4.1).

In what follows, we need another estimate: if \(t|x| \leq 1/e\) and \(b|x| \geq 1\), then by (4.3) we have
\[
(4.4) \quad |h_{t,b}(x)| \leq \int_t^{1/|x|} \frac{1}{u} \, du + |h_{1/|x|,b}(x)| \leq \ln \frac{1}{t|x|} + 4 \leq 5 \ln \frac{1}{t|x|}.
\]

Set
\[
(4.5) \quad h_t(x) := \lim_{b \to \infty} h_{t,b}(x) = \int_t^\infty \frac{e^{-iux}}{u} \, du.
\]

Clearly, \(h_t(x)\) exists as an improper integral. In fact, if \(t < b < b_1 < \infty\), then by (4.2) and (4.3) we have
\[
|h_{t,b}(x) - h_{t,b_1}(x)| = \left| \int_b^{b_1} \frac{e^{-iux}}{u} \, du \right| \leq \frac{5}{b|x|} \to 0 \quad \text{as } b \to \infty.
\]

Furthermore, inequalities (4.3) and (4.4) (the latter for \(t|x| \leq 1/e\)) remain valid for \(h_t\) in place of \(h_{t,b}\).

Second, we claim that letting \(b\) tend to \(\infty\) in (4.1) gives
\[
(4.6) \quad \sqrt{2\pi} \mathcal{C}(\mathcal{F})(t) := \sqrt{2\pi} \int_t^\infty \frac{\mathcal{F}(u)}{u} \, du = \int_{\mathbb{R}} f(x)h_t(x) \, dx,
\]
where 
\[
\mathcal{F}(u) = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}(u) \cos(xu) \, dx.
\]
where \( h_t(x) \) is defined in (4.5). To see the validity of (4.6), first we notice that

\[
\lim_{b \to \infty} \int_{t}^{b} \frac{\hat{f}(u)}{u} du = \int_{t}^{\infty} \frac{\hat{f}(u)}{u} du,
\]

since \( \hat{f} \in L^{p^*}(\mathbb{R}) \) and \( 1/u \in L^{p}(t, \infty) \); consequently, \( \hat{f}(u)/u \in L^{1}(t, \infty) \) for all \( t > 0 \).

On the other hand, by (4.2)–(4.4) and Hölder’s inequality we obtain

\[
\int_{\mathbb{R}} f(x) h_t(x) dx - \int_{\mathbb{R}} f(x) h_{t,b}(x) dx = \left| \int_{\mathbb{R}} f(x) h_b(x) dx \right|
\]

\[
\leq 5 \int_{|x|<1/(be)} |f(x)| \ln \frac{1}{b|x|} dx + \frac{4}{b} \int_{|x|>1/(be)} \left| \frac{f(x)}{x} \right| dx
\]

\[
\leq 5 \left\{ \int_{|x|<1/(be)} |f(x)|^p dx \right\}^{1/p} \left\{ \int_{|x|>1/(be)} \left( \ln \frac{1}{b|x|} \right)^{p^*} dx \right\}^{1/p^*}
\]

\[
+ \frac{4}{b} \left\{ \int_{|x|>1/(be)} |f(x)|^p dx \right\}^{1/p} \left\{ \int_{|x|>1/(be)} \frac{dx}{|x|^{p^*}} \right\}^{1/p^*} \to 0
\]

as \( b \to \infty \), since

\[
\left\{ \int_{0}^{1/(be)} \left( \ln \frac{1}{bx} \right)^{p^*} dx \right\}^{1/p^*} \leq \left\{ \int_{0}^{1/(be)} \left( \ln \frac{1}{b} \right)^{p^*} dx \right\}^{1/p^*} + \int_{1+\ln b}^{\infty} t^{p^*} e^{-t} dt = O(1)
\]

as \( b \to \infty \). This completes the justification of (4.6).

Third, returning to (4.5), we may write

\[
h_t(x) := \int_{t}^{\infty} \frac{e^{-iux}}{u} du = \int_{t}^{\infty} \frac{e^{-iv}}{v} dv = \int_{x}^{\infty} \frac{e^{-itu}}{u} du \quad \text{for } x > 0
\]

(the last integral equals \( h_x(t) \)), and

\[
h_t(x) := - \int_{-\infty}^{tx} \frac{e^{-iv}}{v} dv = - \int_{-\infty}^{-\infty} \frac{e^{-itu}}{u} du \quad \text{for } x < 0.
\]

Substituting these into (4.6), we obtain

\[
\sqrt{2\pi} \mathcal{C}(\hat{f})(t) = \int_{0}^{\infty} f(x) dx \int_{x}^{\infty} \frac{e^{-itu}}{u} du - \int_{-\infty}^{0} f(x) dx \int_{-\infty}^{x} \frac{e^{-itu}}{u} du.
\]
Fourth, we claim that

\[
\sqrt{2\pi} \mathcal{C}(\hat{f})(t) = \lim_{a \to \infty} \left\{ \int_{0}^{a} f(x) \, dx \int_{x}^{\infty} \frac{e^{-itu}}{u} \, du - \int_{-a}^{0} f(x) \, dx \int_{-a}^{x} \frac{e^{-itu}}{u} \, du \right\}.
\]

By (4.8), it is enough to check that

\[
\lim_{a \to \infty} \int_{0}^{a} f(x) \, dx \int_{x}^{\infty} \frac{e^{-itu}}{u} \, du = \int_{0}^{\infty} f(x) \, dx \int_{x}^{\infty} \frac{e^{-itu}}{u} \, du,
\]

\[
\lim_{a \to \infty} \int_{-a}^{0} f(x) \, dx \int_{-a}^{x} \frac{e^{-itu}}{u} \, du = \int_{-\infty}^{0} f(x) \, dx \int_{-\infty}^{x} \frac{e^{-itu}}{u} \, du.
\]

We shall present the proof of (4.10) in detail. It will be done in two steps. By (4.3), (4.5) and Hölder’s inequality, we obtain

\[
\left| \int_{0}^{a} f(x) \, dx \int_{x}^{\infty} \frac{e^{-itu}}{u} \, du - \int_{0}^{a} f(x) \, dx \int_{x}^{\infty} \frac{e^{-itu}}{u} \, du \right| = \left| \int_{0}^{a} f(x) \, dx \int_{a}^{\infty} \frac{e^{-itu}}{u} \, du \right| \leq |h_a(t)| \int_{0}^{a} |f(x)| \, dx
\]

\[
\leq \frac{4}{at} \left\{ \int_{0}^{a} |f(x)|^p \, dx \right\}^{1/p} a^{1/p^*} \to 0 \quad \text{as } a \to \infty.
\]

Again by (4.3), (4.5) and Hölder’s inequality, we find that

\[
\left| \int_{0}^{a} f(x) \, dx \int_{x}^{\infty} \frac{e^{-itu}}{u} \, du - \int_{0}^{\infty} f(x) \, dx \int_{x}^{\infty} \frac{e^{-itu}}{u} \, du \right| = \left| \int_{a}^{\infty} f(x) \, dx \int_{x}^{\infty} \frac{e^{-itu}}{u} \, du \right| = \left| \int_{a}^{\infty} f(x) h_x(t) \, dx \right|
\]

\[
\leq \left\{ \int_{a}^{\infty} |f(x)|^p \, dx \right\}^{1/p} \left\{ \int_{a}^{\infty} |h_x(t)|^p^* \, dx \right\}^{1/p^*} = o(1) \left\{ \int_{a}^{\infty} \left( \frac{4}{tx} \right)^{p^*} \, dx \right\}^{1/p^*} \to 0 \quad \text{as } a \to \infty.
\]

Clearly, (4.10) follows immediately from (4.12) and (4.13).

The limit relation in (4.11) can be proved in a similar manner.

Fifth, by definition we have

\[
(C^*(f))^\wedge := L^p(\mathbb{R}) - \lim_{a \to \infty} ((C^*(f))_a)^\wedge.
\]
Since \((C^*(f))_a \in L^1(\mathbb{R})\), by (1.2) and Fubini’s theorem, we may write
\[
(4.15) \quad \sqrt{2\pi} ((C^*(f))_a)^\wedge (t) := \int_{|u| < a} \left\{ \frac{1}{u} \int_0^u f(x) \, dx \right\} e^{-itu} \, du
\]
\[
= \int_0^a f(x) \, dx \int_x^a \frac{e^{-itu}}{u} \, du - \int_{-a}^0 f(x) \, dx \int_x^{-a} \frac{e^{-itu}}{u} \, du.
\]

A comparison of (4.9), (4.14) and (4.15) completes the proof of (2.2) for \(t > 0\). The proof of (2.2) for \(t < 0\) can be carried out in an analogous way. ■

As a by-product of (2.2), (4.5) and (4.8), we obtain the following representation: If \(f \in L^p(\mathbb{R})\) for some \(1 < p \leq 2\), then
\[
(4.16) \quad \sqrt{2\pi} (C^*(f))^\wedge (t) = \int_0^\infty f(x) \, dx \int_x^\infty \frac{e^{-itu}}{u} \, du - \int_{-\infty}^0 f(x) \, dx \int_x^{-\infty} \frac{e^{-itu}}{u} \, du \quad \text{a.e.}
\]
Both outer integrals on the right-hand side exist in the Lebesgue sense for all \(t \neq 0\), since \(f \in L^p(\mathbb{R})\) and
\[
(4.17) \quad \int_x^{\infty} \frac{e^{-itu}}{u} \, du \in L^q(\mathbb{R}^+, dx), \quad \int_{-\infty}^x \frac{e^{-itu}}{u} \, du \in L^q(\mathbb{R}^-, dx)
\]
for all \(t \neq 0\) and \(1 < q < \infty\). In fact, let \(t > 0\), say; then by (4.3)–(4.5) we have
\[
\int_0^\infty dx \left| \int_x^{\infty} \frac{e^{-itu}}{u} \, du \right|^q \leq \left\{ \int_0^{1/(et)} + \int_{1/(et)}^\infty \right\} |h_t(x)|^q \, dx
\]
\[
\leq 5^q \int_0^{1/(et)} \left( \ln \frac{1}{tx} \right)^q \, dx + \frac{4^q}{t^q} \int_{1/(et)}^\infty \frac{1}{x^q} \, dx < \infty
\]
(cf. the computations in (4.7)). The above claim for \(t < 0\) as well as the second claim in (4.17) can be proved in a similar way.

We make one more remark. Observe that the right-hand side in (2.2) is continuous except possibly at \(t = 0\) and vanishes at infinity. Therefore, we may change the values of \((C^*(f))^\wedge (t)\) (originally defined as a limit in the norm of \(L^p^*(\mathbb{R})\)) on a set of measure zero so that \((C^*(f))^\wedge (t)\) becomes continuous except possibly at \(t = 0\) and vanishes at infinity.

References

Harmonic Cesàro and Copson operators


Bolyai Institute
University of Szeged
Aradi Vértanúk Tere 1
6720 Szeged, Hungary
E-mail: moricz@math.u-szeged.hu

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