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Johnson's projection, Kalton's property (M^*), and M-ideals of compact operators

by

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Abstract. Let X and Y be Banach spaces. We give a "non-separable" proof of the Kalton–Werner–Lima–Oja theorem that the subspace $\mathcal{K}(X,X)$ of compact operators forms an M-ideal in the space $\mathcal{L}(X,X)$ of all continuous linear operators from X to X if and only if X has Kalton's property (M^*) and the metric compact approximation property. Our proof is a quick consequence of two main results. First, we describe how Johnson's projection P on $\mathcal{L}(X,Y)^*$ applies to $f \in \mathcal{L}(X,Y)^*$ when f is represented via a Borel (with respect to the relative weak* topology) measure on $\overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*} \subset \mathcal{L}(X,Y)^*$: If Y^* has the Radon–Nikodým property, then P "passes under the integral sign". Our basic theorem en route to this description—a structure theorem for Borel probability measures on $\overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*}$—also yields a description of $\mathcal{K}(X,Y)^*$ due to Feder and Saphar. Second, we show that property (M^*) for X is equivalent to every functional in $\overline{B_{X^{**}} \otimes B_{X^*}}^{w^*}$ behaving as if $\mathcal{K}(X,X)$ were an M-ideal in $\mathcal{L}(X,X)$.

1. Introduction. Throughout this paper, X and Y will be Banach spaces over the same scalar field $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The closed unit ball and the unit sphere of X will be denoted, respectively, by B_X and S_X, and $\bar{B}(x,r)$ is the closed ball in X with center x and radius r. For a set $A \subset X$, we denote its convex hull by $\operatorname{co} A$, and its linear span by $\operatorname{span} A$. The symbol $\mathcal{L}(X,Y)$ will stand for the space of continuous linear operators from X to Y, and $\mathcal{K}(X,Y)$ for its subspace of compact operators. We shall write $\mathcal{L}(X)$ and $\mathcal{K}(X)$ instead of $\mathcal{L}(X,X)$ and $\mathcal{K}(X,X)$, respectively. The identity operator on X will be denoted by I_X or simply by I.

According to the terminology in [GKS], a closed subspace Z of X is said to be an *ideal* in X if there exists a continuous linear projection P on X^* with $\ker P = Z^\perp = \{x^* \in X^* : x^*|_Z = 0\}$ and $\|P\| = 1$. It is straightforward to verify that if Z is an ideal in X, then, for every $x^* \in X^*$, the functional $Px^* \in X^*$ is a norm-preserving extension of the restriction $x^*|_Z \in Z^*$. If the

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ideal projection P satisfies $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$ for all $x^* \in X^*$, then Z is said to be an *M-ideal* in X (for M-ideals, see the monograph [HWW]).

The space X is said to have the *metric compact approximation property* (briefly, *MCAP*) if there is a net (K_α) in $B_{\mathcal{K}(X)}$ such that $\lim_\alpha K_\alpha x = x$ for all $x \in X$. The net (K_α) is called a *metric compact approximation of the identity* (briefly, *MCAI*). If also $\lim_\alpha K_\alpha^* x^* = x^*$ for all $x^* \in X^*$, then (K_α) is called a *shrinking* MCAI, and X is said to have the *shrinking* MCAP.

Note that (see [J, proof of Lemma 1]) if (K_α) is any weak* convergent (in $\mathcal{K}(Y)^{**}$) MCAI of Y, then $\mathcal{K}(X,Y)$ is an ideal in $L(X,Y)$ with respect to the *Johnson projection* P on $\mathcal{L}(X,Y)^*$ defined by

$$Pf(T) = \lim_\alpha f(K_\alpha T), \qquad T \in \mathcal{L}(X,Y),\ f \in \mathcal{L}(X,Y)^*. \tag{1.1}$$

The space X is said to have *property* (M^*) (see [HWW, p. 296]) if whenever $x^*, u^* \in X^*$, $\|u^*\| \le \|x^*\|$, and $(x_\alpha^*) \subset X^*$ is a bounded net such that $x_\alpha^* \xrightarrow{w^*} 0$, one has

$$\limsup_\alpha \|u^* + x_\alpha^*\| \le \limsup_\alpha \|x^* + x_\alpha^*\|.$$

The following Kalton–Werner–Lima–Oja theorem characterizes M-ideals of compact operators on X.

Theorem 1.1. *The following assertions are equivalent.*

(i) *$\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$.*
(ii) *X has property (M^*) and the MCAP.*

Property (M^*) (in its sequential form) was introduced in [K] where it was proven that, for separable X, $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ if and only if X has property (M^*) and a very strong form of the MCAP; this result was extended to the non-separable case in [O1]. In [KW], Theorem 1.1 was proven for separable X, a simpler proof was given in [L]. Finally, in [O2], it was shown that $\mathcal{K}(X)$ *is an M-ideal in $\mathcal{L}(X)$ if and only if $\mathcal{K}(Z)$ is an M-ideal in $\mathcal{L}(Z)$ for all separable closed subspaces Z of X having the MCAP* (a somewhat simpler proof can be modeled after [P]), thus proving Theorem 1.1 also in the general case (note that if X has property (M^*), then also every closed subspace of X has property (M^*); moreover, X has property (M^*) if and only if every separable closed subspace of X has property (M^*) (see [O3])). The shortest known proof of Theorem 1.1 is given in [O3].

The aim of this paper is to give a direct "non-separable" proof of Theorem 1.1. We develop ideas from [L] and [O3].

Let us fix some more notation, point out some observations, and agree on some conventions.

Recall that, for $x^{**} \in X^{**}$ and $y^* \in Y^*$, the functional $x^{**} \otimes y^* \in \mathcal{L}(X,Y)^*$ is defined by $(x^{**} \otimes y^*)(T) = x^{**}(T^* y^*)$, $T \in \mathcal{L}(X,Y)$. Define further

$$B_{X^{**}} \otimes B_{Y^*} = \{x^{**} \otimes y^* : x^{**} \in B_{X^{**}},\, y^* \in B_{Y^*}\} \subset \mathcal{L}(X,Y)^*.$$

Let $\phi \in \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*}$. Observe that $\phi|_{\mathcal{K}(X,Y)} = x^{**} \otimes y^*|_{\mathcal{K}(X,Y)}$ for some $x^{**} \otimes y^* \in B_{X^{**}} \otimes B_{Y^*}$. Moreover, if $\phi|_{\mathcal{K}(X,Y)} \neq 0$, and $\tilde{x}^{**} \in X^{**}$ and $\tilde{y}^* \in Y^*$ are such that $\phi|_{\mathcal{K}(X,Y)} = \tilde{x}^{**} \otimes \tilde{y}^*|_{\mathcal{K}(X,Y)}$, then $\tilde{x}^{**} = \alpha x^{**}$ and $\tilde{y}^* = \alpha^{-1} y^*$ for some $\alpha \in \mathbb{K}$. Thus the functional $g_\phi := x^{**} \otimes y^* \in \mathcal{L}(X,Y)^*$ is well-defined.

Let us make the convention that, unless explicitly stated otherwise, whenever considering topological properties (such as compactness, openness, Borelness) of subsets of the sets $\overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*} \subset B_{\mathcal{L}(X,Y)^*}$, $B_{X^{**}}$, and B_{Y^*}, the topology we have in mind is the relative weak* topology of the respective set.

Since, for every $T \in \mathcal{L}(X,Y)$, there is some $\phi \in C := \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*} \subset B_{\mathcal{L}(X,Y)^*}$ such that $\operatorname{Re}\phi(T) = \|T\|$, by the Hahn–Banach separation theorem, it quickly follows that $\overline{\operatorname{co}}^{w^*} C = B_{\mathcal{L}(X,Y)^*}$. Thus, for every $f \in S_{\mathcal{L}(X,Y)^*}$, as a consequence of the Riesz representation theorem, there is a regular Borel probability measure μ on C such that $f(T) = \int_C \phi(T)\, d\mu(\phi)$, $T \in \mathcal{L}(X,Y)$. In Section 2, we prove the following characterization of Johnson's projection.

Theorem 1.2. *Let Y^* have the Radon–Nikodým property, let Y have the shrinking MCAP with $(K_\alpha) \subset B_{\mathcal{K}(Y)}$ being a weak* convergent (in $\mathcal{K}(Y)^{**}$) shrinking MCAI, and let μ be a regular Borel (with respect to the relative weak* topology) probability measure on $C := \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*} \subset B_{\mathcal{L}(X,Y)^*}$. Then there is a Borel set $C' \subset C$ such that*

(a) $\int_{C\setminus C'} |\phi(S)|\, d\mu(\phi) = 0$ *for all* $S \in \mathcal{K}(X,Y)$;
(b) *for every* $T \in \mathcal{L}(X,Y)$, *the function* $C \ni \phi \mapsto g_\phi(T)\chi_{C'}(\phi) \in \mathbb{K}$ *is measurable*;
(c) *letting* P *be the Johnson projection defined by* (1.1), *and defining* $f \in \mathcal{L}(X,Y)^*$ *by* $f(T) = \int_C \phi(T)\, d\mu(\phi)$, $T \in \mathcal{L}(X,Y)$, *one has*

$$Pf(T) = \int_{C'} g_\phi(T)\, d\mu(\phi) = \int_{C'} P\phi\,(T)\, d\mu(\phi), \qquad T \in \mathcal{L}(X,Y).$$

If $\mathcal{K}(X,Y)$ were an M-ideal in $\mathcal{L}(X,Y)$, then, for any $\phi \in \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*}$, one would have $\|g_\phi\| + \|\phi - g_\phi\| \le 1$. In Section 3, we prove the following theorem revealing the essence of property (M^*): Every $\phi \in \overline{B_{X^{**}} \otimes B_{X^*}}^{w^*}$ behaves, in a sense, like it would if $\mathcal{K}(X)$ were an M-ideal in $\mathcal{L}(X)$. We write $\mathfrak{L} := \operatorname{span}(\mathcal{K}(X) \cup \{I\}) \subset \mathcal{L}(X)$ and, for $f \in \mathcal{L}(X)^*$, $\|f\|_{\mathfrak{L}} := \|f|_{\mathfrak{L}}\|$.

THEOREM 1.3. *The following assertions are equivalent*:

(i) *X has property (M^*).*

(ii) *For every $\phi \in \overline{B_{X^{**}} \otimes B_{X^*}}^{w^*}$, one has $\|g_\phi\| + \|\phi - g_\phi\| \le 1$.*

(iii) *For every $\phi \in \overline{B_{X^{**}} \otimes B_{X^*}}^{w^*}$, one has $\|g_\phi\| + \|\phi - g_\phi\|_{\mathcal{L}} \le 1$.*

Theorems 1.2 and 1.3 put together easily yield (the implication (ii)⇒(i) of) Theorem 1.1. We also use Theorem 1.2 to indicate a large class of pairs of Banach spaces X and Y for which $\mathcal{K}(X,Y)$ has Phelps' property U in $\mathcal{L}(X,Y)$ (i.e., every functional $f \in \mathcal{K}(X,Y)^*$ has a unique norm-preserving extension to $\mathcal{L}(X,Y)$).

2. Proof of Theorem 1.2. Theorem 1.2 follows from

THEOREM 2.1. *Let Y^* (respectively, X^{**}) have the Radon–Nikodým property, and let μ be a regular Borel (with respect to the relative weak* topology) probability measure on $C := \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*} \subset B_{\mathcal{L}(X,Y)^*}$. Denote by $\mathcal{C}$ the collection of compact subsets A of C with the following property*:

- *there is a norm compact set $Y^*_A \subset S_{Y^*}$ (respectively, $X^{**}_A \subset S_{X^{**}}$) such that, for every $\phi \in A$, there are $y^* \in Y^*_A$ and $x^{**} \in B_{X^{**}}$ (respectively, $y^* \in B_{Y^*}$ and $x^{**} \in X^{**}_A$) with $g_\phi = x^{**} \otimes y^*$.*

Then there are pairwise disjoint Borel sets $C_j \subset C$, $j \in \{0\} \cup \mathbb{N}$, such that $C = \bigcup_{j=0}^{\infty} C_j$, where $\int_{C_0} |\phi(S)|\, d\mu(\phi) = 0$ for all $S \in \mathcal{K}(X,Y)$, and $C_j \in \mathcal{C}$, $j \in \mathbb{N}$.

Proof. Let $D \subset C$ be a Borel subset such that $\int_D |\phi(S)|\, d\mu(\phi) > 0$ for some $S \in S_{\mathcal{K}(X,Y)}$. By a standard exhaustion argument, it suffices to show that there is a subset $A \subset D$ with $A \in \mathcal{C}$ and $\mu(A) > 0$. Without loss of generality, we may assume that $|\phi(S)| = |g_\phi(S)| \ge 2\delta$ for some $\delta > 0$ and all $\phi \in D$, and that D is (weak*) compact. We consider only the case when Y^* has the Radon–Nikodým property. (The proof is symmetric if X^{**} has the Radon–Nikodým property.) Let $\mathcal{Y} \subset B_Y$ be a finite δ-net for $S^{**}[B_{X^{**}}]$. For each $y \in \mathcal{Y}$, define $L_y := B_{X^{**}} \cap (S^{**})^{-1}[\overline{B}(y,\delta)]$; then L_y is (weak*) compact, and thus the set

$$D_y := \{\phi \in D \colon g_\phi = x^{**} \otimes z^* \text{ for some } x^{**} \in L_y \text{ and } z^* \in B_{Y^*}\}$$

is also (weak*) compact. Moreover, for some $y \in \mathcal{Y}$, one must have $\mu(D_y) > 0$. For simplicity, we relabel L_y and D_y, respectively, as L and D.

Denote by $\mathcal{K}$ the collection of compact (in the relative weak* topology) subsets of B_{Y^*}, and let $K_\delta := \{y^* \in B_{Y^*} : y^*(y) = \delta\} \in \mathcal{K}$. For each $K \in \mathcal{K}$ and each compact subset $H \subset D$, define

$$C_K := \{\phi \in D \colon g_\phi = x^{**} \otimes ty^* \text{ for some } x^{**} \in L,\ y^* \in K \cap K_\delta, \text{ and } t \in \mathbb{K} \text{ with } ty^* \in B_{Y^*}\}$$

and

$$K_H := \{y^* \in K_\delta \colon g_\phi = x^{**} \otimes ty^* \\ \text{for some } \phi \in H,\ x^{**} \in L, \text{ and } t \in \mathbb{K} \text{ with } ty^* \in B_{Y^*}\}.$$

Observe that C_K is a compact (and thus Borel) (with respect to the relative weak* topology) subset of C, and $K_H \in \mathcal{K}$. Indeed, let $\phi \in D$, $x^{**} \in L$, $y^* \in K_\delta$, and $t \in \mathbb{K}$ with $ty^* \in B_{Y^*}$ be such that $g_\phi = x^{**} \otimes ty^*$. One has $\delta \le \|y^*\| \le 1$, and since

$$1 \ge |t|\,|y^*(y)| \ge |\phi(S)| - |\phi(S) - ty^*(y)| \ge 2\delta - \|ty^*\|\,\|S^{**}x^{**} - y\| \ge \delta,$$

we obtain $1 \le |t| \le 1/\delta$. The (weak*) compactness of both C_K and K_H now quickly follows. Notice also that $H \subset C_{K_H}$.

Observe that $\varrho\colon \mathcal{K} \ni K \mapsto \int_{C_K} |\phi(S)|\,d\mu(\phi) \in [0,1]$ is a regular content. To see that ϱ is regular, let $K \in \mathcal{K}$ and $\varepsilon > 0$. We have to find a $K' \in \mathcal{K}$ with $K'^\circ \supset K$ such that $\varrho(K') < \varrho(K) + \varepsilon$. To this end, choose a compact set $H \subset D \setminus C_K$ such that $\int_H |\phi(S)|\,d\mu(\phi) > \int_{D\setminus C_K} |\phi(S)|\,d\mu(\phi) - \varepsilon$. Since $K_H \cap K = \emptyset$, there are disjoint open (in the relative weak* topology) sets $U, V \subset B_{Y^*}$ such that $K \subset U$ and $K_H \subset V$. Letting $K' := B_{Y^*} \setminus V \in \mathcal{K}$ one has $K'^\circ \supset U \supset K$, and

$$\begin{aligned}\varrho(K') &= \int_{C_{K'}} |\phi(S)|\,d\mu(\phi) \le \int_{D\setminus C_{K_H}} |\phi(S)|\,d\mu(\phi) \le \int_{D\setminus H} |\phi(S)|\,d\mu(\phi) \\ &= \int_{C_K} |\phi(S)|\,d\mu(\phi) + \int_{D\setminus C_K} |\phi(S)|\,d\mu(\phi) - \int_H |\phi(S)|\,d\mu(\phi) < \varrho(K) + \varepsilon,\end{aligned}$$

as desired.

Let ν be the regular Borel (with respect to the relative weak* topology) measure on B_{Y^*} induced by the regular content ϱ, i.e., for a Borel set $E \subset B_{Y^*}$, $\nu(E) = \inf\{\lambda(U)\colon E \subset U \in \mathcal{U}\}$ where $\mathcal{U}$ is the collection of open subsets of B_{Y^*} and $\lambda(U) = \sup\{\varrho(K)\colon U \supset K \in \mathcal{K}\}$, $U \in \mathcal{U}$, is the inner content induced by ϱ. Since $C_{K_D} = D$, one has $\nu(K_D) \ge \varrho(K_D) = \int_D |\phi(S)|\,d\mu(\phi) > 0$. Since Y^* has the Radon–Nikodým property, by [B, Theorem 4.3.11,(a)⇒(b), and Lemmas 4.3.6 and 4.3.10], there is a norm compact set $K_0 \subset K_D$ such that $\nu(K_0) > 0$. Now we can take C_{K_0} to be the desired A, because $C_{K_0} \subset D$, $C_{K_0} \in \mathcal{C}$ (one can take $Y^*_{C_{K_0}} = \{y^*/\|y^*\|\colon y^* \in K_0\}$), and since by the regularity of ϱ, $\int_{C_{K_0}} |\phi(S)|\,d\mu(\phi) = \varrho(K_0) = \nu(K_0) > 0$, also $\mu(C_{K_0}) > 0$. ∎

Proof of Theorem 1.2. Let the sets C_j, $j \in \{0\}\cup\mathbb{N}$, be as in Theorem 2.1. Put $C' = \bigcup_{j=1}^\infty C_j$. Let $T \in S_{\mathcal{L}(X,Y)}$. Choose an increasing sequence of indices $(j_n)_{n=1}^\infty \subset \mathbb{N}$ so that $\mu(\bigcup_{j=j_n+1}^\infty C_j) < 1/n$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $A_n \subset S_{Y^*}$ be a finite $1/n$-net for $\bigcup_{j=1}^{j_n} Y^*_{C_j}$ where the sets $Y^*_{C_j}$ are as in Theorem 2.1. Choose an increasing sequence of indices $(\alpha_n)_{n=1}^\infty$ so that, whenever $n \in \mathbb{N}$, for each $\alpha \succeq \alpha_n$, one has $\|K^*_\alpha y^* - y^*\| < 1/n$ for all $y^* \in A_n$.

Now let $n \in \mathbb{N}$ be fixed and let $\alpha \succeq \alpha_n$. Suppose that $\phi \in \bigcup_{j=1}^{j_n} C_j$, and let $x^{**} \in B_{X^{**}}$ and $y^* \in Y^*_{C_j}$ ($j \in \{1, \ldots, j_n\}$) be such that $g_\phi = x^{**} \otimes y^*$. For some $y^*_\phi \in A_n$, one has $\|y^* - y^*_\phi\| < 1/n$. Thus

$$\begin{aligned}|g_\phi(T) - \phi(K_\alpha T)| &= |g_\phi(T) - g_\phi(K_\alpha T)| \le \|T^{**}x^{**}\| \, \|y^* - K^*_\alpha y^*\| \\ &\le \|y^* - y^*_\phi\| + \|y^*_\phi - K^*_\alpha y^*_\phi\| + \|K^*_\alpha\| \, \|y^*_\phi - y^*\| < \frac{3}{n}.\end{aligned}$$

It follows that $\phi(K_{\alpha_n} T) \to g_\phi(T)$ for each $\phi \in C'$; thus the function $C \ni \phi \mapsto g_\phi(T)\chi_{C'}(\phi) \in \mathbb{K}$ is measurable.

Letting again $n \in \mathbb{N}$ be fixed and $\alpha \succeq \alpha_n$, one has

$$\begin{aligned}\Big|\int_{C'} g_\phi(T)\, d\mu(\phi) - f(K_\alpha T)\Big| &\le \int_{C'} |g_\phi(T) - \phi(K_\alpha T)| \, d\mu(\phi) \\ &= \int_{\bigcup_{j=1}^{j_n} C_j} |g_\phi(T) - \phi(K_\alpha T)| \, d\mu(\phi) + \int_{\bigcup_{j=j_n+1}^{\infty} C_j} |g_\phi(T) - \phi(K_\alpha T)| \, d\mu(\phi) \\ &< \frac{3}{n} + \frac{2}{n} = \frac{5}{n},\end{aligned}$$

and it follows that $Pf(T) = \lim_\alpha f(K_\alpha T) = \int_{C'} g_\phi(T)\, d\mu(\phi)$. ∎

Remark 2.1. The assumption in Theorem 1.2 that (K_α) is weak* convergent (in $\mathcal{K}(Y)^{**}$) is, in fact, superfluous: A description of $\mathcal{K}(X,Y)^*$ due to Feder and Saphar (see [FS, Theorem 1] or Corollary 2.2 below) implies that if Y^* has the Radon–Nikodým property, then every shrinking MCAI of Y is weak* convergent (in $\mathcal{K}(Y)^{**}$).

Remark 2.2. Suppose that, in Theorem 1.2, Y is separable. Then Y has a shrinking MCAI which is a sequence, label it $(K_n)_{n=1}^\infty$. By [FS, Theorem 1] (or Corollary 2.2 below), one has $Pg\,(T) = \lim_{n\to\infty} Pg(K_n T) = \lim_{n\to\infty} g(K_n T)$ for every $g \in \mathcal{L}(X,Y)^*$ and every $T \in \mathcal{L}(X,Y)$ (for details, see [P, Lemma 1.2]). Thus, for any $T \in \mathcal{L}(X,Y)$, by Lebesgue's bounded convergence theorem,

$$\begin{aligned}Pf(T) &= \lim_{n\to\infty} f(K_n T) = \lim_{n\to\infty} \int_C \phi(K_n T)\, d\mu(\phi) = \lim_{n\to\infty} \int_C g_\phi(K_n T)\, d\mu(\phi) \\ &= \int_C \lim_{n\to\infty} g_\phi(K_n T)\, d\mu(\phi) = \int_C g_\phi(T)\, d\mu(\phi) = \int_C P\phi(T)\, d\mu(\phi).\end{aligned}$$

Notice that the Feder–Saphar description of $\mathcal{K}(X,Y)^*$ which was used in Remarks 2.1 and 2.2 is, in fact, a consequence of Theorem 2.1.

Corollary 2.2 (see [FS, Theorem 1]). *Suppose that X^{**} or Y^* has the Radon–Nikodým property, and let $g \in \mathcal{K}(X,Y)^*$ and $\varepsilon > 0$. Then there are $x_j^{**} \in X^{**}$ and $y_j^* \in Y^*$, $j \in \mathbb{N}$, such that $g = \sum_{j=1}^\infty x_j^{**} \otimes y_j^*$ and $\sum_{j=1}^\infty \|x_j^{**}\| \, \|y_j^*\| < \|g\| + \varepsilon$.*

Proof. It suffices to show that there are $n \in \mathbb{N}$, $x_1^{**}, \ldots, x_n^{**} \in X^{**}$, and $y_1^*, \ldots, y_n^* \in Y^*$ such that $\|g - \sum_{j=1}^n x_j^{**} \otimes y_j^*\| < \varepsilon$ and $\sum_{j=1}^n \|x_j^{**}\|\,\|y_j^*\| \le \|g\|$. One may clearly assume that $\|g\| = 1$.

Let $f \in S_{\mathcal{L}(X,Y)^*}$ be some extension of g. As explained in the Introduction, there is a regular Borel (with respect to the relative weak* topology) probability measure μ on $C := \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*} \subset B_{\mathcal{L}(X,Y)^*}$ such that $f(T) = \int_C \phi(T)\, d\mu(\phi)$, $T \in \mathcal{L}(X,Y)$. Now, in Theorem 2.1, one has $\mu(C_0) = 0$, and one may also assume that $\hat{C} := C \setminus C_0 \in \mathcal{C}$.

We only consider the case when Y^* has the Radon–Nikodým property. (The proof is symmetric if X^{**} has the Radon–Nikodým property.) Let $\{y_1^*, \ldots, y_n^*\} \subset S_{Y^*}$ ($n \in \mathbb{N}$) be an $\varepsilon/3$-net for the set $Y_{\hat{C}}^*$ from Theorem 2.1. Choose $y_j \in S_Y$ such that $|y_j^*(y_j) - 1| < \varepsilon/3$, $j \in \{1, \ldots, n\}$. For each $j \in \{1, \ldots, n\}$, the set

$$B_j := \{\phi \in \hat{C} : g_\phi = x_\phi^{**} \otimes y_\phi^* \text{ for some } x_\phi^{**} \in B_{X^{**}} \text{ and } y_\phi^* \in Y_{\hat{C}}^* \text{ with } \|y_\phi^* - y_j^*\| \le \varepsilon/3\}$$

is (weak*) compact; thus the set $E_j := B_j \setminus \bigcup_{i=1}^{j-1} B_i$ is Borel, and we may define $x_j^{**} \in X^{**}$ by $x_j^{**}(x^*) = \int_{E_j} \phi(x^* \otimes y_j)\, d\mu(\phi) = \int_{E_j} g_\phi(x^* \otimes y_j)\, d\mu(\phi)$, $x^* \in X^*$. Now, whenever $j \in \{1, \ldots, n\}$, one has $\|x_j^{**}\|\,\|y_j^*\| \le \mu(E_j)$, and since, for all $\phi \in E_j$,

$$\|y_\phi^* - y_\phi^*(y_j)\, y_j^*\| \le |1 - y_j^*(y_j)|\, \|y_\phi^*\| + |y_j^*(y_j) - y_\phi^*(y_j)|\, \|y_\phi^*\| + |y_\phi^*(y_j)|\, \|y_\phi^* - y_j^*\| < \varepsilon,$$

one has, for every $S \in B_{\mathcal{K}(X,Y)}$,

$$\begin{aligned} \Big| \int_{E_j} \phi(S)\, d\mu(\phi) - (x_j^{**} \otimes y_j^*)(S) \Big| &= \Big| \int_{E_j} g_\phi(S)\, d\mu(\phi) - x_j^{**}(S^* y_j^*) \Big| \\ &= \Big| \int_{E_j} g_\phi(S)\, d\mu(\phi) - \int_{E_j} g_\phi(S^* y_j^* \otimes y_j)\, d\mu(\phi) \Big| \\ &= \Big| \int_{E_j} x_\phi^{**}(S^* y_\phi^*)\, d\mu(\phi) - \int_{E_j} x_\phi^{**}(S^* y_j^*) y_\phi^*(y_j)\, d\mu(\phi) \Big| \\ &\le \int_{E_j} \|S^{**} x_\phi^{**}\|\, \|y_\phi^* - y_\phi^*(y_j) y_j^*\|\, d\mu(\phi) < \mu(E_j)\varepsilon. \end{aligned}$$

It follows that $\sum_{j=1}^n \|x_j^{**}\|\,\|y_j^*\| \le \|g\|$, and, for every $S \in B_{\mathcal{K}(X,Y)}$,

$$\begin{aligned} \Big| g(S) - \sum_{j=1}^n (x_j^{**} \otimes y_j^*)(S) \Big| &= \Big| \sum_{j=1}^n \int_{E_j} \phi(S)\, d\mu(\phi) - \sum_{j=1}^n (x_j^{**} \otimes y_j^*)(S) \Big| \\ &\le \sum_{j=1}^n \Big| \int_{E_j} \phi(S)\, d\mu(\phi) - (x_j^{**} \otimes y_j^*)(S) \Big| < \varepsilon, \end{aligned}$$

as desired. ∎

3. Proofs of Theorems 1.3 and 1.1. The implication (i)⇒(ii) of Theorem 1.3 is contained in

Proposition 3.1. *Let both X and Y have property (M^*). Then, for any $\phi \in \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*}$, one has $\|g_\phi\| + \|\phi - g_\phi\| \le 1$.*

Proof. Let $\phi \in \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*}$ and let $\phi_\alpha = x_\alpha^{**} \otimes y_\alpha^* \in B_{X^{**}} \otimes B_{Y^*}$ be such that w^*-$\lim_\alpha \phi_\alpha = \phi$ in $\mathcal{L}(X,Y)^*$. We may assume that w^*-$\lim_\alpha x_\alpha^{**} = x^{**}$ in X^{**} and w^*-$\lim_\alpha y_\alpha^* = y^*$ in Y^* for some $x^{**} \in B_{X^{**}}$ and $y^* \in B_{Y^*}$. Write $g = g_\phi = x^{**} \otimes y^*$ and $h = \phi - g$. We must show that $\|g\| + \|h\| \le 1$. The case $y^* = 0$ is trivial, so assume that $y^* \ne 0$. Fix arbitrary $S \in S_{\mathcal{K}(X,Y)}$ with $S^* y^* \ne 0$ and $T \in S_{\mathcal{L}(X,Y)}$. It suffices to show that $|g(S) + h(T)| \le 1$. To this end, pick $y_n \in S_Y$, $n \in \mathbb{N}$, such that $y^*(y_n) \to \|y^*\|$ and denote $K_n = (y^*/\|y^*\|) \otimes y_n \in B_{\mathcal{K}(Y)}$, $n \in \mathbb{N}$. Then $K_n^* y^* = y^*(y_n) y^*/\|y^*\| \to y^*$, thus

$$g(K_n T) = x^{**}(T^* K_n^* y^*) = T^{**} x^{**}(K_n^* y^*) \to T^{**} x^{**}(y^*) = g(T).$$

Fix an arbitrary $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that $\|K_n^* y^* - y^*\| < \varepsilon$ and $|g(K_n T) - g(T)| < \varepsilon$. Find $v^* \in B_{Y^*}$ with $\|v^*\| \le \|y^*\|$ such that $\|T^* v^*\| > \|S^* y^*\|/(1+\varepsilon)$ and $x \in B_X$ such that $(T^* v^*)(x) = \|S^* y^*\|/(1+\varepsilon)$, and put $U = (S^* y^*/\|S^* y^*\|) \otimes x \in B_{\mathcal{K}(X)}$. Then

$$U^* T^* v^* = T^* v^*(x) \frac{S^* y^*}{\|S^* y^*\|} = \frac{1}{1+\varepsilon} S^* y^*,$$

thus $S^* y^* = (1+\varepsilon) U^* T^* v^*$. Now, since X^* and Y^* have property (M^*), one sees that

$$\begin{aligned}
&|g(S) + h(T)| = |\phi(S) + h(T) + g(T) - g(T) + g(K_n T) - g(K_n T)| \\
&\quad \le |\phi(S + T - K_n T)| + |g(K_n T) - g(T)| \\
&\quad < \lim_\alpha |x_\alpha^{**}(S^* y_\alpha^* + T^* y_\alpha^* - T^* K_n^* y_\alpha^*)| + \varepsilon \\
&\quad \le \limsup_\alpha \|S^* y^* + T^* y_\alpha^* - T^* K_n^* y^*\| + \varepsilon \\
&\quad \le \limsup_\alpha \|U^* T^* v^* + T^* y_\alpha^* - T^* y^*\| + \varepsilon \|U^* T^* v^*\| + \|T^* y^* - T^* K_n^* y^*\| + \varepsilon \\
&\quad \le \limsup_\alpha \|T^* v^* + T^*(y_\alpha^* - y^*)\| + \|T^*\| \|y^* - K_n^* y^*\| + 2\varepsilon \\
&\quad \le \limsup_\alpha \|v^* + y_\alpha^* - y^*\| + 3\varepsilon \le \limsup_\alpha \|y^* + y_\alpha^* - y^*\| + 3\varepsilon \le 1 + 3\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ yields $|g(S) + h(T)| \le 1$, as desired. ∎

Observe that, if X is infinite-dimensional, then whenever $S \in \mathcal{K}(X)$ and $\lambda \in \mathbb{K}$ are such that $\|S + \lambda I\| < 1$, one has $|\lambda| < 1$ (because otherwise $\|(1/\lambda) S + I\| < 1$ and thus $(1/\lambda) S$ would be invertible). Hence, for all h in

$\mathcal{K}(X)^\perp \subset \mathcal{L}(X)^*$, one has $\|h\|_{\mathcal{L}} = |h(I)|$ because

$$\begin{aligned}\|h\|_{\mathcal{L}} &= \sup\{|h(S+\lambda I)| \colon S \in \mathcal{K}(X),\ \lambda \in \mathbb{K},\ \|S+\lambda I\| < 1\}\\ &= \sup\{|\lambda||h(I)| \colon S \in \mathcal{K}(X),\ \lambda \in \mathbb{K},\ \|S+\lambda I\| < 1\} \le |h(I)| \le \|h\|_{\mathcal{L}}.\end{aligned}$$

Proof of Theorem 1.3. (i)⇒(ii) is obvious from Proposition 3.1.

(ii)⇒(iii) is more than obvious.

(iii)⇒(i). Let (iii) hold, let $x^*, u^* \in X^*$ be such that $\|u^*\| \le \|x^*\|$, and let $(x^*_\alpha) \subset X^*$ be a bounded weak* null net. We must show that

$$\limsup_\alpha \|u^* + x^*_\alpha\| \le \limsup_\alpha \|x^* + x^*_\alpha\|.$$

We may assume that $\|u^*\| < \|x^*\|$ and $\limsup_\alpha \|u^* + x^*_\alpha\| = \lim_\alpha \|u^* + x^*_\alpha\|$. In this case $M := \limsup_\alpha \|x^* + x^*_\alpha\| > 0$ (because otherwise we would have $x^*_\alpha \to -x^*$ in norm, hence also $x^*_\alpha \to -x^*$ weak* and thus $x^* = 0$ implying that $\|u^*\| < 0$); thus we may assume that $M_\alpha := \|x^* + x^*_\alpha\| > 0$ for all α and also that $M_\alpha \to M$. Pick $S \in B_{\mathcal{K}(X)}$ such that $S^*x^* = u^*$ (note that such a rank one S exists). By passing to product index, we may assume that there is a net $(x_\alpha) \subset S_X$ such that

$$\lim_\alpha \|S^*x^* + x^*_\alpha\| = \lim_\alpha |S^*x^*(x_\alpha) + x^*_\alpha(x_\alpha)|.$$

Considering $\phi_\alpha := x_\alpha \otimes M_\alpha^{-1}(x^* + x^*_\alpha) \in B_{X^{**}} \otimes B_{X^*}$, we may assume that $w^*\text{-}\lim_\alpha \phi_\alpha = \phi$ in $\mathcal{L}(X)^*$ for some $\phi \in \overline{B_{X^{**}} \otimes B_{X^*}}^{w^*}$ and that $w^*\text{-}\lim_\alpha x_\alpha = x^{**}$ in X^{**} for some $x^{**} \in B_{X^{**}}$. Then $g_\phi = M^{-1}x^{**} \otimes x^*$ and $\phi - g_\phi = M^{-1}\, w^*\text{-}\lim_\alpha x_\alpha \otimes x^*_\alpha$. By (iii), one has

$$\begin{aligned}\limsup_\alpha \|u^* + x^*_\alpha\| &= \lim_\alpha |S^*x^*(x_\alpha) + x^*_\alpha(x_\alpha)| = |Mg_\phi(S) + M(\phi - g_\phi)(I)|\\ &\le M(\|g_\phi\| + \|\phi - g_\phi\|_{\mathcal{L}}) \le M = \limsup_\alpha \|x^* + x^*_\alpha\|. \ \blacksquare\end{aligned}$$

REMARK 3.1. In [L, Theorem 2.2] Å. Lima proved, combining knowledge on weak* strongly exposed points of B_{X^*} with a clever slice-cutting technique, that if $\mathcal{K}(X)$ is a semi-M-ideal in $\operatorname{span}(\mathcal{K}(X) \cup \{I\})$, then X has property (M^*). This result is an immediate consequence of our Theorem 1.3(iii)⇒(i), whose proof was more or less elementary.

The following corollary is well known. Our Theorem 1.3 yields a very simple proof for it.

COROLLARY 3.2 (see [HWW, p. 297]). *Let X have property (M^*). Then X is an M-ideal in X^{**}.*

Proof. Let $x^{***} = x^* + x^\perp \in S_{X^{***}}$ (with $x^* \in X^*$, $x^\perp \in X^\perp$), and let $\varepsilon > 0$. It suffices to show that $\|x^*\| + \|x^\perp\| \le 1 + \varepsilon$. To this end, pick

$x^{**} \in S_{X^{**}}$ satisfying $|x^{\perp}(x^{**})| \geq \|x^{\perp}\| - \varepsilon$, and observe that the functional

$$\phi = x^{**} \otimes x^{***} \colon \mathcal{L}(X) \ni T \mapsto x^{***}(T^{**}x^{**})$$

is in $\overline{B_{X^{**}} \otimes B_{X^*}}^{w^*}$ (because whenever a net $(x^*_\alpha) \subset B_{X^*}$ is such that $x^*_\alpha \to x^{***}$ weak* in X^{***}, then $x^{**} \otimes x^*_\alpha \to \phi$ weak* in $\mathcal{L}(X)^*$). Clearly, $g_\phi = x^{**} \otimes x^*$ and thus, since

$$\|\phi - g_\phi\| \geq |(\phi - g_\phi)(I)| = |x^{\perp}(x^{**})| \geq \|x^{\perp}\| - \varepsilon,$$

by Theorem 1.3,

$$\|x^*\| + \|x^{\perp}\| = \|g_\phi\| + \|x^{\perp}\| \leq \|g_\phi\| + \|\phi - g_\phi\| + \varepsilon \leq 1 + \varepsilon. \ ■$$

Now we are in a position to prove Theorem 1.1. The implication (ii)⇒(i) is the particular case with $Y = X$ of the known

Proposition 3.3 (cf. Theorem 1.1 combined with [O1, Theorem 8]). *Let both X and Y have property (M^*), and let Y have the MCAP. Then $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$.*

Proof. Let $(K_\alpha) \subset B_{\mathcal{K}(Y)}$ be an MCAI. By Corollary 3.2, Y is an M-ideal in its bidual; hence B_{Y^*} is the norm closed convex hull of its weak* strongly exposed points (see [HWW, p. 127, Corollary 3.2]). It easily follows that (K_α) is shrinking; thus (1.1) defines an ideal projection P on $\mathcal{L}(X,Y)^*$ by Remark 2.1.

Let $f \in S_{\mathcal{L}(X,Y)^*}$, and let $T_1, T_2 \in B_{\mathcal{L}(X,Y)}$ be arbitrary. It suffices to show that $|Pf(T_1)| + |(I-P)f(T_2)| \leq 1$. As explained in the Introduction, there is a regular Borel probability measure μ on $C := \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*} \subset B_{\mathcal{L}(X,Y)^*}$ such that $f(T) = \int_C \phi(T)\, d\mu(\phi)$, $T \in \mathcal{L}(X,Y)$. By Theorem 1.3, one has $|g_\phi(T_1)| + |(\phi - g_\phi)(T_2)| \leq \|g_\phi\| + \|\phi - g_\phi\| \leq 1$ for all $\phi \in C$. Thus, letting the set C' be as in Theorem 1.2 (notice that, since Y is an M-ideal in its bidual, the dual Y^* enjoys the Radon–Nikodým property—see [HWW, p. 126, Theorem 3.1]), one has

$$\begin{aligned}
&|Pf(T_1)| + |(I-P)f(T_2)| \\
&\quad = \Big|\int_{C'} g_\phi(T_1)\, d\mu(\phi)\Big| + \Big|\int_{C'} (\phi - g_\phi)(T_2)\, d\mu(\phi) + \int_{C\setminus C'} \phi(T_2)\, d\mu(\phi)\Big| \\
&\quad \leq \int_{C'} (|g_\phi(T_1)| + |(\phi - g_\phi)(T_2)|)\, d\mu(\phi) + \mu(C \setminus C') \\
&\quad \leq \mu(C') + \mu(C \setminus C') = 1. \ ■
\end{aligned}$$

Proof of Theorem 1.1. (ii)⇒(i) is immediate from Proposition 3.3.

(i)⇒(ii). Let $\mathcal{K}(X)$ be an M-ideal in $\mathcal{L}(X)$ with $P \in \mathcal{L}(X)^*$ being the ideal projection. Property (M^*) for X follows immediately from the implication (ii)⇒(i) of Theorem 1.3. The argument to obtain the MCAP for X is well known: By Goldstine's theorem (or by the bipolar theorem), $B_{\mathcal{K}(X)}$ is dense in $B_{\mathcal{L}(X)}$ in the weak topology $\sigma(\mathcal{L}(X), \operatorname{ran} P)$. Thus there is a net $(K_\alpha) \subset B_{\mathcal{K}(X)}$ such that $Pf(K_\alpha) \to Pf(I_X)$ for all $f \in \mathcal{L}(X)^*$. In particular, $x^*(K_\alpha x) \to x^*(I_X x)$ for all $x \in X$ and all $x^* \in X^*$ (because $P(x \otimes x^*) = x \otimes x^*$), i.e., $K_\alpha \to I_X$ in the weak operator topology of $\mathcal{L}(X)$. Since the weak and strong operator topologies yield the same dual space (see, e.g., [DSch, Theorem VI.1.4]), after passing to convex combinations, we may assume that $K_\alpha x \to x$ for all $x \in X$, and thus X has the MCAP. ∎

We conclude by showing how Theorem 1.2 yields a result which produces multiple examples of pairs of Banach spaces X and Y for which $\mathcal{K}(X, Y)$ has Phelps' property U in $\mathcal{L}(X, Y)$. Recall that a closed subspace Z of X is said to have (*Phelps*') *property* U in X if every $z^* \in Z^*$ admits a unique norm-preserving extension $x^* \in X^*$.

THEOREM 3.4. *Let Y^* have the Radon–Nikodým property, let Y have the shrinking MCAP, and suppose that, for every $x^{**} \in S_{X^{**}}$ and every $y^* \in S_{Y^*}$, the functional $x^{**} \otimes y^* \in \mathcal{L}(X, Y)^*$ itself is the only norm-preserving extension of its restriction to $\mathcal{K}(X, Y)$. Then $\mathcal{K}(X, Y)$ has property U in $\mathcal{L}(X, Y)$.*

REMARK 3.2. By a result of Å. Lima (see [L, Lemma 3.4]; see also [OP] for a recent easier proof), $x^{**} \otimes y^* \in \mathcal{L}(X, Y)^*$ itself is the only norm-preserving extension of its restriction to $\mathcal{K}(X, Y)$ whenever $x^{**} \in B_X \subset B_{X^{**}}$ is a denting point of B_X or $y^* \in B_{Y^*}$ is a weak* denting point of B_{Y^*}. It is known (see [LLT1] and [LLT2]) that *a point $x \in B_X$ is a denting point of B_X if and only if it is both an extreme point and a point of weak-to-norm continuity of B_X*; moreover, *a point $y^* \in B_{Y^*}$ is a weak* denting point of B_{Y^*} if and only if it is both an extreme point and a point of weak*-to-norm continuity of B_{Y^*}.*

Proof of Theorem 3.4. Let $(K_\alpha) \subset B_Y$ be a shrinking MCAI of Y, and let P be the Johnson projection on $\mathcal{L}(X, Y)^*$ defined by (1.1) (notice that (K_α) is weak* convergent (in $\mathcal{K}(Y)^{**}$) by Remark 2.1). Then $P\phi = g_\phi$ for all $\phi \in \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*} =: C$.

Let $f \in S_{\mathcal{L}(X,Y)^*}$ be such that $\|Pf\| = \|f\| = 1$. It suffices to show that $Pf = f$. As explained in the Introduction, there is a regular Borel (with respect to the relative weak* topology) probability measure μ on C representing f, i.e., $f(T) = \int_C \phi(T)\, d\mu(\phi)$ for all $T \in \mathcal{L}(X, Y)$. Since $\|Pf\| = \|f|_{\mathcal{K}(X,Y)}\|$, one has, in Theorem 1.2, $\mu(C \setminus C') = 0$. Set $C_1 := \{\phi \in C' : \|g_\phi\| = 1\}$. Then $\mu(C' \setminus C_1) = 0$ (the function $C \ni \phi \mapsto \|g_\phi\|$ is measurable since it is lower semicontinuous) because otherwise $\int_{C' \setminus C_1} \|g_\phi\|\, d\mu(\phi) <$

$\mu(C' \setminus C_1)$ and thus

$$\begin{aligned}\|Pf\| &= \sup_{T\in B_{\mathcal{L}(X,Y)}} |Pf(T)| = \sup_{T\in B_{\mathcal{L}(X,Y)}} \Big|\int_{C'} g_\phi(T)\,d\mu(\phi)\Big| \\ &\le \sup_{T\in B_{\mathcal{L}(X,Y)}} \int_{C'} |g_\phi(T)|\,d\mu(\phi) \le \int_{C'} \|g_\phi\|\,d\mu(\phi) \\ &= \int_{C_1} \|g_\phi\|\,d\mu(\phi) + \int_{C'\setminus C_1} \|g_\phi\|\,d\mu(\phi) < \mu(C_1) + \mu(C'\setminus C_1) = 1.\end{aligned}$$

By our assumption, for any $\phi \in C_1$, one has $g_\phi = \phi$. From Theorem 1.2 it now follows that, for any $T \in \mathcal{L}(X,Y)$,

$$Pf(T) = \int_{C_1} g_\phi(T)\,d\mu(\phi) = \int_{C_1} \phi(T)\,d\mu(\phi) = f(T). \ \blacksquare$$

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