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Beurling algebra analogues of theorems of Wiener–Lévy–Żelazko and Żelazko

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Abstract. Let $0 < p \leq 1$, let $\omega : \mathbb{Z} \to [1, \infty)$ be a weight on $\mathbb{Z}$ and let f be a nowhere vanishing continuous function on the unit circle Γ whose Fourier series satisfies $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^p \omega(n) < \infty$. Then there exists a weight ν on $\mathbb{Z}$ such that $\sum_{n\in\mathbb{Z}} |\widehat{(1/f)}(n)|^p \nu(n) < \infty$. Further, ν is non-constant if and only if ω is non-constant; and $\nu = \omega$ if ω is non-quasianalytic. This includes the classical Wiener theorem ($p = 1$, $\omega = 1$), Domar theorem ($p = 1$, ω is non-quasianalytic), Żelazko theorem ($\omega = 1$) and a recent result of Bhatt and Dedania ($p = 1$). An analogue of the Lévy theorem at the present level of generality is also developed. Given a locally compact group G with a continuous weight ω and $0 < p < 1$, the locally bounded space $L^p(G, \omega)$ is closed under convolution if and only if G is discrete if and only if G admits an atom. This generalizes and refines another result of Żelazko.

Let f be a continuous function on the unit circle Γ. Let f have absolutely convergent Fourier series. The celebrated Wiener theorem [12] implies that if $f(z) \neq 0$ for all $z \in \Gamma$, then $1/f$ has absolutely convergent Fourier series. Lévy's generalization [8] of Wiener's theorem implies that, for every function φ holomorphic on some neighbourhood of the range of f, the function $\varphi \circ f$ has absolutely convergent Fourier series if so does f. Żelazko proved in [13] that both these theorems hold if absolute convergence is replaced by pth power absolute convergence for $0 < p < 1$.

Let ω be a *weight* on $\mathbb{Z}$, that is, $\omega : \mathbb{Z} \to [1, \infty)$ satisfies $\omega(m + n) \leq \omega(m)\omega(n)$ $(m, n \in \mathbb{Z})$. A complex sequence $(\lambda_n)_{n\in\mathbb{Z}}$ is *ω-absolutely convergent* if $\sum_n |\lambda_n|\omega(n) < \infty$.

Domar proved in [4, Theorem 2.11] that if f has ω-absolutely convergent Fourier series and is nowhere vanishing on Γ, then $1/f$ has ω-absolutely convergent Fourier series provided ω is non-quasianalytic in the sense that $\sum_{n\in\mathbb{Z}} (\log \omega(n))/(1 + n^2) < \infty$. On the other hand, given f having ω-absolutely convergent Fourier series, it is established in [2] that there ex-

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ists a weight ν on $\mathbb{Z}$ such that $1/f$ (and analogously $\varphi \circ f$) has ν-absolutely convergent Fourier series.

We prove the following theorem that includes all these results, which also gives a pth power analogue of Domar's theorem.

THEOREM 1. *Let $0 < p \leq 1$, let ω be a weight on $\mathbb{Z}$, and let $f \in C(\Gamma)$ have pth power ω-absolutely convergent Fourier series.*

(I) *If $f(z) \neq 0$ $(z \in \Gamma)$, then there exists a weight ν on $\mathbb{Z}$ such that*

  (i) *$1/f$ has pth power ν-absolutely convergent Fourier series*;

  (ii) *ν is non-constant if and only if ω is non-constant*;

  (iii) *$\nu(n) \leq \omega(n)$ $(n \in \mathbb{Z})$.*

(II) *If φ is a holomorphic function on some neighbourhood of the range of f, then there exists a weight χ on $\mathbb{Z}$ such that*

  (i) *$\varphi \circ f$ has pth power χ-absolutely convergent Fourier series*;

  (ii) *χ is non-constant if and only if ω is non-constant*;

  (iii) *$\chi(n) \leq \omega(n)$ $(n \in \mathbb{Z})$.*

COROLLARY 1. *Let $0 < p \leq 1$, let ω be a non-quasianalytic weight on $\mathbb{Z}$, and let $f \in C(\Gamma)$ be nowhere vanishing. If f has pth power ω-absolulely convergent Fourier series, then $1/f$ has pth power ω-absolutely convergent Fourier series.*

Analogous to Gel'fand's proof of the Wiener theorem [7, p. 33], which is based on Banach algebras, we shall use Gel'fand theory of p-Banach algebras developed by Żelazko in the framework of locally bounded Beurling algebras.

Let $\mathcal{A}$ be a (complex) algebra and let $0 < p \leq 1$. Then a mapping $\|\cdot\| : \mathcal{A} \to \mathbb{R}$ is a *p-norm* on $\mathcal{A}$ if, for $x, y \in \mathcal{A}$ and for $\alpha \in \mathbb{C}$,

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|x + y\| \leq \|x\| + \|y\|$;
(iii) $\|\alpha x\| = |\alpha|^p \|x\|$;
(iv) $\|xy\| \leq \|x\|\,\|y\|$.

If $\mathcal{A}$ is complete in the p-norm, then $(\mathcal{A}, \|\cdot\|)$ is a *p-Banach algebra.* Among unital algebras, p-Banach algebras are precisely the complete locally bounded algebras [13, Theorem 2.3]. Given a continuous weight $\omega : G \to [1, \infty)$ on a locally compact group G satisfying $\omega(st) \leq \omega(s)\omega(t)$ $(s, t \in G)$, let $L^p(G, \omega)$ be the set of all measurable functions $f : G \to \mathbb{C}$ such that

$$\|f\|_{p,\omega} := \int_G |f(t)|^p \omega(t)\, dm(t) = \int_G |f(t)|^p\, dm_\omega(t) < \infty,$$

where m is the left invariant Haar measure on G, and $dm_\omega = \omega dm$. Though a complete locally bounded space can be dual-less, i.e., have no non-zero continuous linear functionals (which could be a hurdle in construction of vector-valued integrals), Żelazko [13] constructed a functional calculus in p-Banach algebras; and an offshot of his Gel'fand theory is that a semisimple, commutative, complete locally bounded algebra has sufficiently many continuous linear functionals. For a discrete abelian group G and for $0 < p \le 1$, the space $\ell^p(G)$ is a p-Banach algebra with convolution. In fact, Żelazko proved that, for a locally compact group G and for $0 < p < 1$, the complete locally bounded space $L^p(G)$ is closed under convolution if and only if G is discrete. The following theorem gives a Beurling algebra analogue of this.

THEOREM 2. *Let* $0 < p < 1$, *let* G *be a non-compact, locally compact group, and let* ω *be a continuous weight on* G. *Then the following are equivalent.*

(i) $L^p(G,\omega)$ *is closed under convolution.*
(ii) G *is discrete.*
(iii) $L^p(G,\omega)$ *admits a non-zero continuous linear functional.*
(iv) *The set of continuous linear functionals on* $L^p(G,\omega)$ *separates the points of* $L^p(G,\omega)$.
(v) G *admits an atom.*
(vi) G *admits sufficiently many atoms.*

In this case, if G *is abelian or* ω *is symmetric,* $L^p(G,\omega)$ *is semisimple.*

Note that, in general, it is not known whether $L^1(G,\omega)$ or $\ell^p(G,\omega)$ $(0 < p < 1)$ is semisimple or not [3, p. 175].

Rolewicz [11] has discussed multi-dimensional generalizations of Wiener–Żelazko and Lévy–Żelazko theorems. In the same spirit, it would be interesting to investigate multi-dimensional analogues of Domar's theorem [4, Theorem 2.11] as well as the theorem in [2].

Recall that a measurable set $E \subset G$ is an *atom* if $0 < m(E) < \infty$, and, for any measurable set $F \subset E$, $m(F) = 0$ or $m(F) = m(E)$. A Banach $*$-algebra A is an *A^*-algebra* if it admits a C^*-norm (not necessarily complete).

Proof of Theorem 1. Let

$$\ell^p(\mathbb{Z},\omega) := \Big\{\lambda = (\lambda_n) : |\lambda|_{p,\omega} := \sum_{n\in\mathbb{Z}} |\lambda_n|^p \omega(n) < \infty\Big\}.$$

Then $\ell^p(\mathbb{Z},\omega)$ is a convolution algebra. It is a p-Banach algebra with the p-norm $|\cdot|_{p,\omega}$. Let $A_p(\omega) = \{g \in C(\varGamma) : \widehat{g} \in \ell^p(\mathbb{Z},\omega)\}$. It is a p-Banach algebra with the pointwise operations and the p-norm being $\|g\|_{p,\omega} = |\widehat{g}|_{p,\omega}$. Thus $g \in C(\varGamma)$ has pth power ω-absolulely convergent Fourier series if and only if $g \in A_p(\omega)$ if and only if $\widehat{g} \in \ell^p(\mathbb{Z},\omega)$.

We *claim* that the Gel'fand space $\triangle(A_p(\omega))$ of $A_p(\omega)$ is homeomorphic to the closed annulus $\varGamma(\omega) = \{z \in \mathbb{C} : \rho(2,\omega) \le |z| \le \rho(1,\omega)\}$, where

$$\rho(1,\omega) = \inf\{\omega(n)^{1/n} : n \ge 1\} \quad \text{and} \quad \rho(2,\omega) = \sup\{\omega(n)^{1/n} : n \le -1\}.$$

For $z \in \Gamma(\omega)$, define

$$\varphi_z(g) = \sum_{n\in\mathbb{Z}} \widehat{g}(n) z^n \quad (g \in A_p(\omega)).$$

Then, for large n_0,

$$\sum_{|n|\geq n_0} |\widehat{g}(n) z^n| \leq \sum_{|n|\geq n_0} |\widehat{g}(n)|^p |z|^n \leq \sum_{|n|\geq n_0} |\widehat{g}(n)|^p \omega(n) < \infty.$$

It is routine to check that φ_z is a complex homomorphism on $A_p(\omega)$. Thus $\varphi_z \in \triangle(A_p(\omega))$ $(z \in \Gamma(\omega))$. Let $\varphi \in \triangle(A_p(\omega))$. Then $\|\varphi\| \leq 1$. Let $e_n(z) = z^n$ $(n \in \mathbb{Z}, z \in \Gamma)$. So $|\varphi(e_n)| \leq \|e_n\|_{p,\omega} = \omega(n)$ $(n \in \mathbb{Z})$. Set $\varphi(e_1) = z_0$. Then, for each $n \in \mathbb{Z}$, $\varphi(e_n) = \varphi(e_1)^n = z_0^n$. It is clear that $\rho(2,\omega) \leq |z_0| \leq \rho(1,\omega)$. So, for any $g \in A_p(\omega)$, we have

$$\varphi(g) = \varphi\Big(\sum_{n\in\mathbb{Z}} \widehat{g}(n) e_n\Big) = \sum_{n\in\mathbb{Z}} \widehat{g}(n) \varphi(e_n) = \sum_{n\in\mathbb{Z}} \widehat{g}(n) z_0^n = \varphi_{z_0}(g).$$

Thus $\varphi = \varphi_{z_0}$. This quickly gives the desired homeomorphism and establishes our claim. Thus each function $g \in A_p(\omega)$ extends uniquely as an element (denoted by g itself) in the set $B(\omega)$ consisting of all continuous functions on $\Gamma(\omega)$ which are holomorphic in the interior of $\Gamma(\omega)$. Now the construction of desired weights ν and χ is exactly as in [2]. ∎

Proof of Corollary 1. By the hypothesis, $\sum_{n\in\mathbb{N}} (\log\omega(n))/n^2$ is convergent. Notice that $\sum_{n\geq 2} 1/(n\log n)$ is divergent. For infinitely many $n \in \mathbb{N}$,

$$\frac{\log\omega(n)}{n^2} \leq \frac{1}{n\log n};$$

hence $\inf\{(\log\omega(n))/n : n \in \mathbb{N}\} = 0$, and $\rho(1,\omega) = 1$. Similarly, $\rho(2,\omega) = 1$. Then, from Theorem 1(I), $\nu = \omega$; and the result follows. ∎

Proof of Theorem 2. (i)⇒(ii). We prove that if G is not discrete, then $L^p(G,\omega)$ is not closed under convolution. Let V be a symmetric, open subset of G containing the identity of G such that $\overline{V}$ is compact. Since ω is continuous on G,

(a) $m_\omega(V^2) = \int_G \chi_{V^2}(t)\omega(t)\,dm(t) < \infty$.

Now choose a sequence (V_n) of measurable subsets of G such that

(b) $m_\omega(V_n) > 0$;
(c) $V_n \subseteq V$;
(d) $V_i \cap V_j = \emptyset$ whenever $i \neq j$.

From the properties (a), (c) and (d), we have $\sum_{n=1}^{\infty} m_\omega(V_n) < \infty$. We may assume that $m_\omega(V_n) < 2^{-n}$ $(n \in \mathbb{N})$. Define $f, g : G \to \mathbb{C}$ as

$$f = \chi_{V^2} \quad\text{and}\quad g = \sum_{n=1}^{\infty} [m_\omega(V_n) n^2]^{-1/p} \chi_{V_n}.$$

Then $\|f\|_{p,\omega} = m_\omega(V^2) < \infty$ and

$$\|g\|_{p,\omega} = \sum_{n=1}^{\infty}[m_\omega(V_n)n^2]^{-1}\int_G \chi_{V_n}(t)\omega(t)\,dm(t) = \sum_{n=1}^{\infty} n^{-2} < \infty.$$

Thus $f, g \in L^p(G,\omega)$. On the other hand,

$$(f\star g)(t) = \int_G f(s^{-1}t)g(s)\,dm(s) = \int_{tV^2} g(s)\,dm(s).$$

If $t \in V$, then $tV^2 \subseteq V$ and so

$$(f\star g)(t) = \sum_{n=1}^{\infty}[m_\omega(V_n)n^2]^{-1/p}m(V_n) = \sum_{n=1}^{\infty}[m_\omega(V_n)n^2]^{-1/p}m_\omega(V_n)\,\frac{m(V_n)}{m_\omega(V_n)}.$$

Since ω is bounded on compact subsets of G, there exists $K > 0$ such that $\omega(t) \le K$ $(t \in V)$. Now

$$m_\omega(V_n) = \int_{V_n} \omega(t)\,dm(t) \le \int_{V_n} K\,dm(t) = Km(V_n).$$

Thus

$$\frac{m(V_n)}{m_\omega(V_n)} \ge \frac{1}{K} \qquad (n \in \mathbb{N}).$$

Therefore

$$(f\star g)(t) \ge \frac{1}{K}\sum_{n=1}^{\infty} 2^{n(1/p-1)}n^{-2/p} = \infty.$$

Hence $f\star g$ cannot be in $L^p(G,\omega)$ as it becomes infinite on a set V of positive measure.

(ii)⇒(i). If G is discrete, then it is easy to verify that $L^p(G,\omega)$ is closed under convolution.

(ii)⇒(iv). Let G be discrete. Let $f, g \in L^p(G,\omega)$ and $f \ne g$. Therefore $f(t) \ne g(t)$ for some $t \in G$. Define $\varphi_t : L^p(G,\omega) \to \mathbb{C}$ as $\varphi_t(h) = h(t)$ $(h \in L^p(G,\omega))$. It is easy to verify that φ_t is a continuous linear functional on $L^p(G,\omega)$. Also $\varphi_t(f) = f(t) \ne g(t) = \varphi_t(g)$. Thus (iv) follows.

(v)⇔(vi). Since ω is continuous, $E \subset G$ is an m_ω-atom if and only if E is m-atom; and due to left translation invariance of m, this happens if and only if tE is an atom for any $t \in G$.

(iii)⇒(vi). Let $L^p(G,\omega)$ have a non-zero continuous linear functional. Then, by [10, Corollary 4.2.3], G admits an atom, say E. Since the Haar measure on G is left translation invariant, we may assume that E contains the identity e of G.

Next we *claim* that every set of positive measure contains an atom (measure-theoretically). Let $F \subset G$ be of positive measure. If $F = G$ a.e. m, we are done. Let $m(G\backslash F) > 0$. Now, either $F\cap E = \emptyset$ a.e. m or $m(F\cap E) > 0$. Since E is an atom, in the latter case $F \cap E = E \subset F$ a.e. m. Suppose that

$F \cap E = \emptyset$ a.e. m. Then $G\backslash F = E$ a.e. m. Now, for any $t \in G$, either $tE \cap E = \emptyset$ or $tE = E$ a.e. m. If $tE = E$ for all $t \in G$, then $G = E$ a.e. m, which is not possible as $m(F) = m(G\backslash E) > 0$. Therefore there exists $t \in G$ such that $tE \cap E = \emptyset$ a.e. m. Then $tE \subset G\backslash E = F$ a.e. m. Thus tE is an atom contained in F. This proves our claim. Now G is a union of its atomic part, which is a union of atoms, and a non-atomic part. In this case the measure of the non-atomic part is zero; otherwise it will contain an atom. Thus (vi) follows.

(v)$\Rightarrow$(ii). Suppose that G admits an atom, say E. We may assume that $e \in E$. First we *assert* that E is an atom if and only if E^{-1} is an atom. Since m is left invariant, it follows from [6, 2.32, p. 48] that $m(E) = 0$ if and only if $m(E^{-1}) = 0$. Let $F \subset E^{-1}$ be measurable. Then $F^{-1} \subset E$ is measurable. If $m(F) > 0$, then $m(F^{-1}) > 0$. Since $F^{-1} \subset E$, we have $F^{-1} = E$. Therefore $F = E^{-1}$. This proves our assertion.

Now define $g = m(E^{-1})^{-1}\chi_E$. Then $g \in L^1(G)$. We *claim* that g is the identity of $L^1(G)$. Let $f \in L^1(G)$ and $t \in G$. Then

$$(f \star g)(t) = \int_G f(s^{-1}t)g(s)\,dm(s) = \frac{1}{m(E^{-1})}\int_E f(s^{-1}t)\,dm(s)$$
$$= \frac{1}{m(E^{-1})}\int_{tE^{-1}} f(s)\,dm(s).$$

Note that tE^{-1} is an atom, $t \in tE^{-1}$, and every measurable function is constant on an atom. Therefore

$$(f \star g)(t) = \frac{1}{m(E^{-1})}f(t)\int_{tE^{-1}} dm(s) = f(t).$$

Hence $f \star g = f$; similarly, $g \star f = f$. Thus g is the identity of $L^1(G)$. It follows that G is discrete.

(vi)$\Rightarrow$(iv). Assume that G admits sufficiently many atoms. Now let $f, g \in L^p(G,\omega)$ be such that $f \neq g$. Then there exists an atom, say E, on which $f \neq g$. Also since every measurable function on an atom is constant, $f = c_f \neq c_g = g$ on E. Define $\varphi_E : L^p(G,\omega) \to \mathbb{C}$ as

$$\varphi_E(h) = \int_E h(s)\,dm_\omega(s) \qquad (h \in L^p(G,\omega)).$$

Then φ_E is a continuous linear functional on $L^p(G,\omega)$. Moreover, $\varphi_E(f) = m_\omega(E)c_f \neq m_\omega(E)c_g = \varphi_E(g)$. Thus (iv) follows.

(i)$\Rightarrow$(vi). Since G is discrete, it is clear that G is a union of atoms.

(iv)$\Rightarrow$(iii) is clear.

Since G is discrete, $L^p(G,\omega) = \ell^p(G,\omega)$. If ω is symmetric, then $\ell^p(G,\omega)$ is a $*$-subalgebra of $\ell^1(G)$. Since $\ell^1(G)$ admits a C^*-norm, it is an A^*-algebra. So, by [9, Theorem 4.1.19], $\ell^p(G,\omega)$ is semisimple. If G is abelian, then

$\ell^p(G,\omega)$ is a subalgebra of the commutative Banach algebra $\ell^1(G,\omega)$. Since $\ell^1(G,\omega)$ is semisimple [1], $\ell^p(G,\omega)$ is semisimple [9, Corollary 2.3.7]. ∎

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