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Strict u-ideals in Banach spaces

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Abstract. We study strict u-ideals in Banach spaces. A Banach space X is a strict u-ideal in its bidual when the canonical decomposition $X^{***} = X^* \oplus X^\perp$ is unconditional. We characterize Banach spaces which are strict u-ideals in their bidual and show that if X is a strict u-ideal in a Banach space Y then X contains c_0. We also show that ℓ_∞ is not a u-ideal.

1. Introduction. Strict u-ideals were introduced by Godefroy, Kalton and Saphar in [9]. Let X be a subspace of a Banach space Y. We will say that X is a *summand* of Y if it is the range of a contractive projection and that X is an *ideal* in Y if $X^\perp$ is the kernel of a contractive projection on Y^*.

A norm one operator $\phi : X^* \to Y^*$ such that $\phi(x^*)(x) = x^*(x)$ is said to be a *Hahn–Banach extension operator*. The set of all such ϕ is denoted by $\mathrm{HB}(X, Y)$. For every $\phi \in \mathrm{HB}(X, Y)$ we have

$$Y^* = X^\perp \oplus \phi(X^*).$$

Let i_X be the natural embedding $i_X : X \to Y$. Then $P_\phi = \phi \circ i_X^*$ is a norm one projection on Y^* with $\ker P_\phi = X^\perp$. X is an ideal in Y if and only if $\mathrm{HB}(X, Y) \neq \emptyset$ (see [8, Theorem 2.4]). If we have $\|x^\perp + \phi(x^*)\| = \|x^\perp - \phi(x^*)\|$ for all $x^\perp \in X^\perp$ and $x^* \in X^*$ we say that X is a *u-ideal* in Y and that ϕ is *unconditional*. Note that ϕ is unconditional if and only if $\|I - 2P_\phi\| = 1$. We get the well-known notion of an *M-ideal* ([3], [12]) if $\|x^\perp + \phi(x^*)\| = \|x^\perp\| + \|\phi(x^*)\|$ for all $x^\perp \in X^\perp$ and $x^* \in X^*$.

We get another useful viewpoint by defining a norm one operator $T_\phi : Y \to X^{**}$ by

$$\langle i_X^* y^*, T_\phi(y)\rangle = \langle y, P_\phi(y^*)\rangle \tag{1.1}$$

for all $y \in Y$ and $y^* \in Y^*$. Then $T_\phi(x) = x$ for all $x \in X$. Note that $T_\phi = \phi^*|_Y$.

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X is a *strict ideal* in Y if there is a $\phi \in \mathrm{HB}(X,Y)$ such that $\phi(X^*)$ is norming. In this case ϕ is called *strict*. That ϕ is strict is equivalent to the existence for every $y \in Y$ and $\varepsilon > 0$ of an $x^* \in B_{X^*}$ such that

$$\|y\| - \varepsilon < \langle \phi(x^*), y\rangle = \langle x^*, T_\phi(y)\rangle.$$

Since $|\langle x^*, T_\phi(y)\rangle| \le \|T_\phi\|\,\|x^*\|\,\|y\|$ we see that ϕ is strict if and only if $T_\phi : Y \to X^{**}$ is isometric.

In this paper we study strict u-ideals, i.e. ideals for which the Hahn–Banach extension operator is both strict and unconditional. Godefroy, Kalton and Saphar note in the introduction to their paper [9] that the theory of u-ideals is much less satisfactory and complete than in the complex case of h-ideals (which we will not discuss). We aim to fill a few of the gaps in the theory of u-ideals.

In Section 2 we use the local and geometric description of u-ideals the authors obtained in [15] to develop similar tools needed to study strict u-ideals. We obtain a characterization of when a space of codimension one is a strict u-ideal (see Theorem 2.4), and when a Banach space is a strict u-ideal in its bidual (see Theorems 2.8 and 2.9). Some of these results were first shown by Godefroy, Kalton and Saphar under the assumption that X was separable or did not contain ℓ_1. We also show that if X is a non-trivial subspace of a Banach space Y, then X contains a copy of c_0 whenever X is a strict u-ideal in Y. In Theorem 2.12 we show that if a dual space X^* is a u-ideal in its bidual, then it is in fact a u-summand. In particular, it can never be a strict u-ideal. The proof relies on the fact that ℓ_∞ is not a u-ideal in its bidual (see Theorem 2.11).

In Section 3 we look at denting points and strongly exposed points in the unit ball of the dual of X when X is a strict u-ideal in its bidual.

We use standard Banach space notation. For a Banach space X, B_X is the closed unit ball and S_X is the unit sphere. The canonical embedding $X \to X^{**}$ is denoted by k_X. If A is a subset of X, $\mathrm{span}(A)$ is the linear span of A and $\mathrm{conv}(A)$ is the convex hull of A.

We consider real Banach spaces only.

2. Strict u-ideals. First we show that to check whether a u-ideal is strict or not it is enough to check one direction at a time.

PROPOSITION 2.1. *Assume X is a u-ideal in Y. Then X is a strict u-ideal in Y if and only if X is a strict u-ideal in* $\mathrm{span}(X,\{y\})$ *for all $y \in Y$.*

Proof. Let $\phi \in \mathrm{HB}(X,Y)$ be unconditional. As noted in the introduction, ϕ is strict if and only if T_ϕ is isometric (notation of (1.1)). But by Lemma 2.2 and 3.1 in [15], $T_\phi(y)$ is uniquely and locally determined. ∎

Recall that an element c in a convex set K is a *center of symmetry* if $2c - x \in K$ for all $x \in K$.

PROPOSITION 2.2. *If X is a strict u-ideal in Y then every element of $\mathrm{HB}(X,Y)$ is strict.*

Proof. Assume $\phi \in \mathrm{HB}(X,Y)$ is unconditional and strict. Then ϕ is a center of symmetry in $\mathrm{HB}(X,Y)$ (see e.g. [1, Proposition 2.2]) so that $2\phi - \psi \in \mathrm{HB}(X,Y)$ for all $\psi \in \mathrm{HB}(X,Y)$. Let $\psi \in \mathrm{HB}(X,Y)$ and $y \in Y$. Then

$$\|y\| \geq \|(2T_\phi - T_\psi)(y)\| \geq 2\|T_\phi(y)\| - \|T_\psi(y)\| = 2\|y\| - \|T_\psi(y)\|.$$

Hence $\|T_\psi(y)\| = \|y\|$ and ψ is strict. ∎

Let us introduce some more notation. Assume X is a closed subspace of a Banach space Y. For each $y \in Y \setminus X$ define

$$D_y = X^{**} \cap \bigcap_{x \in X} B_{X^{**}}(x, \|x - y\|). \tag{2.1}$$

It is a convex and weak*-compact subset of X^{**}. Let $Z = \mathrm{span}(X, \{y\})$. There is a one-to-one correspondence between D_y and $\mathrm{HB}(X,Z)$ given by $\phi \leftrightarrow T_\phi(y)$. (If $d_y \in D_y$ define $T : Z \to X^{**}$ by $T(ay + x) = ad_y + x$. See also Lemma 2.2 in [14].) Note that $D_{ay} = aD_y$ for $a \in \mathbb{R}$.

In view of the previous two propositions the following corollary is obvious.

COROLLARY 2.3. *Assume X is a u-ideal in Y. Then it is a strict u-ideal if and only if $D_y \subset S_{X^{**}}$ for all $y \in S_Y$.*

In Proposition 2.1 we saw that it is enough to check strictness of a u-ideal one direction at a time. Next we characterize strict u-ideals of codimension one.

THEOREM 2.4. *Let X be a closed subspace of a Banach space Y. Let $y \in Y \setminus X$ and $Z = \mathrm{span}(X, \{y\})$. Assume that X is a u-ideal in Z. The following statements are equivalent.*

(a) *X is a strict u-ideal in Z.*
(b) *For every $z \in S_Z$ we have* $\inf_{x \in S_X} \|z - 2x\| = 1$.
(c) *For every $z \in S_Z$ and $\varepsilon > 0$ there exists $x \in S_X$ such that*

$$B_X(0, 1-\varepsilon) \cap B_X(2x, \|z - 2x\|) = \emptyset.$$

Proof. (a)⇒(b). Let $\phi \in \mathrm{HB}(X,Z)$ be unconditional and strict and let $z \in S_Z$. Then $\|T_\phi(z)\| = 1$ and by Lemma 2.2 in [9] there exists a net (x_α) in X such that $\omega^*\text{-}\lim x_\alpha = T_\phi(z)$ and $\limsup_\alpha \|z - 2x_\alpha\| \leq 1$. Then

$$2\|T_\phi(z)\| \leq 2\liminf_\alpha \|x_\alpha\| \leq \limsup_\alpha \|2x_\alpha\| \leq \limsup_\alpha \|z - 2x_\alpha\| + \|z\| \leq 2$$

so we may assume that $x_\alpha \in S_X$ for all α.

For all $x \in S_X$ we have $\|z - 2x\| \geq 2\|x\| - \|z\| = 1$ but for $\varepsilon > 0$ there is an x_α such that $\|z - 2x_\alpha\| < 1 + \varepsilon$.

(b)⇒(c). Let $z \in S_Z$ and $\varepsilon > 0$. Choosing $x \in S_X$ with $\|z - 2x\| < 1 + \varepsilon$ we get $B_X(0, 1 - \varepsilon) \cap B_X(2x, \|z - 2x\|) = \emptyset$.

(c)⇒(a). We use Corollary 2.3. For all $z \in Z$ we have $D_z \subseteq B_{X^{**}}(0, \|z\|)$ by definition. Let $z \in S_Z$. By (c) and the principle of local reflexivity we must have $B_{X^{**}}(0, 1-\varepsilon) \cap B_{X^{**}}(2x, \|z-2x\|) = \emptyset$ and hence $D_z \cap B_{X^{**}}(0, 1-\varepsilon) = \emptyset$ for all $\varepsilon > 0$. ∎

Proposition 2.5. *If X is a (non-trivial) strict u-ideal in Y and $P : Y \to X$ is a projection then $\|P\| \geq 2$.*

Proof. Assume that $P : Y \to X$ is a projection with norm $\|P\| = \lambda$. Let $y \in S_Y \cap \ker P$, let $\varepsilon > 0$ and choose $x \in S_X$ such that

$$B_X(0, 1 - \varepsilon) \cap B_X(2x, \|y - 2x\|) = \emptyset$$

using Theorem 2.4. We then get $\|2x\| = \|P(y - 2x)\| \leq \lambda \|y - 2x\|$ so that $(2 - 2/\lambda)x \in B_X(2x, \|y - 2x\|)$. Then $(2 - 2/\lambda)x \notin B_X(0, 1 - \varepsilon)$ and since ε is arbitrary we get $2 - 2/\lambda \geq 1$ or $\lambda \geq 2$. ∎

Since dual spaces are 1-complemented in their biduals they can never be strict u-ideals in their biduals. In fact, they cannot be a strict u-ideal in any superspace.

Corollary 2.6. *Assume that X is a (non-trivial) u-ideal in Y. If X is λ-complemented in its bidual with $\lambda < 2$ then X is not a strict u-ideal in Y.*

Proof. Let $P : X^{**} \to X$ be a projection with norm $\|P\| = \lambda$. Let $y \in Y \setminus X$ and $Z = \operatorname{span}(X, \{y\})$. Let $x^{**} \in D_y$. Note that D_y is non-empty since $\mathrm{HB}(X, Y)$ is. Then for $x \in X$,

$$\|P(x^{**}) - x\| \leq \lambda \|x^{**} - x\| \leq \lambda \|x - y\|.$$

Hence X is λ-complemented in Z by the projection $Q : Z \to X$ defined by $Q(y) = P(x^{**})$ and $Q(x) = x$. From Propositions 2.5 and 2.1 we conclude that X cannot be a strict u-ideal in Y. ∎

Harmand and Lima [11, Theorem 3.5] showed that if X is an M-ideal in its bidual then X contains almost isometric copies of c_0 (i.e. X has a subspace isomorphic to c_0). Next we generalize this to strict u-ideals. Note that the discussion regarding *isometric* copies of c_0 in [12, p. 79] also applies to strict u-ideals.

Theorem 2.7. *If X is a (non-trivial) strict u-ideal in Y, then X contains a copy of c_0.*

Proof. If X does not contain a copy of c_0 then X is a u-summand in Y by Theorem 3.5 in [9]. Using Proposition 2.5 gives us a contradiction. ∎

The following is proved for separable Banach spaces and Banach spaces not containing ℓ_1 in Proposition 5.2 in [9]. For every X the natural embedding

$k_{X^*} : X^* \to X^{***}$ is an element of $\mathrm{HB}(X, X^{**})$. We let $\pi : X^{***} \to X^{***}$ denote the associated ideal projection with $\ker \pi = X^{\perp}$.

THEOREM 2.8. *X is a strict u-ideal in X^{**} if and only if $\|I - 2\pi\| = 1$.*

Proof. Assume that X is a strict u-ideal in its bidual. Let $x^{**} \in X^{**} \setminus X$. We have $X \cap \bigcap_{x\in X} B_{X^{**}}(x, \|x - x^{**}\|) = \emptyset$ since any element in the intersection would define a norm one projection from $\operatorname{span}(X, \{x^{**}\})$ onto X, contradicting Proposition 2.5 (and Proposition 2.1).

By Lemma 2.4 in [10] we get $\bigcap_{x\in X} B_{X^{**}}(x, \|x - x^{**}\|) = \{x^{**}\}$ and so the only element in $\mathrm{HB}(X, X^{**})$ is k_{X^*}.

The other direction is trivial as X^* is norming for X^{**}. ∎

REMARK 2.1. The above proof shows that if X is a strict u-ideal in its bidual then $\mathrm{HB}(X, X^{**})$ has only one element, i.e. the only extension operator is the trivial one k_{X^*}. In particular, the set $D_{x^{**}} = \{x^{**}\}$ is a singleton for every $x^{**} \in X^{**}$ (see (2.1), page 277).

The following theorem was inspired by Theorem 5.5 in [9]. The main improvement is that we remove the assumption that the space does not contain ℓ_1.

THEOREM 2.9. *Let X be a Banach space. The following statements are equivalent.*

(a) *X is a strict u-ideal in its bidual.*
(b) *Every subspace Y of X is a strict u-ideal in its bidual.*
(c) *For every subspace Y of X and $y^{**} \in S_{Y^{**}}$,*

$$\inf_{y\in S_Y} \|y^{**} - 2y\| = 1.$$

(d) *Every separable subspace Y of X is a strict u-ideal in its bidual.*
(e) *For every separable subspace Y of X and $y^{**} \in S_{Y^{**}}$,*

$$\inf_{y\in S_Y} \|y^{**} - 2y\| = 1.$$

Proof. (b)⇒(d) and (c)⇒(e) are trivial. (b)⇒(c) and (d)⇒(e) follow from Theorem 2.4.

(a)⇒(b). We use Theorem 2.8. Let Y be a closed subspace of X with natural embedding $i_Y : Y \to X$. By assumption $\|I - 2\pi_X\| = 1$ where $\pi_X = k_{X^*}k_X^*$. We need to show that $\|I - 2\pi_Y\| = 1$ where $\pi_Y = k_{Y^*}k_Y^*$. It is easy to check $i_Y^{**}k_Y = k_X i_Y$ and $i_Y^{***}k_{X^*} = k_{Y^*}i_Y^*$ so that $i_Y^{***}\pi_X = \pi_Y i_Y^{***}$. We get

$$1 \geq \|i_Y^{***}(I - 2\pi_X)\| = \|(I - 2\pi_Y)i_Y^{***}\|.$$

Since $i_Y^{**} : Y^{**} \to X^{**}$ is isometric, i_Y^{***} is onto Y^{***} and hence $\|I - 2\pi_Y\| = 1$.

(e)⇔(d) is proved in Theorem 5.5 in [9].

Finally, (d)⇔(a) follows from Proposition 2.3 in [9] which characterizes strict u-ideals using sequences. Hence strict u-ideals are separably determined. ∎

A quick look at Theorem 2.7 gives the following corollary.

Corollary 2.10. *Assume that X is non-reflexive. If X is a strict u-ideal in its bidual then every non-reflexive subspace of X contains a copy of c_0.*

Remark 2.2. From Theorem 5.1 in [9] we know that a Banach space is not a strict u-ideal in its bidual if it contains ℓ_1. The above corollary gives an alternative proof of this fact.

From Proposition 2.5 we know that ℓ_∞ is not a strict u-ideal in its bidual. The next theorem shows that it is not even a u-ideal. We will also look at some consequences below.

Theorem 2.11. *ℓ_∞ is not a u-ideal in its bidual.*

Before giving the proof of this theorem we need to introduce some more notation.

It is well-known that ℓ_∞ is isometrically isomorphic to $C(\beta\mathbb{N})$ where $\beta\mathbb{N}$ is the Stone–Čech compactification of the natural numbers (see e.g. Corollary 15.2 in [6]). The Riesz representation theorem identifies the dual with the measures on $\beta\mathbb{N}$. The state space of $C(\beta\mathbb{N})$ is the set

$$S = \{x^* \in \ell_\infty^* : \|x^*\| = x^*(1) = 1\},$$

which is a weak*-closed subset of the dual unit ball. S can be identified with the probability measures on $\beta\mathbb{N}$; the set of extreme points of S, $\operatorname{ext} S$, is homeomorphic to $\beta\mathbb{N}$; and S is a Bauer simplex (see e.g. [2, Corollary II.4.2]). $C(\beta\mathbb{N})$ is isometrically isomorphic to $A(S)$, the continuous affine functions on S (see e.g. [2, Theorem II.1.8]). Thus for $f \in A(S)$ and $s \in S$ there is a unique probability measure μ on $\operatorname{ext} S$ such that $f(s) = \int_{\operatorname{ext} S} f\,d\mu$. We will write $s = r(\mu)$ where r is the resultant (or barycenter) function. It is well-known that S is a simplex (see e.g. [18, p. 53]) so μ is unique, i.e. r is 1-1 ([18, Proposition 11.1]).

We say that a measure μ on $\beta\mathbb{N}$ is *discrete* if there is a countable set $\{z_j\}_{j=1}^\infty \subset \beta\mathbb{N}$ and numbers $\{a_j\}_{j=1}^\infty$ such that $\mu = \sum_{j=1}^\infty a_j\delta_{z_j}$. On the other hand, μ is *continuous* if $\mu(\{z\}) = 0$ for all $z \in \beta\mathbb{N}$. Any measure μ can be written uniquely as $\mu = \mu_d + \mu_c$ where μ_d is discrete and μ_c is continuous by letting $E = \{z : \mu(\{z\}) \neq 0\}$ and defining $\mu_d(A) = \mu(A \cap E)$ and $\mu_c(A) = \mu(A \setminus E)$. Since $\mathbb{N}$ is countable we can write $\mu_d = \mu_{nd} + \mu_{bd}$ where $\mu_{nd}(A) = \mu_d(A \cap \mathbb{N})$ and $\mu_{bd}(A) = \mu_d(A \setminus \mathbb{N})$.

We will define the following faces of S:

$$S_1 = \{s \in S : s = r(\mu_{nd}),\ \text{a discrete measure on } \mathbb{N}\},$$
$$S_2 = \{s \in S : s = r(\mu_{bd}),\ \text{a discrete measure on } \beta\mathbb{N}\},$$
$$S_3 = \{s \in S : s = r(\mu_c),\ \text{a continuous measure on } \beta\mathbb{N}\}.$$

We have $S = \operatorname{conv}(\bigcup_{i=1}^3 S_i)$ and $S_i \cap S_j = \emptyset$ for $i \neq j$. We will also need the complementary face of S_i, namely $S_i' = \operatorname{conv}(\bigcup_{i\neq j} S_i)$. (Here we have used that closed faces in a simplex are split; see [2, pp. 132–133, Proposition II.6.7 and Corollary II.6.8] and [4, p. 140, Theorem 8.3].) Also note that $S_3 \neq \emptyset$ since we can pull back Lebesgue measure from $C[0,1]^*$.

Proof of Theorem 2.11. We identify ℓ_∞ with $A(S)$ and $A(S)^{**}$ with the bounded affine functions on S, $A_b(S)$. (This is "easy to check" [4, p. 43].) Each $s \in S$ can be written uniquely as $s = \alpha_i s_i + (1-\alpha_i)s_i'$ where $\alpha_i \in [0,1]$, $s_i \in S_i$ and $s_i' \in S_i'$. Thus the functions $f_i(s) = 2\alpha_i - 1$ are well-defined and $f_i \in A_b(S)$. We will use that $f_i = 1$ on S_i and $f_i = -1$ on S_i'.

Assume for contradiction that ℓ_∞ is a u-ideal in its bidual. Define $H = \operatorname{span}(f_i)_{i=1}^3$, a subspace of ℓ_∞^{**}, and let $\varepsilon > 0$.

By the local characterization of u-ideals (Proposition 3.6 in [9]), there is an operator $L : H \to \ell_\infty$ such that $\|L\| \le 1+\varepsilon$, $\|h - 2L(h)\| \le (1+\varepsilon)\|h\|$ for all $h \in H$ and $L(x) = x$ for all $x \in H \cap \ell_\infty$. Since $L(1) = 1$ we get $\sum_{i=1}^3 L(f_i) = -1$.

Using $\|f_i - 2L(f_i)\| \le (1+\varepsilon)\|f_i\| \le 1+\varepsilon$ we see that on S, $-(1+\varepsilon) \le -f_i + 2L(f_i) \le 1+\varepsilon$ or $f_i - 1 - \varepsilon \le 2L(f_i) \le f_i + 1 + \varepsilon$. So on S_i we have $-\varepsilon/2 \le L(f_i) \le 1 + \varepsilon/2$.

By density of $\mathbb{N}$ in its compactification $\beta\mathbb{N}$ we must have $L(f_1) \ge -\varepsilon/2$ on $\beta\mathbb{N}$ since $L(f_1) \ge -\varepsilon/2$ on $\mathbb{N}$. Also, we have $L(f_2) \ge -\varepsilon/2$ on $\beta\mathbb{N} \setminus \mathbb{N}$. Since $\mathbb{N}$ is countable the continuous measure μ corresponding to $s \in S_3$ has support on $\beta\mathbb{N} \setminus \mathbb{N}$ so

$$Lf_i(s) = \int_{\operatorname{ext} S} Lf_i\, d\mu = \int_{\beta\mathbb{N}\setminus\mathbb{N}} Lf_i\, d\mu \ge -\varepsilon/2$$

for $i = 1, 2$. Thus on S_3 we have

$$-\varepsilon/2 \le L(f_3) = -1 - L(f_1) - L(f_2) \le -1 + \varepsilon,$$

or $0 \le -1 + 3\varepsilon/2$. Since $\varepsilon > 0$ is arbitrary this is a contradiction. ∎

REMARK 2.3. Since ℓ_∞ is injective, ℓ_∞ is never a strict u-ideal in $Z = \operatorname{span}\{\ell_\infty, f\}$ for $f \in \ell_\infty^{**}$. In some cases it is a u-ideal, however.

In the notation above, set $f = 1$ on S_1 and $f = -1$ on $S_1' = \operatorname{conv}(S_2 \cup S_3)$. Let $\varepsilon > 0$, $x_i \in \ell_\infty$ and $r_i = \|f - x_i\|$ for $i = 1,2,3$. Without loss of generality we may assume that $x_i = \sum_{k=1}^m a_{i,k}\chi_{A_k}$ where A_k is a partition of $\mathbb{N}$ (use an

ε-net on the set $(x_i(n))_{n=1}^{\infty}$ if necessary). We may assume that $A_1, \ldots, A_p$ are finite sets and that $A_{p+1}, \ldots, A_m$ are infinite.

Define an element $x \in \ell_\infty$ by setting $x_n = 2$ for $n \in \bigcup_{k=1}^{p} A_k$ and $x_n = 0$ for $n \in \bigcup_{k=p+1}^{m} A_k$. Then $x \in \ell_\infty \cap \bigcap_{i=1}^{3} B_Z(f+x_i, r_i+\varepsilon)$ and by Theorem 1.3 in [15], ℓ_∞ is a u-ideal in Z.

As noted in Proposition 2.5, a non-reflexive dual space can never be a strict u-ideal. Using that ℓ_∞ is not a u-ideal in its bidual we can say even more.

THEOREM 2.12. *Let X be a Banach space such that X^* is a u-ideal in its bidual. Then X^* is a u-summand.*

Proof. If X^* contains a copy of c_0 then it contains a copy of ℓ_∞ by Bessaga and Pełczyński [5]. By Partington [16] and Talagrand [19, Theorem 6] (and injectivity) it has $(1+\varepsilon)$-complemented copies of ℓ_∞ for every $\varepsilon > 0$. The local characterization of u-ideals (Proposition 4.1 in [9]) would then imply that ℓ_∞ is a u-ideal in its bidual, which is impossible by Theorem 2.11. Hence X^* is a u-ideal not containing c_0, so it is a u-summand by Theorem 3.5 in [9]. ∎

REMARK 2.4. Assume X is a strict u-ideal in its bidual. Then $\|I - 2\pi\| = 1$ and considering the adjoint projection $P = \pi^*$ on $X^{(4)}$ we have $\ker P = (\operatorname{im}\pi)^\perp = (X^*)^\perp$. Since $\|I - 2P\| = \|I - 2\pi\| = 1$ we conclude that X^* is a u-ideal in its bidual and by the above theorem even a u-summand.

We do not know whether X a u-ideal in its bidual and X^* a u-summand in its bidual implies that X is a strict u-ideal.

3. Geometric properties. A *slice* of a bounded, closed, convex subset C of X is a subset $S(C, x^*, \alpha)$ of C defined by

$$S(C, x^*, \alpha) = \{x \in C : x^*(x) > \sup_{y \in C} x^*(y) - \alpha\},$$

where $x^* \in X^* \setminus \{0\}$ and $\alpha > 0$. If X is a dual space we can speak of a *weak*-slice* when the defining functional is weak*-continuous. A bounded, closed, convex set C is *dentable* if it has slices of arbitrarily small diameter. Recall that the *diameter* of a non-empty set A is given by $\operatorname{diam}(A) = \sup\{\|x - y\| : x, y \in A\}$. A point $x \in C$ is called a *denting point* in C if there is a sequence of slices S_n of C with $x \in S_n$, for all n, such that $\operatorname{diam}(S_n) \to 0$. If C is a subset of a dual space X^* then $x^* \in C$ is a *weak*-denting point* in C if there is a sequence of weak*-slices S_n of C with $x^* \in S_n$ for all n such that $\operatorname{diam}(S_n) \to 0$. A point $x \in C$ is called a *strongly exposed point* in C if there is an $x^* \in X^*$ such that $x^*(x) > x^*(y)$ for all $x \neq y \in C$ and $\operatorname{diam}(S(C, x^*, \alpha)) \to 0$ as $\alpha \to 0$. Weak* strongly exposed points are defined in the obvious way.

By definition ω^*-str.exp. $B_{X^*} \subset \omega^*$-dent. B_{X^*}. When X is a strict u-ideal in its bidual we can say much more. The next proposition highlights that this is a really strong geometric property.

PROPOSITION 3.1. *Assume that X is a strict u-ideal in its bidual. Then*

$$\text{str.exp.} B_{X^*} \subset \omega^*\text{-dent.}\, B_{X^*}.$$

Proof. Let $x^* \in$ str.exp.B_{X^*} and let $x^{**} \in S_{X^{**}}$ be a strongly exposing functional for x^*. Let $\varepsilon > 0$ and choose $\delta_0 > 0$ such that $\{u^* \in B_{X^*} : x^{**}(u^*) > 1 - \sqrt{\delta_0}\} \subset B_{X^*}(x^*, \varepsilon)$ and $1 + \varepsilon\delta_0 > 2\sqrt{\delta_0}(1 + \varepsilon)$.

Let $\delta \in (0, \delta_0)$. Then $1 + \varepsilon\delta > 2\sqrt{\delta}(1 + \varepsilon)$, which is equivalent to $2(1 - \delta)/(1 + \varepsilon\delta) - 2 + \sqrt{\delta} > 0$. Choose $\eta > 0$ with $0 < \eta < 2(1 - \delta)/(1 + \varepsilon\delta) - 2 + \sqrt{\delta}$ and $\{u^* \in B_{X^*} : x^{**}(u^*) > 1 - \eta\} \subset B_{X^*}(x^*, \varepsilon\delta/(1 + \varepsilon\delta))$.

Since X is a strict u-ideal we have $1 = \inf_{x \in S_X} \|x^{**} - 2x\|$. Choose $x \in S_X$ such that $\|x^{**} - 2x\| < 1 + \eta$. Choose $u^* \in B_{X^*}$ such that $u^*(x) = 1$. Then

$$1 + \eta > \|x^{**} - 2x\| \geq u^*(2x - x^{**}) = 2 - x^{**}(u^*).$$

Thus $x^{**}(u^*) > 1 - \eta$. It follows that $\|u^* - x^*\| < \varepsilon\delta/(1 + \varepsilon\delta)$.

Let $u = x/x^*(x)$. Then $x^*(x) \geq u^*(x) - \|x^* - u^*\| > 1/(1 + \varepsilon\delta)$ so $\|u\| = 1/x^*(x) \leq 1 + \varepsilon\delta$. If $z^* \in B_{X^*}$ and $z^*(u) > 1 - \delta$, then $z^*(x) = z^*(u)x^*(x) > (1 - \delta)x^*(x)$. Hence

$$1 + \eta > \|x^{**} - 2x\| \geq z^*(2x - x^{**}) \geq 2(1 - \delta)x^*(x) - x^{**}(z^*),$$

and $x^{**}(z^*) > 2(1 - \delta)x^*(x) - 1 - \eta \geq 2(1 - \delta)/(1 + \varepsilon\delta) - 1 - \eta$. But then $x^{**}(z^*) > 1 - \sqrt{\delta}$, from which it follows that $\|z^* - x^*\| < \varepsilon$. Thus x^* is contained in weak*-slices of arbitrarily small diameter, i.e. x^* is weak*-denting. ∎

Next we use the weak*-denting points in the unit ball to characterize when a u-ideal is a strict u-ideal. For Banach spaces not containing ℓ_1 the equivalence of (a) and (d) was proved in Theorem 7.4 in [9].

PROPOSITION 3.2. *Let X be a Banach space. Assume that X is a u-ideal in its bidual. Then the following are equivalent.*

(a) *X is a strict u-ideal in its bidual.*
(b) $B_{X^*} = \overline{\text{conv}}(\omega^*\text{-str.exp.}\, B_{X^*})$.
(c) $B_{X^*} = \overline{\text{conv}}(\omega^*\text{-dent.}\, B_{X^*})$.
(d) *X^* contains no proper norming subspaces.*
(e) *$T_\phi = I_{X^{**}}$ where $\phi \in \mathrm{HB}(X, X^{**})$ is the unconditional extension operator.*

Proof. (a)⇒(b) follows from Theorem 2.8 and Proposition 4.1 in [13].

(b)⇒(c) is trivial.

(c)⇒(a). The weak*-denting points have unique norm-preserving extension so $\mathrm{HB}(X, X^{**}) = \{k_{X^*}\}$. X is a strict u-ideal by Theorem 2.8.

(a)⇒(d). Follows from Theorem 2.8 and Proposition 2.7 in [9].

(d)⇒(e). By Proposition 2.5 in [10], X has the unique extension property and by definition the only contractive operator $T : X^{**} \to X^{**}$ with $T|_X = I_X$ is $T = I_{X^{**}}$.

(e)⇒(a). X is a strict u-ideal by Theorem 2.8. ∎

Remark 3.1. The dual of a Banach space X has the Radon–Nikodým property if and only if every separable subspace of X has separable dual (see e.g. [7, Corollary VII.2.8]). This is the case if X is a strict u-ideal in its bidual (see e.g. Proposition 4.1 in [13] or Proposition 2.8 in [9]).

On the other hand, if X^* has the Radon–Nikodým property then $B_{X^*} = \overline{\mathrm{conv}}^{w^*}(\omega^*\text{-str.exp. } B_{X^*})$ [17, Theorem 5.12]. We do not know if this is enough to ensure that a u-ideal is strict.

It is also an open problem whether a u-ideal is strict if the space does not contain ℓ_1 (see Question 5 in [9]).

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References

[1] T. A. Abrahamsen, V. Lima, and Å. Lima, *Unconditional ideals of finite rank operators*, Czechoslovak Math. J. 58 (2008), 1257–1278.

[2] E. M. Alfsen, *Compact Convex Sets and Boundary Integrals*, Springer, Berlin, 1971.

[3] E. M. Alfsen and E. G. Effros, *Structure in real Banach spaces. Parts I and II*, Ann. of Math. 96 (1972), 98–173.

[4] L. Asimow and A. J. Ellis, *Convexity Theory and its Applications in Functional Analysis*, London Math. Soc. Monogr. 16, Academic Press, London, 1980.

[5] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), 151–164.

[6] N. L. Carothers, *A Short Course on Banach Space Theory*, London Math. Soc. Student Texts 64, Cambridge Univ. Press, Cambridge, 2005.

[7] J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, RI, 1977.

[8] H. Fakhoury, *Sélections linéaires associées au théorème de Hahn–Banach*, J. Funct. Anal. 11 (1972), 436–452.

[9] G. Godefroy, N. J. Kalton, and P. D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. 104 (1993), 13–59.

[10] G. Godefroy and P. D. Saphar, *Duality in spaces of operators and smooth norms on Banach spaces*, Illinois J. Math. 32 (1988), 672–695.

[11] P. Harmand and Å. Lima, *Banach spaces which are M-ideals in their biduals*, Trans. Amer. Math. Soc. 283 (1984), 253–264.

[12] P. Harmand, D. Werner, and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.

[13] Å. Lima, *Property (wM^*) and the unconditional metric compact approximation property*, Studia Math. 113 (1995), 249–263.

[14] Å. Lima and E. Oja, *Ideals of finite rank operators, intersection properties of balls, and the approximation property*, ibid. 133 (1999), 175–186.

[15] V. Lima and Å. Lima, *A three ball intersection property for u-ideals*, J. Funct. Anal. 251 (2007), 220–232.
[16] J. R. Partington, *Subspaces of certain Banach sequence spaces*, Bull. London Math. Soc. 13 (1981), 163–166.
[17] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Math. 1364, Springer, Berlin, 1989.
[18] —, *Lectures on Choquet's Theorem*, 2nd ed., Lecture Notes in Math. 1757, Springer, Berlin, 2001.
[19] M. Talagrand, *Sur les espaces de Banach contenant $l^1(\tau)$*, Israel J. Math. 40 (1981), 324–330.

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