

Composition in ultradifferentiable classes

by

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Abstract. We characterize stability under composition of ultradifferentiable classes defined by weight sequences M , by weight functions ω , and, more generally, by weight matrices \mathfrak{M} , and investigate continuity of composition $(g, f) \mapsto f \circ g$. In addition, we represent the Beurling space $\mathcal{E}^{(\omega)}$ and the Roumieu space $\mathcal{E}^{\{\omega\}}$ as intersection and union of spaces $\mathcal{E}^{(M)}$ and $\mathcal{E}^{\{M\}}$ for associated weight sequences, respectively.

1. Introduction. This paper arose from our wish to characterize stability under composition of Denjoy–Carleman classes $\mathcal{E}^{\{M\}}$ and $\mathcal{E}^{(M)}$. For these classes we have developed a calculus in infinite dimensions beyond Banach spaces in [24, 26, 25] which is heavily based on composition: A smooth mapping f is of class $\mathcal{E}^{\{M\}}$ if and only if $f \circ p \in \mathcal{E}^{\{M\}}$ for all $\mathcal{E}^{\{M\}}$ Banach plots (i.e., mappings defined in open subsets of Banach spaces); similarly for $\mathcal{E}^{(M)}$. Sometimes curves suffice.

Denjoy–Carleman differentiable functions form classes of smooth functions that are described by growth conditions on the Taylor expansion. The growth is prescribed in terms of a sequence $M = (M_k)$ of positive real numbers which serves as a weight for the iterated derivatives: for compact K the sets

$$\left\{ \frac{f^{(k)}(x)}{\rho^k k! M_k} : x \in K, k \in \mathbb{N} \right\}$$

are required to be bounded. The positive real number ρ is subject to either a universal or an existential quantifier, thereby dividing the Denjoy–Carleman classes into those of Beurling type $\mathcal{E}^{(M)}$ and those of Roumieu type $\mathcal{E}^{\{M\}}$, respectively. We write $\mathcal{E}^{[M]}$ for either $\mathcal{E}^{(M)}$ or $\mathcal{E}^{\{M\}}$.

It is well-known that $\mathcal{E}^{[M]}$ is stable under composition if M is log-convex (see [34], [20], [13]), and usually in the literature log-convexity is assumed in order to have stability under composition; but is log-convexity also necessary? Actually, when proving stability under composition with Faà di Bruno’s formula one needs a weaker condition that we call the (FdB)-

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property. We prove that the (FdB)-property (for the weakly log-convex minorant $M^{b(c)}$) is also a necessary condition for stability under composition if $\mathcal{E}^{[M]}$ is stable under derivation (see Theorem 3.2). More precisely, if $\mathcal{E}^{[M]}$ is stable under derivation, then stability under composition is in turn equivalent to being holomorphically closed, being inverse closed, $(M_k^{b(c)})^{1/k}$ being almost increasing, and $M^{b(c)}$ having the (FdB)-property. For further equivalent stability properties we refer to [33]. Inverse closedness has been studied intensively, e.g. in [35], [10], [39]. In this context we prove that, as in the Roumieu case [11], one has $\mathcal{E}^{(M)} = \mathcal{E}^{(M^{b(c)})}$ if $C^\omega \subseteq \mathcal{E}^{(M)}$ (see Theorem 2.15). Finally, we demonstrate that log-convexity is not necessary for stability under composition. We construct classes $\mathcal{E}^{[M]}$ which are stable under composition and such that there is no log-convex $N = (N_k)$ with $\mathcal{E}^{[M]} = \mathcal{E}^{[N]}$ (see Example 3.6).

Another common way to define ultradifferentiable classes is by means of a weight function ω which controls the decay of the Fourier transform (see [5] and [6]). We shall use the following equivalent description due to [9]: for compact K the sets

$$\left\{ f^{(k)}(x) \exp\left(-\frac{1}{\rho}\varphi^*(\rho k)\right) : x \in K, k \in \mathbb{N} \right\},$$

where φ^* is the Young conjugate of $\varphi(t) = \omega(e^t)$, are required to be bounded either for all $\rho > 0$ in the Beurling case $\mathcal{E}^{(\omega)}$ or for some $\rho > 0$ in the Roumieu case $\mathcal{E}^{\{\omega\}}$. Again $\mathcal{E}^{[\omega]}$ stands for either $\mathcal{E}^{(\omega)}$ or $\mathcal{E}^{\{\omega\}}$. For these classes stability under composition was characterized in [16] under the additional assumption of non-quasianalyticity. Note that the sets $\{\mathcal{E}^{[M]} : M \text{ a weight sequence}\}$ and $\{\mathcal{E}^{[\omega]} : \omega \text{ a weight function}\}$ have a *large* intersection but neither of them contains the other (see [8]). We stress that the usual requirements on the weight function ω ensure that the spaces $\mathcal{E}^{[\omega]}$ come with incorporated stability properties, for instance stability under derivation (see Corollary 5.15).

We prove that $\mathcal{E}^{(\omega)}$ and $\mathcal{E}^{\{\omega\}}$ can be represented (as locally convex spaces with their natural topologies) as intersections and unions of ultradifferentiable classes defined by means of associated weight sequences (see Theorem 5.14). For each open subset $U \subseteq \mathbb{R}^n$, compact $K \subseteq U$, and for $\Omega^\rho = (\Omega_k^\rho)$ defined by

$$\Omega_k^\rho := \frac{1}{k!} \exp\left(\frac{1}{\rho}\varphi^*(\rho k)\right)$$

we have

$$(1.1) \quad \mathcal{E}^{(\omega)}(U) = \bigcap_{\rho>0} \mathcal{E}^{(\Omega^\rho)}(U) \quad \text{and} \quad \mathcal{E}^{\{\omega\}}(U) = \bigcap_{K \subseteq U} \bigcup_{\rho>0} \mathcal{E}^{\{\Omega^\rho\}}(K).$$

We use this representation to characterize stability under composition, and believe that it is also of independent interest.

In fact, inspired by (1.1), we characterize stability under composition for more general ultradifferentiable classes defined by weight matrices $\mathfrak{M} = \{M^\lambda \in \mathbb{R}_{>0}^{\mathbb{N}} : \lambda \in \Lambda\}$, where Λ is an ordered subset of \mathbb{R} :

$$\mathcal{E}^{(\mathfrak{M})}(U) := \bigcap_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(U) \quad \text{and} \quad \mathcal{E}^{\{\mathfrak{M}\}}(U) := \bigcap_{K \subseteq U} \bigcup_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(K),$$

endowed with their natural topologies. Among the spaces $\mathcal{E}^{(\mathfrak{M})}$ and $\mathcal{E}^{\{\mathfrak{M}\}}$, together denoted by $\mathcal{E}^{[\mathfrak{M}]}$, are all the spaces defined by means of weight sequences and weight functions, but not exclusively (see Theorem 5.22). For instance, the intersection, resp. the union, of all non-quasianalytic Gevrey classes is an autonomous $\mathcal{E}^{(\mathfrak{M})}$ -space, resp. $\mathcal{E}^{\{\mathfrak{M}\}}$ -space, with suitable \mathfrak{M} . Intersections of non-quasianalytic ultradifferentiable classes have been studied by Rudin [35], Boman [7], Chaumat and Chollet [12], Beaugendre [3, 4], and Schmets and Valdivia [37, 38] (among others). It seems, however, that unions of ultradifferentiable classes have not been investigated before.

Given that $\mathcal{E}^{[\mathfrak{M}]}$ is stable under composition, the nonlinear composition operators

$$\begin{aligned} \text{comp}^{(\mathfrak{M})} : \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^p, \mathbb{R}^q) \times \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^q, \mathbb{R}^r) &\rightarrow \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^p, \mathbb{R}^r) : (g, f) \mapsto f \circ g, \\ \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, f) : \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, \mathbb{R}^q) &\rightarrow \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, \mathbb{R}^r) : g \mapsto f \circ g, \quad f \in \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^q, \mathbb{R}^r), \end{aligned}$$

turn out to be continuous. This is proved in Theorem 4.13. The special case of $\mathcal{E}^{[\omega]}$ was treated in [16] (see also [1]). Under suitable assumptions we expect $\text{comp}^{[\mathfrak{M}]}$ to be of class $\mathcal{E}^{[\mathfrak{M}]}$ (see Remark 4.23).

The paper is structured as follows. We first treat the weight sequence case in Sections 2 and 3. In Section 4 we introduce ultradifferentiable classes defined by weight matrices \mathfrak{M} , characterize their stability under composition, and show that composition is continuous. We discuss classes defined by weight functions ω and identify them as classes defined by weight matrices \mathfrak{M} in Section 5, and characterize their stability under composition in Section 6.

Notation and conventions. The notation $\mathcal{E}^{[*]}$ for $* \in \{M, \omega, \mathfrak{M}\}$ stands for either $\mathcal{E}^{(*)}$ or $\mathcal{E}^{\{*\}}$ with the following restriction: Statements that involve more than one $\mathcal{E}^{[*]}$ symbol must not be interpreted by mixing $\mathcal{E}^{(*)}$ and $\mathcal{E}^{\{*\}}$. This convention will be used broadly, but self-evidently: For example, $\mathfrak{M}[\preceq]\mathfrak{N} \Leftrightarrow \mathcal{E}^{[\mathfrak{M}]} \subseteq \mathcal{E}^{[\mathfrak{N}]}$ in Proposition 4.6 means $\mathfrak{M}(\preceq)\mathfrak{N} \Leftrightarrow \mathcal{E}^{(\mathfrak{M})} \subseteq \mathcal{E}^{(\mathfrak{N})}$ and $\mathfrak{M}\{\preceq\}\mathfrak{N} \Leftrightarrow \mathcal{E}^{\{\mathfrak{M}\}} \subseteq \mathcal{E}^{\{\mathfrak{N}\}}$.

Let $\mathbb{N} = \mathbb{N}_{>0} \cup \{0\}$. For $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{N}^q$ and $x = (x_1, \dots, x_q) \in \mathbb{R}^q$ we write $\alpha! = \alpha_1! \cdots \alpha_q!$, $|\alpha| = \alpha_1 + \cdots + \alpha_q$, and $x^\alpha = x_1^{\alpha_1} \cdots x_q^{\alpha_q}$. We use $\partial_i = \partial/\partial x_i$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_q^{\alpha_q}$ and write $d^k f$ or $f^{(k)}$ for the k th order Fréchet derivative of f , and $d_v f$ for the directional derivative in direction v . For sequences of reals $M = (M_k)$ and $N = (N_k)$ we write $M \leq N$ if $M_k \leq N_k$ for all k .

$L(E_1, \dots, E_k; F)$ is the space of k -linear bounded mappings $E_1 \times \dots \times E_k \rightarrow F$ (between topological vector spaces); if $E_i = E$ for all i , we also write $L^k(E, F)$.

Let \mathcal{F} and \mathcal{G} denote classes of mappings. We write $\mathcal{F} \subseteq \mathcal{G}$ if $\mathcal{F}(U, \mathbb{R}^m) \subseteq \mathcal{G}(U, \mathbb{R}^m)$ for all open subsets $U \subseteq \mathbb{R}^n$ and all $n, m \in \mathbb{N}_{>0}$. We say that \mathcal{F} is *stable under composition* if $g \in \mathcal{F}(U, V)$ and $f \in \mathcal{F}(V, W)$ implies $f \circ g \in \mathcal{F}(U, W)$, for all open subsets $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$, $W \subseteq \mathbb{R}^r$, and all $p, q, r \in \mathbb{N}_{>0}$. A class \mathcal{F} is called *holomorphically closed* if $f \circ g \in \mathcal{F}(U, \mathbb{C})$ for each $g \in \mathcal{F}(U) = \mathcal{F}(U, \mathbb{R})$ and each f which is holomorphic in a complex neighborhood of the range of g , and \mathcal{F} is *inverse closed* if $1/f \in \mathcal{F}(U)$ for each non-vanishing $f \in \mathcal{F}(U)$. That \mathcal{F} is *derivation closed* means that $f \in \mathcal{F}(U)$ implies $\partial_i f \in \mathcal{F}(U)$ for all open $U \subseteq \mathbb{R}^n$, $n \in \mathbb{N}_{>0}$, and $1 \leq i \leq n$. A class \mathcal{F} of smooth mappings is *quasianalytic* if for each open connected $U \subseteq \mathbb{R}^n$ and each $x \in U$ the Borel mapping $\mathcal{F}(U) \ni f \mapsto (\partial^\alpha f(x))_\alpha$ is injective.

2. Weight sequences and $[M]$ -ultradifferentiable functions

2.1. Weight sequences. A sequence $M = (M_k) \in \mathbb{R}_{>0}^{\mathbb{N}}$ of positive real numbers is said to be *log-convex* if $k \mapsto \log M_k$ is convex, i.e.,

$$(M_{lc}) \quad \forall k : \quad M_k^2 \leq M_{k-1}M_{k+1};$$

weakly log-convex if

$$(M_{wlc}) \quad (k!M_k)_k \text{ is log-convex;}$$

of moderate growth if

$$(M_{mg}) \quad \exists C > 0 \forall j, k \geq 1 : \quad M_{j+k} \leq C^{j+k} M_j M_k;$$

derivation closed if

$$(M_{dc}) \quad \exists C > 0 \forall k \geq 1 : \quad M_{k+1} \leq C^k M_k;$$

almost increasing if

$$(M_{ai}) \quad \exists C > 0 \forall j \leq k : \quad M_j \leq C M_k;$$

have the (FdB)-*property* if

$$(M_{FdB}) \quad \exists C > 0 \forall \alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \dots + \alpha_j = k : \quad M_j M_{\alpha_1} \dots M_{\alpha_j} \leq C^k M_k;$$

and be *quasianalytic* if

$$(M_{qa}) \quad \sum_{k=1}^{\infty} (k!M_k)^{-1/k} = \infty.$$

Obviously (M_{lc}) implies (M_{wlc}) , and (M_{mg}) implies (M_{dc}) . If M is log-convex, we further have $M_j M_k \leq M_0 M_{j+k}$ for all j, k and $(M_k/M_0)^{1/k}$ is increasing. Moreover:

2.2. LEMMA. For $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ having the (FdB)-property, each of the following conditions is sufficient:

- (1) M is log-convex.
- (2) M is derivation closed and $(M_k^{1/k})_k$ is almost increasing.
- (3) $M_j M_k \leq M_1 M_{j+k-1}$ for all $j, k \geq 1$.

Proof. (1) We show (M_{FdB}) with $C := \max\{M_1, 1\}$ by induction on k . The assertion is trivial for $k = j$. Assume that $j < k$. Then $\alpha'_i := \alpha_i - 1 \geq 1$ for some i , and we have

$$M_j M_{\alpha_1} \cdots M_{\alpha_j} = M_j M_{\alpha_1} \cdots M_{\alpha'_i} \cdots M_{\alpha_j} \frac{M_{\alpha_i}}{M_{\alpha'_i}} \leq C^{k-1} M_{k-1} \frac{M_k}{M_{k-1}} \leq C^k M_k,$$

by induction hypothesis and by (M_{lc}) .

(2) This is proved in more general terms in 4.9[(3) \Rightarrow (4)] and 4.11[(3) \Rightarrow (4)].

(3) This is readily seen by iteration. ■

For $M, N \in \mathbb{R}_{>0}^{\mathbb{N}}$ we define:

$$M \preceq N \quad :\Leftrightarrow \quad \exists C, \rho > 0 \quad \forall k : M_k \leq C \rho^k N_k \quad \Leftrightarrow \quad \sup_k \left(\frac{M_k}{N_k} \right)^{1/k} < \infty,$$

$$M \approx N \quad :\Leftrightarrow \quad M \preceq N \text{ and } N \preceq M,$$

$$M \triangleleft N \quad :\Leftrightarrow \quad \forall \rho > 0 \quad \exists C > 0 \quad \forall k : M_k \leq C \rho^k N_k \quad \Leftrightarrow \quad \lim_{k \rightarrow \infty} \left(\frac{M_k}{N_k} \right)^{1/k} = 0.$$

The following lemma is a variant of [21, Lemma 6].

2.3. LEMMA. Let $L, M \in \mathbb{R}_{>0}^{\mathbb{N}}$ satisfy $L \triangleleft M$ and $M_k^{1/k} \rightarrow \infty$. Then there exist sequences $N^i \in \mathbb{R}_{>0}^{\mathbb{N}}$, $i = 1, 2$, satisfying $(N_k^i)^{1/k} \rightarrow \infty$ such that $L \leq N^1 \triangleleft N^2 \triangleleft M$.

Proof. It suffices to show that there exists $N^1 \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $L \leq N^1 \triangleleft M$ and $(N_k^1)^{1/k} \rightarrow \infty$; for $N^2 = (N_k^2)$ we may then choose $N_k^2 := \sqrt{N_k^1 M_k}$.

The sequence $N^1 = (N_k^1)$ defined by $N_k^1 := \max\{\sqrt{M_k}, L_k\}$ is as required. We have $L \leq N^1 \triangleleft M$, since

$$\left(\frac{N_k^1}{M_k} \right)^{1/k} = \max \left\{ M_k^{-1/2k}, \left(\frac{L_k}{M_k} \right)^{1/k} \right\} \rightarrow 0$$

as $M_k^{1/k} \rightarrow \infty$ and $L \triangleleft M$. Moreover, $N_k^1 \geq \sqrt{M_k}$ implies $(N_k^1)^{1/k} \rightarrow \infty$. ■

2.4. REMARK. The lemma remains true if we replace $M_k^{1/k} \rightarrow \infty$ by $(k! M_k)^{1/k} \rightarrow \infty$ and $(N_k^i)^{1/k} \rightarrow \infty$ by $(k! N_k^i)^{1/k} \rightarrow \infty$; set $N_k^1 := \max\{\sqrt{M_k/k!}, L_k\}$ in the above proof. But in this case it is unclear if $\underline{\lim} M_k^{1/k} > 0$ implies $\underline{\lim} (N_k^i)^{1/k} > 0$ which we need in Theorem 2.15.

2.5. Regularizations (cf. [2], [27], or [22]). For $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $(k!M_k)^{1/k} \rightarrow \infty$ set

$$T_M(t) := \sup_{k \in \mathbb{N}} \frac{t^k}{k!M_k}, \quad t > 0, \quad \text{and} \quad M_k^{b(c)} := \frac{1}{k!} \sup_{t>0} \frac{t^k}{T_M(t)}.$$

Then $T_M = T_{M^{b(c)}}$. The sequence $(k!M_k^{b(c)})_k$ is the largest log-convex minorant of $(k!M_k)_k$; in particular, M is weakly log-convex if and only if $M = M^{b(c)}$. The condition $(k!M_k)^{1/k} \rightarrow \infty$ guarantees that $M_k = M_k^{b(c)}$ for infinitely many k .

We shall also use

$$S_M(t) := \max_{k \leq t} \frac{t^k}{k!M_k} \quad \text{and} \quad M_k^{b(o)} := \frac{1}{k!} \sup_{t \geq k} \frac{t^k}{S_M(t)},$$

and again have $S_M = S_{M^{b(o)}}$.

2.6. LEMMA. *Let $M, N \in \mathbb{R}_{>0}^{\mathbb{N}}$ satisfy $(k!M_k)^{1/k} \rightarrow \infty$ and $(k!N_k)^{1/k} \rightarrow \infty$. Then $M \preceq N$ implies $M^{b(c)} \preceq N^{b(c)}$, and $M \triangleleft N$ implies $M^{b(c)} \triangleleft N^{b(c)}$.*

Proof. For $\rho > 0$ set $N^\rho = (N_k^\rho) := (\rho^k N_k)$. Easy computations show $T_{N^\rho}(t) = T_N(t/\rho)$ and thus $(N^\rho)^{b(c)} = (N^{b(c)})^\rho$. Both assertions follow immediately. ■

2.7. $[M]$ -ultradifferentiable functions. Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ and let $U \subseteq \mathbb{R}^n$ be open. Define

$$\begin{aligned} \mathcal{E}^{(M)}(U) &:= \{f \in C^\infty(U, \mathbb{R}) : \forall K \subseteq U \text{ compact } \forall \rho > 0 : \|f\|_{K, \rho}^M < \infty\}, \\ \mathcal{E}^{\{M\}}(U) &:= \{f \in C^\infty(U, \mathbb{R}) : \forall K \subseteq U \text{ compact } \exists \rho > 0 : \|f\|_{K, \rho}^M < \infty\}, \\ \|f\|_{K, \rho}^M &:= \sup \left\{ \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^n, \mathbb{R})}}{k! \rho^k M_k} : x \in K, k \in \mathbb{N} \right\}, \end{aligned}$$

and endow $\mathcal{E}^{(M)}(U)$ with its natural Fréchet space topology and $\mathcal{E}^{\{M\}}(U)$ with the projective limit topology over K of the inductive limit topology over ρ ; note that it suffices to take countable limits. We write $\mathcal{E}^{[M]}$ for either $\mathcal{E}^{(M)}$ or $\mathcal{E}^{\{M\}}$. The elements of $\mathcal{E}^{[M]}(U)$ are called $[M]$ -ultradifferentiable functions; an $(M)/\{M\}$ -ultradifferentiable function is said to be of *Beurling/Roumieu type*, respectively. For compact $K \subseteq U$ with smooth boundary,

$$\mathcal{E}_\rho^M(K) := \{f \in C^\infty(K) : \|f\|_{K, \rho}^M < \infty\}$$

is a Banach space, and we have

$$\mathcal{E}^{(M)}(U) = \varprojlim_{K \subseteq U} \varinjlim_{m \in \mathbb{N}} \mathcal{E}_{1/m}^M(K) \quad \text{and} \quad \mathcal{E}^{\{M\}}(U) = \varprojlim_{K \subseteq U} \varinjlim_{m \in \mathbb{N}} \mathcal{E}_m^M(K);$$

we also set

$$\mathcal{E}^{(M)}(K) := \{f \in C^\infty(K) : \forall \rho > 0 : \|f\|_{K,\rho}^M < \infty\} = \varprojlim_{m \in \mathbb{N}} \mathcal{E}_{1/m}^M(K),$$

$$\mathcal{E}^{\{M\}}(K) := \{f \in C^\infty(K) : \exists \rho > 0 : \|f\|_{K,\rho}^M < \infty\} = \varinjlim_{m \in \mathbb{N}} \mathcal{E}_m^M(K).$$

The definitions work as well for mappings $f : U \rightarrow \mathbb{R}^m$, and so we shall use also $\mathcal{E}^{[M]}(U, V)$, $\mathcal{E}^{[M]}(K, V)$, and $\mathcal{E}_\rho^M(K, V)$ for open subsets $V \subseteq \mathbb{R}^m$.

By the Denjoy–Carleman theorem, $\mathcal{E}^{[M]}$ is quasianalytic if and only if $M^{b(c)}$ satisfies (M_{qa}) ; this is in turn equivalent to

$$\sum_{k=0}^{\infty} \frac{M_k^{b(c)}}{(k+1)M_{k+1}^{b(c)}} = \infty \quad \text{and} \quad \int_1^{\infty} \frac{\log T_M(t)}{t^2} dt = \infty.$$

For modern proofs see for instance [18, 1.3.8], [36, 19.11], and [20, 4.2].

2.8. EXAMPLES. For $s \in \mathbb{R}_{\geq 0}$ the sequence $G^s = (G_k^s) = ((k!)^s)$ is log-convex and has moderate growth; it is quasianalytic if and only if $s = 0$. The elements of $\mathcal{E}^{\{G^0\}}(U)$ are exactly the real analytic functions $C^\omega(U)$ and the elements of $\mathcal{E}^{(G^0)}(U)$ are exactly the restrictions of entire functions $\mathcal{H}(\mathbb{C}^n)$. The class $\mathcal{E}^{\{G^s\}}$ coincides with the Gevrey class \mathcal{G}^{1+s} .

2.9. LEMMA. Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ be weakly log-convex. Then there exists a function $f \in \mathcal{E}_{\text{global}}^{\{M\}}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : \exists \rho > 0 : \|f\|_{\mathbb{R},\rho}^M < \infty\}$ such that $|f^{(k)}(0)| \geq k!M_k$ for all k .

Such a function is called a *characteristic $\mathcal{E}^{\{M\}}$ -function*.

Proof. The complex valued function

$$(2.10) \quad g(t) := \sum_{k=0}^{\infty} \frac{k!M_k}{(2\mu_k)^k} e^{2i\mu_k t}, \quad \text{where} \quad \mu_k := \frac{(k+1)M_{k+1}}{M_k},$$

belongs to $\mathcal{E}_{\text{global}}^{\{M\}}(\mathbb{R}, \mathbb{C})$ and satisfies

$$(2.11) \quad g^{(j)}(0) = i^j h_j, \quad \text{where} \quad h_j \geq j!M_j,$$

thus

$$|g^{(j)}(0)| \geq j!M_j,$$

for all j (see [40, Thm. 1]). Setting $f := \text{Re } g + \text{Im } g$ we obtain a real valued function with the required properties. ■

2.12. PROPOSITION. Let $L, M, N \in \mathbb{R}_{>0}^{\mathbb{N}}$, let $U \subseteq \mathbb{R}^n$ be open, and let $K \subseteq U$ be compact. We have:

- (1) $M \preceq N \Rightarrow \mathcal{E}^{[M]} \subseteq \mathcal{E}^{[N]}$ and $M \triangleleft N \Rightarrow \mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{(N)}$ with continuous inclusions. If M is weakly log-convex, then also the converse

implications hold; more precisely, $\mathcal{E}^{[M]}(\mathbb{R}) \subseteq \mathcal{E}^{[N]}(\mathbb{R}) \Rightarrow M \preceq N$ and $\mathcal{E}^{\{M\}}(\mathbb{R}) \subseteq \mathcal{E}^{(N)}(\mathbb{R}) \Rightarrow M \triangleleft N$.

(2) We have

$$\mathcal{E}^{\{M\}}(U, \mathbb{R}^m) = \bigcap_{M \triangleleft N} \mathcal{E}^{(N)}(U, \mathbb{R}^m) = \bigcap_{M \triangleleft N} \mathcal{E}^{\{N\}}(U, \mathbb{R}^m).$$

If M is (weakly) log-convex, then the intersections may be taken over all (weakly) log-convex $M \triangleleft N$.

(3) If $M_k^{1/k} \rightarrow \infty$ then

$$\mathcal{E}^{(M)}(K, \mathbb{R}^m) = \bigcup_{\substack{L \triangleleft M \\ L_k^{1/k} \rightarrow \infty}} \mathcal{E}^{\{L\}}(K, \mathbb{R}^m) = \bigcup_{\substack{L \triangleleft M \\ L_k^{1/k} \rightarrow \infty}} \mathcal{E}^{(L)}(K, \mathbb{R}^m).$$

If $(k!M_k)^{1/k} \rightarrow \infty$ then the unions may be taken over all $L \triangleleft M$ with $(k!L_k)^{1/k} \rightarrow \infty$. If M is log-convex and $M_{k+1}/M_k \rightarrow \infty$ then the unions may be taken over all log-convex $L \triangleleft M$ with $L_{k+1}/L_k \rightarrow \infty$.

Proof. (1) The directions “ \Rightarrow ” are clear by definition (see also [24, 2.3]). If M is weakly log-convex, then the implications $\mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{\{N\}} \Rightarrow M \preceq N$ and $\mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{(N)} \Rightarrow M \triangleleft N$ follow from the existence of a characteristic $\mathcal{E}^{\{M\}}$ -function (see Lemma 2.9). That $\mathcal{E}^{(M)} \subseteq \mathcal{E}^{(N)}$ implies $M \preceq N$ is shown in [10, Thm. 2.2] and in more general terms in Proposition 4.6.

(2) See [24, 2.4 and 8.2].

(3) follows from (1), Lemma 2.3, Remark 2.4, and [21, Lemma 6]. ■

As the elements of $\mathcal{E}^{\{1\}}(U)$ are exactly the real analytic functions $C^\omega(U)$ and the elements of $\mathcal{E}^{(1)}(U)$ are exactly the restrictions of entire functions $\mathcal{H}(\mathbb{C}^n)$, we may conclude:

(4) $C^\omega \subseteq \mathcal{E}^{\{M\}} \Leftrightarrow \mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}^{(M)}(U) \forall U \subseteq \mathbb{R}^n \Leftrightarrow \underline{\lim} M_k^{1/k} > 0$.

(5) $C^\omega \subseteq \mathcal{E}^{(M)}$ if and only if $\lim M_k^{1/k} = \infty$.

(6) $\mathcal{E}^{[M]}$ is derivation closed if M satisfies (M_{dc}) . If M is weakly log-convex, then (M_{dc}) is also necessary for $\mathcal{E}^{[M]}$ being derivation closed; indeed, for $M^{+1} = (M_k^{+1}) := (M_{k+1})$ we find that $\mathcal{E}^{[M^{+1}]}(\mathbb{R}) = \{f' : f \in \mathcal{E}^{[M]}(\mathbb{R})\}$.

In particular, if $L \triangleleft M$ with $\underline{\lim} L_k^{1/k} > 0$ then necessarily $\lim M_k^{1/k} = \infty$, by (1), (4), and (5).

Note that $\lim M_{k+1}/M_k = \infty$ implies $\lim M_k^{1/k} = \infty$ and thus $C^\omega \subseteq \mathcal{E}^{(M)}$. Indeed, there exists k_0 with $M_{k_0} \geq 1$, and for every $C > 0$ there exists $k_1 \geq k_0$ such that $M_k \geq CM_{k-1}$ for all $k > k_1$, whence $M_k^{1/k} \geq M_{k_0}^{1/k} C^{1-k_0/k} \geq C^{1/2}$ as $k > 2k_1$. If $M_k^{1/k}$ is increasing, we also have the converse: $\lim M_k^{1/k} = \infty$ implies $\lim M_{k+1}/M_k = \infty$.

2.13. LEMMA ([11, Lemme 3]). *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ and $\lambda > 0$. If $M_0 \leq \lambda^k M_k$ for all k and*

$$|f^{(k)}(t)| \leq k! M_k \quad \text{for all } t \in [-\lambda, \lambda], k \in \mathbb{N},$$

then

$$|f^{(k)}(0)| \leq 2e^k k! M_k^{b(c)} \quad \text{for all } k \in \mathbb{N}.$$

2.14. PROPOSITION. *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ satisfy $\underline{\lim} M_k^{1/k} > 0$ and $M_0 = 1$, let $K \subseteq \mathbb{R}^n$ be compact, and let $K_\lambda := \bigcup_{x \in K} \overline{B_\lambda(x)}$, $\lambda > 0$, be a λ -neighborhood of K . Then $\mathcal{E}^{\{M\}}(K_\lambda) \subseteq \mathcal{E}^{\{M^{b(c)}\}}(K)$ via restriction.*

Proof. By the assumption $\underline{\lim} M_k^{1/k} > 0$ there exists $\tau > 0$ such that $M_k \geq \tau^k$ for all k . If $f \in \mathcal{E}^{\{M\}}(K_\lambda)$, then $C := \|f\|_{K_\lambda, \rho}^M < \infty$, where we may assume that ρ is such that $\rho\lambda\tau \geq 1$. The function $f_{x,v}(t) := f(x+tv)$ satisfies $\|f_{x,v}\|_{[-\lambda, \lambda], \rho}^M \leq \|f\|_{K_\lambda, \rho}^M = C$ for all $x \in K$ and $v \in S^{n-1}$. By Lemma 2.13, we have

$$|d_v^k f(x)| = |f_{x,v}^{(k)}(0)| \leq 2C(e\rho)^k k! M_k^{b(c)} \quad \text{for all } x \in K, v \in S^{n-1}, k \in \mathbb{N},$$

since $(C\rho^k M_k)^{b(c)} = C\rho^k M_k^{b(c)}$ (see 2.6). Thus $f|_K \in \mathcal{E}^{\{M^{b(c)}\}}(K)$ (see e.g. [23, 7.13.1]). ■

2.15. THEOREM. *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ and let $U \subseteq \mathbb{R}^n$ be open. We have:*

- (1) *If $\underline{\lim} M_k^{1/k} > 0$ then $\mathcal{E}^{\{M\}}(U) = \mathcal{E}^{\{M^{b(c)}\}}(U)$.*
- (2) *If $\lim M_k^{1/k} = \infty$ then $\mathcal{E}^{\{M\}}(U) = \mathcal{E}^{\{M^{b(c)}\}}(U)$.*

Under these assumptions $\mathcal{E}^{\{M\}}(U)$ is an algebra.

(1) is due to [11, Thm. I & Appendix].

Proof. (1) Apply Propositions 2.12(1) and 2.14.

(2) Proposition 2.12(1) implies $\mathcal{E}^{\{M^{b(c)}\}}(U) \subseteq \mathcal{E}^{\{M\}}(U)$. Conversely, let $K \subseteq U$ be compact and let $K_\lambda := \bigcup_{x \in K} \overline{B_\lambda(x)} \subseteq U$ be a λ -neighborhood of K in U . By Proposition 2.12(3), Proposition 2.14, and Lemma 2.6,

$$\mathcal{E}^{\{M\}}(K_\lambda) = \bigcup \mathcal{E}^{\{L\}}(K_\lambda) \subseteq \bigcup \mathcal{E}^{\{L^{b(c)}\}}(K) \subseteq \mathcal{E}^{\{M^{b(c)}\}}(K),$$

where the unions are taken over all $L \triangleleft M$ with $L_k^{1/k} \rightarrow \infty$. As K was arbitrary, we have $\mathcal{E}^{\{M\}}(U) \subseteq \mathcal{E}^{\{M^{b(c)}\}}(U)$.

The last statement is a well-known consequence of weak log-convexity. ■

As a consequence, $C^\omega \subseteq \mathcal{E}^{\{M\}} = \mathcal{E}^{\{N\}}$ is impossible. Assume the contrary. Then, by 2.12(4)&(5) and Theorem 2.15, we may assume that M and N are weakly log-convex, and by Proposition 2.12(1), we have $M \triangleleft N$. Setting $L = (L_k)$ with $L_k := \sqrt{M_k N_k}$ we obtain $M \triangleleft L \triangleleft N$, and, by Lemma 2.6, we may assume that L is weakly log-convex. But then $\mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{\{L\}} \subseteq \mathcal{E}^{\{L\}} \subseteq \mathcal{E}^{\{N\}} = \mathcal{E}^{\{M\}}$ and thus $M \approx L \approx N$, a contradiction.

3. Stability under composition of $\mathcal{E}^{[M]}$. For $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ we define $M^\circ = (M_k^\circ)$ by setting

$$M_k^\circ := \max\{M_j M_{\alpha_1} \dots M_{\alpha_j} : \alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \dots + \alpha_j = k\}, \quad M_0^\circ := 1.$$

Clearly, $M \preceq M^\circ$. We have $M^\circ \preceq M$ if and only if M has the (FdB)-property.

3.1. PROPOSITION. *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ and let $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$, and $W \subseteq \mathbb{R}^r$ be open.*

- (1) *If $g \in \mathcal{E}^{[M]}(U, V)$ and $f \in \mathcal{E}^{[M]}(V, W)$, then $f \circ g \in \mathcal{E}^{[M^\circ]}(U, W)$.*
- (2) *If M has the (FdB)-property, then $\mathcal{E}^{[M]}$ is stable under composition.*

Proof. (1) Let $K \subseteq U$ be compact. There exist $C_g, \rho_g > 0$ (resp. for each $\rho_g > 0$ there exists $C_g > 0$) such that

$$\frac{\|g^{(k)}(x)\|_{L^k(\mathbb{R}^p, \mathbb{R}^q)}}{k!} \leq C_g \rho_g^k M_k \quad \text{for all } x \in K, k \in \mathbb{N},$$

and there exist $C_f, \rho_f > 0$ (resp. for each $\rho_f > 0$ there exists $C_f > 0$) such that

$$\frac{\|f^{(k)}(y)\|_{L^k(\mathbb{R}^q, \mathbb{R}^r)}}{k!} \leq C_f \rho_f^k M_k \quad \text{for all } y \in g(K), k \in \mathbb{N}.$$

By Faà di Bruno’s formula (see [15] for the 1-dimensional version; the second sum below is over all $\alpha \in \mathbb{N}_{>0}^j$ with $\alpha_1 + \dots + \alpha_j = k$)

$$\begin{aligned} & \frac{\|(f \circ g)^{(k)}(x)\|_{L^k(\mathbb{R}^p, \mathbb{R}^q)}}{k!} \\ & \leq \sum_{j \geq 1} \sum_{\alpha} \frac{\|f^{(j)}(g(x))\|_{L^j(\mathbb{R}^q, \mathbb{R}^r)}}{j!} \prod_{i=1}^j \frac{\|g^{(\alpha_i)}(x)\|_{L^{\alpha_i}(\mathbb{R}^p, \mathbb{R}^q)}}{\alpha_i!} \\ & \leq \sum_{j \geq 1} \sum_{\alpha} C_f \rho_f^j C_g^j \rho_g^k M_j \prod_{i=1}^j M_{\alpha_i} \leq C_f \rho_g^k \left(\sum_{j \geq 1} \binom{k-1}{j-1} (\rho_f C_g)^j \right) M_k^\circ \\ & \leq C_f C_g \rho_f (\rho_g (1 + \rho_f C_g))^k M_k^\circ. \end{aligned}$$

This implies the assertion in the Roumieu case. For the Beurling case, let $\tau > 0$ be arbitrary, and choose $\sigma > 0$ such that $\tau = \sqrt{\sigma} + \sigma$. If we set $\rho_g = \sqrt{\sigma}$ and $\rho_f = \sqrt{\sigma}/C_g$, then $\|f \circ g\|_{K, \tau}^{M^\circ} < \infty$.

(2) follows immediately from (1) and Proposition 2.12(1). ■

We get a nice characterization of stability under composition if we assume that $\mathcal{E}^{[M]}$ is stable under derivation.

3.2. THEOREM. *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ and assume that $\mathcal{E}^{[M]}$ is stable under derivation. Consider the following conditions:*

- (1) $\mathcal{E}^{[M]}$ is stable under composition.
- (2) $\mathcal{E}^{[M]}$ is holomorphically closed.
- (3) $\mathcal{E}^{[M]}$ is inverse closed.
- (4) $(M_k^{b(c)})^{1/k}$ is almost increasing.
- (5) $(M_k^{b(o)})^{1/k}$ is almost increasing.
- (6) $M^{b(c)}$ has the (FdB)-property.
- (7) $M^{b(o)}$ has the (FdB)-property.

If $\underline{\lim} M_k^{1/k} > 0$ then all conditions are equivalent in the Roumieu case $\mathcal{E}^{[M]} = \mathcal{E}^{\{M\}}$. If $\lim M_k^{1/k} = \infty$ then all conditions are equivalent in any case.

Proof. Under the assumption $\underline{\lim} M_k^{1/k} > 0$ we have $\mathcal{E}^{\{M\}} = \mathcal{E}^{\{M^{b(c)}\}}$, by Theorem 2.15. The equivalences (4) \Leftrightarrow (5) and (6) \Leftrightarrow (7) follow from the fact that $\mathcal{E}^{\{M\}}(I) = \mathcal{E}^{\{M^{b(o)}\}}(I)$ for open intervals I (see [27, 6.5.1]), which implies $M^{b(c)} \approx M^{b(o)}$, by [39, Lemma II]. Lemma 2.2 and 2.12(6) imply (4) \Rightarrow (6).

Let us prove the remaining implications in the Roumieu case $\mathcal{E}^{[M]} = \mathcal{E}^{\{M\}}$. Since $C^\omega \subseteq \mathcal{E}^{\{M\}}$ by 2.12(4), we clearly have (1) \Rightarrow (2) \Rightarrow (3). The implication (3) \Rightarrow (5) follows from [39], and (6) \Rightarrow (1) follows from Proposition 3.1. Note that (3) \Rightarrow (4) is shown in greater generality in the proof of Theorem 4.9 below.

Now let us assume the stronger condition $\lim M_k^{1/k} = \infty$ and show the remaining implications in the Beurling case $\mathcal{E}^{[M]} = \mathcal{E}^{(M)}$. Since $C^\omega \subseteq \mathcal{E}^{(M)}$, by 2.12(5), we have (1) \Rightarrow (2) \Rightarrow (3). The implication (3) \Rightarrow (4) follows from [10] since $\mathcal{E}^{(M)}(\mathbb{R}) = \mathcal{E}^{(M^{b(c)})}(\mathbb{R})$ is a Fréchet algebra, by Theorem 2.15, and (6) \Rightarrow (1) follows from Proposition 3.1. ■

3.3. Log-convexity is not necessary for stability under composition. There exist classes $\mathcal{E}^{[M]}$ (containing C^ω) which are closed under composition and there is no log-convex $N \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\mathcal{E}^{[M]} = \mathcal{E}^{[N]}$. We need the following lemma.

3.4. LEMMA. *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ be such that $C^\omega \subseteq \mathcal{E}^{[M]}$ (i.e., $\underline{\lim} M_k^{1/k} > 0$ in the Roumieu case and $\lim M_k^{1/k} = \infty$ in the Beurling case). If there exists a log-convex $N \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\mathcal{E}^{[M]} = \mathcal{E}^{[N]}$, then the sequence k_{i+1}/k_i is bounded, where the k_i are precisely those k with $M_k = M_k^{b(c)}$.*

Proof. This is a special case of [11, Appendix Prop. 3]. For the reader's convenience we give a short proof. By Theorem 2.15, we have $\mathcal{E}^{[M^{b(c)}]} = \mathcal{E}^{[N]}$ and thus $M^{b(c)} \approx N$, by Proposition 2.12(1). Since N is weakly log-convex,

we have $N \leq M^{b(c)} \leq M$. Set

$$L := \left\{ \begin{array}{ll} N_k, & k = k_i, \\ +\infty, & \text{otherwise} \end{array} \right\}^{b(c)}.$$

For $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ consider the graph $\Gamma_M := \{(k, \log(k!M_k)) : k \in \mathbb{N}\}$. Then $\Gamma_{M^{b(c)}}$ and Γ_L lie on piecewise linear curves with vertices $\{(k_i, \log(k_i!M_{k_i})) : i \in \mathbb{N}\}$ and $\{(k_i, \log(k_i!N_{k_i})) : i \in \mathbb{N}\}$, respectively. Since N is weakly log-convex and since Γ_L lies below $\Gamma_{M^{b(c)}}$, we have $N \leq L \leq M^{b(c)} \approx N$ and hence $L \approx N$. As N is log-convex, we have, for $k_i \leq k \leq k_{i+1}$,

$$\begin{aligned} \log(k!L_k) &= \frac{k_{i+1} - k}{k_{i+1} - k_i} \log(k_i!N_{k_i}) + \frac{k - k_i}{k_{i+1} - k_i} \log(k_{i+1}!N_{k_{i+1}}) \\ &\geq \frac{k_{i+1} - k}{k_{i+1} - k_i} \log k_i! + \frac{k - k_i}{k_{i+1} - k_i} \log k_{i+1}! + \log N_k, \end{aligned}$$

and therefore

$$(3.5) \quad \log \left(\frac{k!L_k}{k!N_k} \right)^{1/k} \geq \frac{1}{k} \frac{k_{i+1} - k}{k_{i+1} - k_i} \log k_i! + \frac{1}{k} \frac{k - k_i}{k_{i+1} - k_i} \log k_{i+1}! - \frac{1}{k} \log k!.$$

By Stirling's formula, for $k_{i+1}/k_i =: a_i$ and $k := 2k_i$ the right-hand side of (3.5) is greater than

$$\begin{aligned} \frac{1}{2} \frac{a_i - 2}{a_i - 1} (\log k_i - 1) + \frac{1}{2} \frac{a_i}{a_i - 1} (\log a_i + \log k_i - 1) - \log(2k_i) \\ = \frac{1}{2} \frac{a_i}{a_i - 1} \log a_i - \log 2 - 1, \end{aligned}$$

and so $L \approx N$ implies that a_i is bounded. ■

3.6. EXAMPLE. Choose $r \in \mathbb{R}_{\geq 4}$. Set $k_i := k_{i-1} \lceil \log(i+1) \rceil$, $i \geq 2$, $k_1 := 3$, where $\lceil x \rceil$ denotes the smallest integer $n \geq x$, and define

$$\mu_k = \mu(r)_k := \begin{cases} 1, & k = 1, 2, \\ r^k, & k = k_i, \\ r^{k_i-1}, & k_i < k < k_{i+1}, \end{cases}$$

$$M_k = M(r)_k := \frac{1}{k!} \prod_{j=1}^k \mu_j.$$

Then $M = (M_k)$ is derivation closed, since $\mu_k/k \leq r^k$ for all k , and M is not weakly log-convex, since $\mu = (\mu_k)$ is not increasing. By construction we have $M_j M_k \leq M_1 M_{j+k-1}$ for all $j, k \geq 1$, i.e.,

$$\frac{\mu_1 \dots \mu_k}{1 \dots k} \leq \frac{\mu_1}{1} \frac{\mu_{j+1}}{j+1} \dots \frac{\mu_{j+k-1}}{j+k-1}, \quad j, k \geq 1.$$

Indeed, since μ_k/k is decreasing for $k_i \leq k < k_{i+1}$ and since

$$\frac{\mu_{k_i+1}}{k_i+1} \leq \frac{\mu_{k_{i+2}-1}}{k_{i+2}-1} \quad \text{for all } i,$$

it suffices to check that, for all i ,

$$\frac{\mu_{k_{i+1}-1}}{k_{i+1}-1} \frac{\mu_{k_{i+1}}}{k_{i+1}} \leq \frac{\mu_{k_{i+2}-2}}{k_{i+2}-2} \frac{\mu_{k_{i+2}-1}}{k_{i+2}-1}$$

which is a straightforward computation. By Lemma 2.2(3) and Proposition 3.1, $\mathcal{E}^{[M]}$ is stable under composition.

Consider the graph $\Gamma_M := \{P_k := (k, \log(k!M_k)) : k \in \mathbb{N}\}$. The subset $\{P_k : k_i \leq k < k_{i+1}\}$ lies on an affine line with slope $(k_i - 1) \log r$. The line that connects $P_{k_{i-1}}$ and P_{k_i} has slope $k_i \log r$, and the line that connects $P_{k_{i-1}}$ and $P_{k_{i+1}-1}$ has slope $(k_i - 1 + (k_{i+1} - k_i)^{-1}) \log r$. All these slopes are strictly increasing to infinity as $i \rightarrow \infty$. We conclude that the graph $\Gamma_{M^{b(c)}} := \{(k, \log(k!M_k^{b(c)})) : k \in \mathbb{N}\}$ lies on the piecewise linear curve with vertices $\{P_{k_{i-1}} : i \in \mathbb{N}\}$ and that $\{k_i - 1\}$ is precisely the set of k with $M_k = M_k^{b(c)}$.

As $M_k/M_{k-1} = \mu_k/k \rightarrow \infty$ we have $M_k^{1/k} \rightarrow \infty$ (see the remarks after 2.12), and, by Lemma 3.4, there is no log-convex $N \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\mathcal{E}^{[M]} = \mathcal{E}^{[N]}$. It is easy to see that the mapping $r \mapsto \mathcal{E}^{[M(r)]}$ is injective.

4. More general spaces of ultradifferentiable functions

4.1. Weight matrices. A *weight matrix* $\mathfrak{M} = \{M^\lambda \in \mathbb{R}_{>0}^{\mathbb{N}} : \lambda \in \Lambda\}$ is a family of weakly log-convex sequences $M^\lambda = (M_k^\lambda)$ satisfying $M_0^\lambda = 1$, $\lim_k (k!M_k^\lambda)^{1/k} = \infty$, and $M^\lambda \leq M^\mu$ if $\lambda \leq \mu$, where Λ is a directed partially ordered set. Let $\mathcal{M} = \mathcal{M}(\Lambda)$ be the set of all weight matrices \mathfrak{M} parameterized by the same set Λ . Consider the following conditions:

$$\begin{aligned} (\mathfrak{M}_{\mathcal{H}}) \quad & \forall \lambda \in \Lambda : \quad \underline{\lim} (M_k^\lambda)^{1/k} > 0. \\ (\mathfrak{M}_{(C^\omega)}) \quad & \forall \lambda \in \Lambda : \quad \lim (M_k^\lambda)^{1/k} = \infty. \\ (\mathfrak{M}_{\{C^\omega\}}) \quad & \exists \lambda \in \Lambda : \quad \underline{\lim} (M_k^\lambda)^{1/k} > 0. \\ (\mathfrak{M}_{(\text{dc})}) \quad & \forall \lambda \in \Lambda \exists \mu \in \Lambda \exists C > 0 \forall k \in \mathbb{N} : \quad M_{k+1}^\mu \leq C^k M_k^\lambda. \\ (\mathfrak{M}_{\{\text{dc}\}}) \quad & \forall \lambda \in \Lambda \exists \mu \in \Lambda \exists C > 0 \forall k \in \mathbb{N} : \quad M_{k+1}^\lambda \leq C^k M_k^\mu. \\ (\mathfrak{M}_{(\text{mg})}) \quad & \forall \lambda \in \Lambda \exists \mu \in \Lambda \exists C > 0 \forall j, k \in \mathbb{N} : \quad M_{j+k}^\mu \leq C^{j+k} M_j^\lambda M_k^\lambda. \\ (\mathfrak{M}_{\{\text{mg}\}}) \quad & \forall \lambda \in \Lambda \exists \mu \in \Lambda \exists C > 0 \forall j, k \in \mathbb{N} : \quad M_{j+k}^\lambda \leq C^{j+k} M_j^\mu M_k^\mu. \\ (\mathfrak{M}_{(\text{alg})}) \quad & \forall \lambda \in \Lambda \exists \mu \in \Lambda \exists C > 0 \forall j, k \in \mathbb{N} : \quad M_j^\mu M_k^\mu \leq C^{j+k} M_{j+k}^\lambda. \\ (\mathfrak{M}_{\{\text{alg}\}}) \quad & \forall \lambda \in \Lambda \exists \mu \in \Lambda \exists C > 0 \forall j, k \in \mathbb{N} : \quad M_j^\lambda M_k^\lambda \leq C^{j+k} M_{j+k}^\mu. \\ (\mathfrak{M}_{(\text{FdB})}) \quad & \forall \lambda \in \Lambda \exists \mu \in \Lambda : \quad (M^\mu)^\circ \preceq M^\lambda. \end{aligned}$$

$$\begin{aligned}
(\mathfrak{M}_{\{\text{FdB}\}}) \quad & \forall \lambda \in \Lambda \exists \mu \in \Lambda : \quad (M^\lambda)^\circ \preceq M^\mu. \\
(\mathfrak{M}_{\{\text{L}\}}) \quad & \forall \lambda \in \Lambda \forall \rho > 0 \exists \mu \in \Lambda \exists C > 0 \forall k \in \mathbb{N} : \quad \rho^k M_k^\mu \leq C M_k^\lambda. \\
(\mathfrak{M}_{\{\text{L}\}}) \quad & \forall \lambda \in \Lambda \forall \rho > 0 \exists \mu \in \Lambda \exists C > 0 \forall k \in \mathbb{N} : \quad \rho^k M_k^\lambda \leq C M_k^\mu. \\
(\mathfrak{M}_{\{\text{BR}\}}) \quad & \forall \lambda \in \Lambda \exists \mu \in \Lambda : \quad M^\mu \triangleleft M^\lambda. \\
(\mathfrak{M}_{\{\text{BR}\}}) \quad & \forall \lambda \in \Lambda \exists \mu \in \Lambda : \quad M^\lambda \triangleleft M^\mu.
\end{aligned}$$

Obviously, $(\mathfrak{M}_{\{C^\omega\}}) \Rightarrow (\mathfrak{M}_{\mathcal{H}}) \Rightarrow (\mathfrak{M}_{\{C^\omega\}})$ and $(\mathfrak{M}_{[\text{mg}]}) \Rightarrow (\mathfrak{M}_{[\text{dc}]})$. Both $(\mathfrak{M}_{\{\text{alg}\}})$ and $(\mathfrak{M}_{\{\text{alg}\}})$ are trivially satisfied since all M^λ are weakly log-convex; but see Remarks 4.5.

Henceforth we assume that Λ is \mathbb{R} or any ordered subset of \mathbb{R} . This will enable us to assume that the limits over $\lambda \in \Lambda$ in the definition of $[\mathfrak{M}]$ -ultradifferentiable functions in 4.2 are countable. Then \mathfrak{M} is in fact an infinite matrix, and the name *weight matrix* is justified. On the other hand it is convenient to admit uncountable index sets Λ .

4.2. $[\mathfrak{M}]$ -ultradifferentiable functions. Let \mathfrak{M} be a weight matrix, let $U \subseteq \mathbb{R}^n$ be open, and let $K \subseteq U$ be compact. We define

$$\begin{aligned}
\mathcal{E}^{(\mathfrak{M})}(K) &:= \bigcap_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(K), & \mathcal{E}^{\{\mathfrak{M}\}}(K) &:= \bigcup_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(K), \\
\mathcal{E}^{(\mathfrak{M})}(U) &:= \bigcap_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(U), & \mathcal{E}^{\{\mathfrak{M}\}}(U) &:= \bigcap_{K \subseteq U} \bigcup_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(K),
\end{aligned}$$

and endow these spaces with their natural topologies:

$$\mathcal{E}^{(\mathfrak{M})}(U) := \varprojlim_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(U), \quad \mathcal{E}^{\{\mathfrak{M}\}}(U) := \varprojlim_{K \subseteq U} \varprojlim_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(K).$$

It is no loss of generality to assume that the limits are countable. We write $\mathcal{E}^{[\mathfrak{M}]}$ for either $\mathcal{E}^{(\mathfrak{M})}$ or $\mathcal{E}^{\{\mathfrak{M}\}}$. The elements of $\mathcal{E}^{[\mathfrak{M}]}(U)$ are called $[\mathfrak{M}]$ -ultradifferentiable functions. Note that $\mathcal{E}^{[\mathfrak{M}]}(U)$ forms an algebra, since all M^λ are weakly log-convex.

We shall use also $\mathcal{E}^{[\mathfrak{M}]}(U, V)$ and $\mathcal{E}^{[\mathfrak{M}]}(K, V)$ for open subsets $V \subseteq \mathbb{R}^m$.

The inductive limit

$$\mathcal{E}^{\{\mathfrak{M}\}}(K, \mathbb{R}^m) = \varinjlim_{\lambda \in \Lambda} \varinjlim_{\rho > 0} \mathcal{E}_\rho^{M^\lambda}(K, \mathbb{R}^m) = \varinjlim_{(\lambda, \rho)} \mathcal{E}_\rho^{M^\lambda}(K, \mathbb{R}^m),$$

where $(\lambda, \rho) \leq (\mu, \sigma)$ if and only if $\lambda \leq \mu$ and $\rho \leq \sigma$, is a Silva space. Indeed, if $\lambda \leq \mu$ and $\rho < \sigma$ then the inclusion

$$\mathcal{E}_\rho^{M^\lambda}(K, \mathbb{R}^m) \rightarrow \mathcal{E}_\rho^{M^\mu}(K, \mathbb{R}^m) \rightarrow \mathcal{E}_\sigma^{M^\mu}(K, \mathbb{R}^m)$$

is compact, since the first inclusion is bounded and the second inclusion is compact, by [20, Prop. 2.2].

If \mathfrak{M} satisfies $(\mathfrak{M}_{(L)})$, respectively $(\mathfrak{M}_{\{L\}})$, we have

$$(4.3) \quad \begin{aligned} \mathcal{E}^{(\mathfrak{M})}(K, \mathbb{R}^m) &= \varprojlim_{(\lambda, \rho)} \mathcal{E}_\rho^{M^\lambda}(K, \mathbb{R}^m) = \varprojlim_{\lambda} \mathcal{E}_1^{M^\lambda}(K, \mathbb{R}^m), \quad \text{respectively} \\ \mathcal{E}^{\{\mathfrak{M}\}}(K, \mathbb{R}^m) &= \varinjlim_{(\lambda, \rho)} \mathcal{E}_\rho^{M^\lambda}(K, \mathbb{R}^m) = \varinjlim_{\lambda} \mathcal{E}_1^{M^\lambda}(K, \mathbb{R}^m) \end{aligned}$$

as locally convex spaces, where the latter is a Silva space. Indeed, for $\rho > 1$ and by $(\mathfrak{M}_{\{L\}})$ the inclusion

$$\mathcal{E}_1^{M^\lambda}(K, \mathbb{R}^m) \rightarrow \mathcal{E}_\rho^{M^\lambda}(K, \mathbb{R}^m) \rightarrow \mathcal{E}_1^{M^\mu}(K, \mathbb{R}^m)$$

is compact. If $(\mathfrak{M}_{(L)})$ then for each $\lambda \in \Lambda$ and each $\rho > 0$ we find $\mu \in \Lambda$ such that $\mathcal{E}_1^{M^\mu}(K, \mathbb{R}^m) \subseteq \mathcal{E}_\rho^{M^\lambda}(K, \mathbb{R}^m)$ with continuous inclusion.

If \mathfrak{M} satisfies $(\mathfrak{M}_{(BR)})$, respectively $(\mathfrak{M}_{\{BR\}})$, we have

$$(4.4) \quad \begin{aligned} \mathcal{E}^{(\mathfrak{M})}(U, \mathbb{R}^m) &= \varprojlim_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(U, \mathbb{R}^m) = \varprojlim_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(U, \mathbb{R}^m), \quad \text{respectively} \\ \mathcal{E}^{\{\mathfrak{M}\}}(K, \mathbb{R}^m) &= \varinjlim_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(K, \mathbb{R}^m) = \varinjlim_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(K, \mathbb{R}^m) \end{aligned}$$

as locally convex spaces.

Among the spaces $\mathcal{E}^{[\mathfrak{M}]}$ we recover the spaces $\mathcal{E}^{[M]}$ defined by weight sequences, if $\mathfrak{M} = \{M\}$ consists just of a single $M \in \mathbb{R}_{>0}^{\mathbb{N}}$, and the spaces $\mathcal{E}^{[\omega]}$ defined by weight functions (see Corollary 5.15 below). We shall see in Theorem 5.22 that in general $\mathcal{E}^{[\mathfrak{M}]}$ is different from $\mathcal{E}^{[M]}$ and from $\mathcal{E}^{[\omega]}$.

4.5. REMARKS. (1) One can replace the condition that the $M^\lambda \in \mathfrak{M}$ are weakly log-convex by the condition $(\mathfrak{M}_{\mathcal{H}})$ (resp. $(\mathfrak{M}_{(C^\omega)})$), and work with the log-convex minorants $(M^\lambda)^{b(c)}$ without changing the space $\mathcal{E}^{\{\mathfrak{M}\}}(U)$ (resp. $\mathcal{E}^{(\mathfrak{M})}(U)$) (see Proposition 2.14 and Theorem 2.15). Alternatively, assuming $(\mathfrak{M}_{[\text{alg}]})$ makes $\mathcal{E}^{[\mathfrak{M}]}$ into an algebra as well. The condition $M^\lambda \leq M^\mu$ if $\lambda \leq \mu$ may be relaxed to $M^\lambda \preceq M^\mu$.

(2) Assuming that $(M_k^\lambda / M_k^\mu)^{1/k}$ is (ultimately) monotonic in k for all λ, μ , we have either $M^\lambda \approx M^\mu$ for all λ, μ or $M^\lambda \triangleleft M^\mu$ for all $\lambda < \mu$. That is, either $\mathcal{E}^{[\mathfrak{M}]} = \mathcal{E}^{[M^\lambda]}$ for all λ or we have the representations in (4.4).

For $\mathfrak{M}, \mathfrak{N} \in \mathcal{M}$ we define

$$\begin{aligned} \mathfrak{M}(\preceq)\mathfrak{N} &:\Leftrightarrow \forall \lambda \in \Lambda \exists \mu \in \Lambda : M^\mu \preceq N^\lambda, \\ \mathfrak{M}\{\preceq\}\mathfrak{N} &:\Leftrightarrow \forall \lambda \in \Lambda \exists \mu \in \Lambda : M^\lambda \preceq N^\mu, \\ \mathfrak{M}[\approx]\mathfrak{N} &:\Leftrightarrow \mathfrak{M}[\preceq]\mathfrak{N} \text{ and } \mathfrak{N}[\preceq]\mathfrak{M}, \\ \mathfrak{M}(\preceq)\mathfrak{N} &:\Leftrightarrow \exists \lambda \in \Lambda \exists \mu \in \Lambda : M^\lambda \preceq N^\mu, \\ \mathfrak{M}\{\triangleleft\}\mathfrak{N} &:\Leftrightarrow \forall \lambda \in \Lambda \forall \mu \in \Lambda : M^\lambda \triangleleft N^\mu. \end{aligned}$$

4.6. PROPOSITION. For $\mathfrak{M}, \mathfrak{N} \in \mathcal{M}$ we have:

- (1) $\mathfrak{M}[\preceq] \mathfrak{N} \Rightarrow \mathcal{E}^{[\mathfrak{M}]} \subseteq \mathcal{E}^{[\mathfrak{N}]}$ and $\mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}) \subseteq \mathcal{E}^{[\mathfrak{N}]}(\mathbb{R}) \Rightarrow \mathfrak{M}[\preceq] \mathfrak{N}$.
- (2) $\mathfrak{M}\{\triangleleft\} \mathfrak{N} \Rightarrow \mathcal{E}^{\{\mathfrak{M}\}} \subseteq \mathcal{E}^{\{\mathfrak{N}\}}$ and $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}) \subseteq \mathcal{E}^{\{\mathfrak{N}\}}(\mathbb{R}) \Rightarrow \mathfrak{M}\{\triangleleft\} \mathfrak{N}$.
- (3) $\mathfrak{M}(\preceq) \mathfrak{N} \Rightarrow \mathcal{E}^{(\mathfrak{M})} \subseteq \mathcal{E}^{(\mathfrak{N})}$ and $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) \subseteq \mathcal{E}^{(\mathfrak{N})}(\mathbb{R}) \Rightarrow \mathfrak{M}(\preceq) \mathfrak{N}$.

All inclusions are continuous.

Proof. (1) That $\mathfrak{M}[\preceq] \mathfrak{N}$ implies $\mathcal{E}^{[\mathfrak{M}]} \subseteq \mathcal{E}^{[\mathfrak{N}]}$ is clear by definition. If $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}) \subseteq \mathcal{E}^{\{\mathfrak{N}\}}(\mathbb{R})$ then $\mathfrak{M}\{\preceq\} \mathfrak{N}$ follows from the existence of characteristic $\mathcal{E}^{\{M^\lambda\}}$ -functions, by Lemma 2.9. If $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) \subseteq \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$ then this inclusion is continuous, by the closed graph theorem, since convergence in $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ implies pointwise convergence; here we follow [10, Thm. 2.2]. Thus for each $\lambda \in \Lambda$, each compact $I \subseteq \mathbb{R}$, and each $\tau > 0$ there exist $\mu \in \Lambda$, $J \subseteq \mathbb{R}$ compact, and constants $C, \rho > 0$ such that

$$\|f\|_{I,\tau}^{N^\lambda} \leq C \|f\|_{J,\rho}^{M^\mu} \quad \text{for } f \in \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}).$$

In particular, for $f_t(x) = e^{itx}$ and $\tau = 1$, we obtain

$$T_{N^\lambda}(t) = \sup_{k \in \mathbb{N}} \frac{t^k}{k! N_k^\lambda} \leq C \sup_{k \in \mathbb{N}} \frac{t^k}{k! \rho^k M_k^\mu} = C T_{M^\mu}(t/\rho),$$

and thus

$$k! N_k^\lambda = \sup_{t > 0} \frac{t^k}{T_{N^\lambda}(t)} \geq \sup_{t > 0} \frac{t^k}{C T_{M^\mu}(t/\rho)} = k! \frac{\rho^k}{C} M_k^\mu,$$

that is, $\mathfrak{M}(\preceq) \mathfrak{N}$.

(2) That $\mathfrak{M}\{\triangleleft\} \mathfrak{N}$ implies $\mathcal{E}^{\{\mathfrak{M}\}} \subseteq \mathcal{E}^{\{\mathfrak{N}\}}$ is clear by definition. The converse follows from the existence of characteristic $\mathcal{E}^{\{M^\lambda\}}$ -functions.

(3) That $\mathfrak{M}(\preceq) \mathfrak{N}$ implies $\mathcal{E}^{(\mathfrak{M})} \subseteq \mathcal{E}^{(\mathfrak{N})}$ is clear by definition. Conversely, if $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) \subseteq \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$ then the closed graph theorem (cf. [19, 5.4.1]) implies that this inclusion is continuous. Indeed, $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ is a Fréchet space, $\mathcal{E}^{\{\mathfrak{N}\}}(\mathbb{R})$ is projective limit of Silva spaces, hence webbed, and convergence implies pointwise convergence. This and Grothendieck's factorization theorem (e.g. [28, 24.33]) imply that for each compact $I \subseteq \mathbb{R}$ there exist $\lambda \in \Lambda$, $\tau > 0$, $\mu \in \Lambda$, $J \subseteq \mathbb{R}$ compact, and constants $C, \rho > 0$ such that

$$\|f\|_{I,\tau}^{N^\lambda} \leq C \|f\|_{J,\rho}^{M^\mu} \quad \text{for } f \in \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}).$$

Applying this to $f_t(x) = e^{itx}$ we obtain, as in (1),

$$M_k^\mu \leq C(\tau/\rho)^k N_k^\lambda,$$

that is, $\mathfrak{M}(\preceq) \mathfrak{N}$. ■

We may conclude:

- (4) $\mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}^{(\mathfrak{M})}(U)$ for all open $U \subseteq \mathbb{R}^n$ if and only if $(\mathfrak{M}_{\mathcal{H}})$ holds.
- (5) $C^\omega \subseteq \mathcal{E}^{[\mathfrak{M}]}$ if and only if $(\mathfrak{M}_{[C^\omega]})$ holds.
- (6) $\mathcal{E}^{[\mathfrak{M}]}$ is derivation closed if and only if $(\mathfrak{M}_{[\text{dc}]})$ holds.

Note that for $L \in \mathbb{R}_{>0}^{\mathbb{N}}$ we have $L(\preceq)\mathfrak{M}$ if and only if $L\{\preceq\}\mathfrak{M}$; in particular, $\mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(U)$ if and only if $C^\omega(U) \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(U)$, for all open $U \subseteq \mathbb{R}^n$. Moreover:

4.7. COROLLARY. *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $\lim M_k^{1/k} = \infty$. Then there is no $N \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that*

$$\mathcal{E}^{(M)}(\mathbb{R}) \subsetneq \mathcal{E}^{[N]}(\mathbb{R}) \subsetneq \mathcal{E}^{\{M\}}(\mathbb{R}).$$

Proof. This follows from Proposition 4.6 and Theorem 2.15. ■

4.8. REMARK. It is easy to see that $\mathcal{E}^{\{\mathfrak{M}\}}$ is non-quasianalytic if and only if there is some $\lambda \in \Lambda$ such that M^λ is non-quasianalytic. Likewise, if $\mathcal{E}^{\{\mathfrak{M}\}}$ is non-quasianalytic then M^λ is non-quasianalytic for all $\lambda \in \Lambda$. Intersections $\bigcap_M \mathcal{E}^{[M]}$, where M runs through a *large* family of non-quasianalytic weakly log-convex weight sequences, can be quasianalytic (see [26] and references therein). But we do not know whether $\mathcal{E}^{\{\mathfrak{M}\}}$ can be quasianalytic if all M^λ are non-quasianalytic and Λ is restricted to a 1-parameter family (as assumed in this paper).

4.9. THEOREM. *For a weight matrix \mathfrak{M} satisfying $(\mathfrak{M}_{\{\text{dc}\}})$ and $(\mathfrak{M}_{\{C^\omega\}})$ the following are equivalent:*

- (1) $\mathcal{E}^{\{\mathfrak{M}\}}$ is stable under composition.
- (2) $\mathcal{E}^{\{\mathfrak{M}\}}$ is holomorphically closed.
- (3) For all $\lambda \in \Lambda$ there are $\mu \in \Lambda$ and $C > 0$ such that

$$(M_j^\lambda)^{1/j} \leq C(M_k^\mu)^{1/k} \quad \text{if } j \leq k.$$

- (4) \mathfrak{M} satisfies $(\mathfrak{M}_{\{\text{FdB}\}})$.

Note that $(\mathfrak{M}_{\{C^\omega\}})$ is only needed for (1) \Rightarrow (2), and $(\mathfrak{M}_{\{\text{dc}\}})$ is only needed for (3) \Rightarrow (4).

Proof. (1) \Rightarrow (2). This is obvious, by 4.6(5).

(2) \Rightarrow (3). We prove that (3) holds if $\mathcal{E}^{\{\mathfrak{M}\}}$ is inverse closed; we follow the idea of [39]. Let $\lambda \in \Lambda$ be fixed and let $g \in \mathcal{E}^{\{M^\lambda\}}(\mathbb{R}, \mathbb{C})$ be as defined by (2.10) (with $M = (M_k)$ replaced by $M^\lambda = (M_k^\lambda)$). Choose $H > 0$ such that $H > 1 + \sup_{t \in \mathbb{R}} |g(t)|$. We have $H - g \in \mathcal{E}^{\{M^\lambda\}}(\mathbb{R}, \mathbb{C})$, and thus

$$f := (H - g)^{-1} \in \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}, \mathbb{C}),$$

as $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}, \mathbb{C})$ is inverse closed, by assumption. Thus, there exist $\mu \in \Lambda$ and constants $C, \rho > 0$ such that

$$(4.10) \quad \|f\|_{[-1,1],\rho}^{M^\mu} < C.$$

By Faà di Bruno's formula and using (2.11), for $k \geq 1$,

$$\begin{aligned} \frac{f^{(k)}(0)}{k!} &= \sum_{j \geq 1} \sum_{\substack{\alpha_1 + \dots + \alpha_j = k \\ a_\ell > 0}} \frac{1}{(H - g(0))^{j+1}} \prod_{\ell=1}^j \frac{g^{(\alpha_\ell)}(0)}{\alpha_\ell!} \\ &= i^k \sum_{j \geq 1} \sum_{\substack{\alpha_1 + \dots + \alpha_j = k \\ a_\ell > 0}} \frac{1}{(H - g(0))^{j+1}} \prod_{\ell=1}^j \frac{h_{\alpha_\ell}}{\alpha_\ell!}. \end{aligned}$$

By (4.10),

$$\begin{aligned} C\rho^k M_k^\mu &\geq \frac{|f^{(k)}(0)|}{k!} = \sum_{j \geq 1} \sum_{\substack{\alpha_1 + \dots + \alpha_j = k \\ a_\ell > 0}} \frac{1}{(H - g(0))^{j+1}} \prod_{\ell=1}^j \frac{h_{\alpha_\ell}}{\alpha_\ell!} \\ &\geq \sum_{j \geq 1} \sum_{\substack{\alpha_1 + \dots + \alpha_j = k \\ a_\ell > 0}} \frac{1}{(H - g(0))^{j+1}} \prod_{\ell=1}^j M_{\alpha_\ell}^\lambda \\ &\geq \frac{1}{(H - g(0))^{k+1}} \prod_{\ell=1}^j M_{\alpha_\ell}^\lambda. \end{aligned}$$

In particular, for $\alpha_1 = \dots = \alpha_j = p$, $p \in \mathbb{N}_{>0}$, we have

$$C_1 \rho_1^{pj} M_{pj}^\mu \geq (M_p^\lambda)^j$$

and hence, for all j and p ,

$$C_2 (M_{pj}^\mu)^{1/pj} \geq (M_p^\lambda)^{1/p}.$$

For arbitrary $p \leq k$ choose j so that $jp \leq k < (j+1)p$. Then

$$(M_k^\mu)^{1/k} \geq (M_{jp}^\mu)^{1/(jp)} \frac{(jp)!^{1/(jp)}}{k!^{1/k}} \geq C_2^{-1} (M_p^\lambda)^{1/p} \frac{(jp)!^{1/(jp)}}{k!^{1/k}} \geq C_2^{-1} (M_p^\lambda)^{1/p},$$

since $(k!M_k^\mu)^{1/k}$ is non-decreasing.

(3) \Rightarrow (4). By $\mathfrak{M}_{\{\text{dc}\}}$, for $\lambda \in \Lambda$ there exist $\mu \in \Lambda$ and $D > 0$ such that $M_{k+1}^\lambda \leq D^k M_k^\mu$ for all $k \geq 1$. The assumption implies that there is $\nu \in \Lambda$ such that $M_{\beta_1}^\mu \dots M_{\beta_j}^\mu \leq C^k M_k^\nu$ for all $\beta_i \in \mathbb{N}_{>0}$ with $\beta_1 + \dots + \beta_j = k$. Let $I := \{i : \alpha_i \geq 2\}$ and set $\alpha'_i := \alpha_i - 1$. Then, as $\mu \geq \lambda$,

$$\begin{aligned} M_j^\lambda M_{\alpha_1}^\lambda \dots M_{\alpha_j}^\lambda &= M_j^\lambda (M_1^\lambda)^{j-|I|} \prod_{i \in I} M_{\alpha_i}^\lambda \leq D^{k-j} M_j^\lambda (M_1^\lambda)^{j-|I|} \prod_{i \in I} M_{\alpha'_i}^\mu \\ &\leq D^{k-j} (M_1^\lambda)^{j-|I|} C^k M_k^\nu \leq \tilde{C}^k M_k^\nu, \end{aligned}$$

which shows (4).

(4) \Rightarrow (1). Let $g \in \mathcal{E}^{\{\mathfrak{m}\}}(U, V)$ and $f \in \mathcal{E}^{\{\mathfrak{m}\}}(V, W)$, for open subsets $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$, $W \subseteq \mathbb{R}^r$, and let $K \subseteq U$ be compact. By defini-

tion, there exist $\lambda_i \in \Lambda$, $i = 1, 2$, such that $g \in \mathcal{E}^{\{M^{\lambda_1}\}}(K, V)$ and $f \in \mathcal{E}^{\{M^{\lambda_2}\}}(g(K), W)$, and there exists $\lambda \geq \lambda_i$, $i = 1, 2$. By $(\mathfrak{M}_{\{\text{FdB}\}})$, there exists $\mu \in \Lambda$ such that $(M^\lambda)^\circ \preceq M^\mu$, and thus, by Proposition 3.1, we have $f \circ g \in \mathcal{E}^{\{M^\mu\}}(K, W)$, which implies the assertion. ■

4.11. THEOREM. *For a weight matrix \mathfrak{M} satisfying $(\mathfrak{M}_{(\text{dc})})$ and $(\mathfrak{M}_{\mathcal{H}})$ the following are equivalent:*

- (1) $\mathcal{E}^{(\mathfrak{M})}$ is stable under composition.
- (2) $\mathcal{E}^{(\mathfrak{M})}$ is holomorphically closed.
- (3) For all $\lambda \in \Lambda$ there are $\mu \in \Lambda$ and $C > 0$ such that

$$(M_j^\mu)^{1/j} \leq C(M_k^\lambda)^{1/k} \quad \text{if } j \leq k.$$

- (4) \mathfrak{M} satisfies $(\mathfrak{M}_{(\text{FdB})})$.

Note that $(\mathfrak{M}_{\mathcal{H}})$ is only needed for (1) \Rightarrow (2), and $(\mathfrak{M}_{(\text{dc})})$ is only needed for (3) \Rightarrow (4).

Proof. (1) \Rightarrow (2). This is obvious, by 4.6(4).

(2) \Rightarrow (3). We follow [10]. Since all M^λ are weakly log-convex, $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ is a Fréchet algebra which is locally m-convex, by [29], i.e., $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ has an equivalent seminorm system $\{p\}$ such that $p(fg) \leq p(f)p(g)$ for all $f, g \in \mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$. So for each $\lambda \in \Lambda$, compact $K \subseteq \mathbb{R}$, and $\rho > 0$ there exist $p, \mu \in \Lambda$, compact $L \subseteq \mathbb{R}$, $\sigma > 0$ and constants $C, D > 0$ such that

$$\|f^m\|_{K, \rho}^{M^\lambda} \leq Cp(f^m) \leq C(p(f))^m \leq CD^m(\|f\|_{L, \sigma}^{M^\mu})^m, \quad f \in \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}), m \in \mathbb{N},$$

in particular, for $f_t(x) = e^{itx}$ and $\rho = 1$, we find

$$T_{M^\lambda}(mt) \leq CD^m(T_{M^\mu}(t/\sigma))^m.$$

Let $j \leq k$ and suppose that $k = jl$ with $l \in \mathbb{N}$. We have, for some constant \tilde{C} ,

$$\begin{aligned} (T_{M^\lambda}(t))^{1/k} &= \left(T_{M^\lambda}\left(\frac{t}{l}\right)\right)^{1/k} \leq C^{1/k} D^{1/j} \left(T_{M^\mu}\left(\frac{t}{\sigma l}\right)\right)^{1/j} \\ &\leq \tilde{C} \left(T_{M^\mu}\left(\frac{t}{\sigma l}\right)\right)^{1/j}, \end{aligned}$$

thus

$$(k!M_k^\lambda)^{1/k} = \sup_{t>0} \frac{t}{(T_{M^\lambda}(t))^{1/k}} \geq \sup_{t>0} \frac{t}{\tilde{C} (T_{M^\mu}(t/\sigma l))^{1/j}} = \frac{\sigma l}{\tilde{C}} (j!M_j^\mu)^{1/j}.$$

In general choose $l \in \mathbb{N}$ such that $lj \leq k < (l+1)j$. Then, as $(k!M_k^\lambda)^{1/k}$ is increasing,

$$(k!M_k^\lambda)^{1/k} \geq ((lj)!M_{lj}^\lambda)^{1/(lj)} \geq \frac{\sigma l}{\tilde{C}} (j!M_j^\mu)^{1/j} \geq \frac{\sigma(l+1)}{2\tilde{C}} (j!M_j^\mu)^{1/j},$$

and, by Stirling's formula, there is a constant $\tilde{C} > 0$ such that $(M_j^\mu)^{1/j} \leq \tilde{C}(M_k^\lambda)^{1/k}$ for all $j \leq k$.

(3) \Rightarrow (4). The assumption implies that $M_{\beta_1}^\mu \cdots M_{\beta_j}^\mu \leq C^k M_k^\lambda$ for all β_i in $\mathbb{N}_{>0}$ with $\beta_1 + \cdots + \beta_j = k$. By $(\mathfrak{M}_{(\text{dc})})$, there exist $\nu \in \Lambda$ and $D > 0$ such that $M_{k+1}^\nu \leq D^k M_k^\mu$ for all $k \geq 1$. Let $I := \{i : \alpha_i \geq 2\}$ and set $\alpha'_i := \alpha_i - 1$. Then, as $\nu \leq \mu$,

$$\begin{aligned} M_j^\nu M_{\alpha_1}^\nu \cdots M_{\alpha_j}^\nu &= M_j^\nu (M_1^\nu)^{j-|I|} \prod_{i \in I} M_{\alpha_i}^\nu \leq D^{k-j} M_j^\nu (M_1^\nu)^{j-|I|} \prod_{i \in I} M_{\alpha'_i}^\mu \\ &\leq D^{k-j} (M_1^\nu)^{j-|I|} C^k M_k^\lambda \leq \tilde{C}^k M_k^\lambda, \end{aligned}$$

which shows (4).

(4) \Rightarrow (1). Let $g \in \mathcal{E}^{(\mathfrak{M})}(U, V)$ and $f \in \mathcal{E}^{(\mathfrak{M})}(V, W)$, for open subsets $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$, $W \subseteq \mathbb{R}^r$, and let $K \subseteq U$ be compact. By definition, for each $\mu \in \Lambda$ we have $g \in \mathcal{E}^{(M^\mu)}(K, V)$ and $f \in \mathcal{E}^{(M^\mu)}(g(K), W)$. By $(\mathfrak{M}_{(\text{FdB})})$ and by Proposition 3.1, we obtain $f \circ g \in \mathcal{E}^{(M^\lambda)}(K, W)$ for each $\lambda \in \Lambda$, which implies the assertion. ■

4.12. Composition operators. Let \mathfrak{M} be a weight matrix. If \mathfrak{M} satisfies $(\mathfrak{M}_{[\text{FdB}]})$, we may consider the nonlinear composition operators

$$\begin{aligned} \text{comp}^{[\mathfrak{M}]} : \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^p, \mathbb{R}^q) \times \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^q, \mathbb{R}^r) &\rightarrow \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^p, \mathbb{R}^r) : (g, f) \mapsto f \circ g, \\ \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^p, f) : \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^p, \mathbb{R}^q) &\rightarrow \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^p, \mathbb{R}^r) : g \mapsto f \circ g, \quad f \in \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^q, \mathbb{R}^r), \end{aligned}$$

by Theorems 4.9 and 4.11.

4.13. THEOREM. *We have:*

- (1) *If \mathfrak{M} satisfies $(\mathfrak{M}_{(\text{FdB})})$, then $\text{comp}^{(\mathfrak{M})}$ is continuous.*
- (2) *If \mathfrak{M} satisfies $(\mathfrak{M}_{\{\text{FdB}\}})$, then $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, f)$, for $f \in \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^q, \mathbb{R}^r)$, is continuous and $\text{comp}^{\{\mathfrak{M}\}}$ is sequentially continuous.*

Proof. We follow [1] and subdivide the proof into several claims.

4.14. CLAIM. *If \mathfrak{M} satisfies $(\mathfrak{M}_{[\text{FdB}]})$, then $\text{comp}^{[\mathfrak{M}]}$ is bounded.*

We treat the cases $\mathcal{E}^{(\mathfrak{M})}$ and $\mathcal{E}^{\{\mathfrak{M}\}}$ separately.

($\mathcal{E}^{[\mathfrak{M}]} = \mathcal{E}^{\{\mathfrak{M}\}}$) Let $\mathcal{B}_1 \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, \mathbb{R}^q)$ and $\mathcal{B}_2 \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^q, \mathbb{R}^r)$ be bounded subsets. Let $K \subseteq \mathbb{R}^p$ be an arbitrary, but fixed, compact subset. Then \mathcal{B}_1 is bounded in $\mathcal{E}^{\{\mathfrak{M}\}}(K, \mathbb{R}^q)$. Since the inductive limit $\mathcal{E}^{\{\mathfrak{M}\}}(K, \mathbb{R}^q) = \varinjlim_{\lambda, \rho} \mathcal{E}_\rho^{M^\lambda}(K, \mathbb{R}^q)$ is regular (see 4.2), \mathcal{B}_1 is contained and bounded in some step $\mathcal{E}_{\rho_1}^{M^{\lambda_1}}(K, \mathbb{R}^q)$, i.e., there exist $\lambda_1 \in \Lambda$ and $\rho_1 > 0$ such that $\sup_{g \in \mathcal{B}_1} \|g\|_{K, \rho_1}^{M^{\lambda_1}} < \infty$. In particular, the closure

$$(4.15) \quad L := \overline{\bigcup_{g \in \mathcal{B}_1} g(K)}$$

is a compact subset of \mathbb{R}^q , and \mathcal{B}_2 is bounded in $\mathcal{E}^{\{\mathfrak{M}\}}(L, \mathbb{R}^r) = \varinjlim_{(\lambda, \rho)} \mathcal{E}_\rho^{M^\lambda}(L, \mathbb{R}^r)$. So there exist $\lambda_2 \in \Lambda$ and $\rho_2 > 0$ such that $\sup_{f \in \mathcal{B}_2} \|f\|_{L, \rho_2}^{M^{\lambda_2}} < \infty$. For $\lambda := \max \lambda_i$, we have

$$(4.16) \quad C_1 := \sup_{g \in \mathcal{B}_1} \|g\|_{K, \rho_1}^{M^\lambda} < \infty \quad \text{and} \quad C_2 := \sup_{f \in \mathcal{B}_2} \|f\|_{L, \rho_2}^{M^\lambda} < \infty.$$

By the proof of Proposition 3.1, we find that

$$(4.17) \quad \sup_{(g, f) \in \mathcal{B}_1 \times \mathcal{B}_2} \|f \circ g\|_{K, \sigma}^{(M^\lambda)^\circ} \leq C_1 C_2 \rho_2 < \infty \quad \text{with} \quad \sigma := \rho_1(1 + \rho_2 C_1),$$

and, by $(\mathfrak{M}_{\{\text{FdB}\}})$, there exist $\mu \in \Lambda$ and $C > 0$ such that

$$(4.18) \quad \sup_{(g, f) \in \mathcal{B}_1 \times \mathcal{B}_2} \|f \circ g\|_{K, C\sigma}^{M^\mu} \leq \sup_{(g, f) \in \mathcal{B}_1 \times \mathcal{B}_2} \|f \circ g\|_{K, \sigma}^{(M^\lambda)^\circ} < \infty.$$

Since K was arbitrary, $\text{comp}^{\{\mathfrak{M}\}}(\mathcal{B}_1 \times \mathcal{B}_2)$ is bounded in $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, \mathbb{R}^r)$.

$(\mathcal{E}^{\{\mathfrak{M}\}} = \mathcal{E}^{\{\mathfrak{M}\}})$ Let $\mathcal{B}_1 \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, \mathbb{R}^q)$ and $\mathcal{B}_2 \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^q, \mathbb{R}^r)$ be bounded. Let $\mu \in \Lambda$, let $K \subseteq \mathbb{R}^p$ be compact, and let $\tau > 0$. By $(\mathfrak{M}_{\{\text{FdB}\}})$, we find $\lambda \in \Lambda$ and $C > 0$ such that $(M^\lambda)_k^\circ \leq C^k M_k^\mu$ for all k . Choose $\rho > 0$ so that $\tau/C = \sqrt{\rho} + \rho$ and set $\rho_1 = \sqrt{\rho}$. Let C_1 be defined by (4.16); \mathcal{B}_1 is bounded in $\mathcal{E}_{\rho_1}^{M^\lambda}(K, \mathbb{R}^q)$. Set $\rho_2 = \sqrt{\rho}/C_1$ and let C_2 be defined by (4.16); \mathcal{B}_2 is bounded in $\mathcal{E}_{\rho_2}^{M^\lambda}(L, \mathbb{R}^r)$, where L is defined by (4.15). As before, we deduce (4.17) and (4.18), where $\sigma = \tau/C$, which completes the proof of the claim.

4.19. CLAIM. *If \mathfrak{M} satisfies $(\mathfrak{M}_{\{\text{FdB}\}})$, then $\text{comp}^{\{\mathfrak{M}\}}$ is sequentially continuous.*

$(\mathcal{E}^{\{\mathfrak{M}\}} = \mathcal{E}^{\{\mathfrak{M}\}})$ Let $(g_n, f_n) \rightarrow (g, f)$ in $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, \mathbb{R}^q) \times \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^q, \mathbb{R}^r)$. Then the sets $\mathcal{B}_1 := \{g_n : n \in \mathbb{N}\}$, $\mathcal{B}_2 := \{f_n : n \in \mathbb{N}\}$, and $\{f_n \circ g_n : n \in \mathbb{N}\} \subseteq \text{comp}^{\{\mathfrak{M}\}}(\mathcal{B}_1 \times \mathcal{B}_2)$ are bounded, by Claim 4.14. Let $K \subseteq \mathbb{R}^p$ be an arbitrary, but fixed, compact subset, and let L be given by (4.15). By regularity of the inductive limit $\mathcal{E}^{\{\mathfrak{M}\}}(K, \mathbb{R}^r) = \varinjlim_{(\lambda, \rho)} \mathcal{E}_\rho^{M^\lambda}(K, \mathbb{R}^r)$, the set $\{f_n \circ g_n : n \in \mathbb{N}\}$ is contained and bounded in some step $\mathcal{E}_\rho^{M^\lambda}(K, \mathbb{R}^r)$ and hence is precompact in $\mathcal{E}_\sigma^{M^\mu}(K, \mathbb{R}^r)$, where $\lambda \leq \mu$ and $\rho < \sigma$ (see 4.2), and so it has an accumulation point $h \in \mathcal{E}_\sigma^{M^\mu}(K, \mathbb{R}^r)$. It is well-known that composition of continuous mappings, i.e., $\text{comp}^0 : C^0(K, L) \times C^0(L, \mathbb{R}^r) \rightarrow C^0(K, \mathbb{R}^r)$, is continuous (see e.g. [14, Thm. 3.4.2]), and thus $f_n \circ g_n \rightarrow f \circ g$ in $C^0(K, \mathbb{R}^r)$. It follows that $f \circ g = h$. As K was arbitrary, the assertion follows.

$(\mathcal{E}^{\{\mathfrak{M}\}} = \mathcal{E}^{\{\mathfrak{M}\}})$ The proof is analogous; note that here $\{f_n \circ g_n : n \in \mathbb{N}\}$ is precompact in every step $\mathcal{E}_\sigma^{M^\mu}(K, \mathbb{R}^r)$.

4.20. CLAIM. *If \mathfrak{M} satisfies $(\mathfrak{M}_{\{\text{FdB}\}})$, then $\text{comp}^{\{\mathfrak{M}\}}$ is continuous.*

This follows from Claim 4.19, since $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^p, \mathbb{R}^q) \times \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^q, \mathbb{R}^r)$ is metrizable.

4.21. CLAIM. *If \mathfrak{M} satisfies $(\mathfrak{M}_{\{\text{FdB}\}})$, then $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, f)$ is continuous.*

Arguments similar to those in the proof of Claim 4.19 show that the restricted mapping $\mathcal{E}^{\{\mathfrak{M}\}}(K, f) : \mathcal{E}^{\{\mathfrak{M}\}}(K, \mathbb{R}^q) \rightarrow \mathcal{E}^{\{\mathfrak{M}\}}(K, \mathbb{R}^r)$ is sequentially continuous, thus continuous, for each compact subset $K \subseteq \mathbb{R}^p$, since $\mathcal{E}^{\{\mathfrak{M}\}}(K, \mathbb{R}^q)$ is sequential, by 4.2 and e.g. [31, 8.5.28]. The projective structure of $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, \mathbb{R}^q) = \varprojlim_K \mathcal{E}^{\{\mathfrak{M}\}}(K, \mathbb{R}^q)$ implies that the mapping $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, f) : \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, \mathbb{R}^q) \rightarrow \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, \mathbb{R}^r)$ is continuous. ■

4.22. COROLLARY. *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ satisfy $(M_{\{\text{FdB}\}})$. Then $\text{comp}^{(M)}$ is continuous, $\mathcal{E}^{\{M\}}(\mathbb{R}^p, f)$, for $f \in \mathcal{E}^{\{M\}}(\mathbb{R}^q, \mathbb{R}^r)$, is continuous, and $\text{comp}^{\{M\}}$ is sequentially continuous.*

Proof. This is a special case of Theorem 4.13; weak log-convexity of M is not needed here. ■

4.23. REMARK. If M additionally has moderate growth, then the mapping $\text{comp}^{[M]}$ is even $\mathcal{E}^{[M]}$ which is a consequence of the $\mathcal{E}^{[M]}$ -exponential law (see [25, 5.5]). We expect that more generally $\text{comp}^{[\mathfrak{M}]}$ is $\mathcal{E}^{[\mathfrak{M}]}$, if \mathfrak{M} satisfies $(\mathfrak{M}_{\{\text{FdB}\}})$ and $(\mathfrak{M}_{\{\text{img}\}})$. Work on this is in progress and will appear in a forthcoming paper.

5. Weight functions and $[\omega]$ -ultradifferentiable functions

5.1. Weight functions. Let \mathscr{W} be the set of all continuous increasing functions $\omega : [0, \infty) \rightarrow [0, \infty)$ with $\omega|_{[0,1]} = 0$, $\lim_{t \rightarrow \infty} \omega(t) = \infty$, and such that the following assumptions (ω_1) , (ω_2) , and (ω_3) are satisfied:

- (ω_1) $\omega(2t) = O(\omega(t))$ as $t \rightarrow \infty$.
- (ω_2) $\log(t) = o(\omega(t))$ as $t \rightarrow \infty$.
- (ω_3) $\varphi : t \mapsto \omega(e^t)$ is convex on $[0, \infty)$.

Occasionally, we shall also consider the following conditions:

- (ω_4) $\omega(t) = O(t)$ as $t \rightarrow \infty$.
- (ω_5) $\omega(t) = o(t)$ as $t \rightarrow \infty$.
- (ω_6) $\exists H \geq 1 \forall t \geq 0 : 2\omega(t) \leq \omega(Ht) + H$.
- (ω_7) $\exists C > 0 \exists t_0 > 0 \forall \lambda \geq 1 \forall t \geq t_0 : \omega(\lambda t) \leq C\lambda\omega(t)$.
- (ω_8) $\exists C > 0 \exists H > 0 \forall t \geq 0 : \omega(t^2) \leq C\omega(Ht) + C$.

Then \mathscr{W} forms an abelian semigroup with respect to pointwise addition, which also preserves all conditions (ω_4) – (ω_8) .

For $\omega \in \mathscr{W}$ the *Young conjugate* of φ , given by

$$\varphi^*(t) := \sup\{st - \varphi(s) : s \geq 0\}, \quad t \geq 0,$$

is convex, increasing, and satisfies $\varphi^*(0) = 0$, $\varphi^{**} = \varphi$, and $\lim_{t \rightarrow \infty} t/\varphi^*(t) = 0$. Moreover, the functions $t \mapsto \varphi(t)/t$ and $t \mapsto \varphi^*(t)/t$ are increasing (cf. [9]). Convexity of φ^* and $\varphi^*(0) = 0$ implies

$$(5.2) \quad \varphi^*(t) + \varphi^*(s) \leq \varphi^*(t+s) \leq \frac{1}{2}\varphi^*(2t) + \frac{1}{2}\varphi^*(2s), \quad t, s \geq 0.$$

Note that $\omega(t) := \max\{0, (\log t)^s\}$, $s > 1$, belongs to \mathscr{W} and satisfies all the listed conditions except (ω_6) .

For $\omega, \sigma \in \mathscr{W}$ we define:

$$\begin{aligned} \omega \preceq \sigma &: \Leftrightarrow \sigma(t) = O(\omega(t)) \text{ as } t \rightarrow \infty, \\ \omega \approx \sigma &: \Leftrightarrow \omega \preceq \sigma \text{ and } \sigma \preceq \omega, \\ \omega \triangleleft \sigma &: \Leftrightarrow \sigma(t) = o(\omega(t)) \text{ as } t \rightarrow \infty. \end{aligned}$$

5.3. $[\omega]$ -ultradifferentiable functions. Let $\omega \in \mathscr{W}$ and let $U \subseteq \mathbb{R}^n$ be open. Define

$$\begin{aligned} \mathcal{E}^{(\omega)}(U) &:= \{f \in C^\infty(U, \mathbb{R}) : \forall K \subseteq U \text{ compact } \forall \rho > 0 : \|f\|_{K, \rho}^\omega < \infty\}, \\ \mathcal{E}^{\{\omega\}}(U) &:= \{f \in C^\infty(U, \mathbb{R}) : \forall K \subseteq U \text{ compact } \exists \rho > 0 : \|f\|_{K, \rho}^\omega < \infty\}, \\ \|f\|_{K, \rho}^\omega &:= \sup \left\{ \|f^{(k)}(x)\|_{L^k(\mathbb{R}^n, \mathbb{R})} \exp\left(-\frac{1}{\rho}\varphi^*(\rho k)\right) : x \in K, k \in \mathbb{N} \right\}, \end{aligned}$$

and endow $\mathcal{E}^{(\omega)}(U)$ with its natural Fréchet space topology and $\mathcal{E}^{\{\omega\}}(U)$ with the projective limit topology over K of the inductive limit topology over ρ ; note that it suffices to take countable limits. We write $\mathcal{E}^{[\omega]}$ for either $\mathcal{E}^{(\omega)}$ or $\mathcal{E}^{\{\omega\}}$. The elements of $\mathcal{E}^{[\omega]}(U)$ are called $[\omega]$ -ultradifferentiable functions; an $(\omega)/\{\omega\}$ -ultradifferentiable function is said to be of Beurling/Roumieu type, respectively. For compact $K \subseteq U$ with smooth boundary, set

$$\begin{aligned} \mathcal{E}_\rho^\omega(K) &:= \{f \in C^\infty(K) : \|f\|_{K, \rho}^\omega < \infty\}, \\ \mathcal{E}^{(\omega)}(K) &:= \{f \in C^\infty(K) : \forall \rho > 0 : \|f\|_{K, \rho}^\omega < \infty\} = \varprojlim_{m \in \mathbb{N}} \mathcal{E}_{1/m}^\omega(K), \\ \mathcal{E}^{\{\omega\}}(K) &:= \{f \in C^\infty(K) : \exists \rho > 0 : \|f\|_{K, \rho}^\omega < \infty\} = \varinjlim_{m \in \mathbb{N}} \mathcal{E}_m^\omega(K). \end{aligned}$$

We shall also use $\mathcal{E}^{[\omega]}(U, V)$, $\mathcal{E}^{[\omega]}(K, V)$, and $\mathcal{E}_\rho^{[\omega]}(K, V)$ for open subsets $V \subseteq \mathbb{R}^m$.

Note that $\mathcal{E}^{[\omega]}$ is quasianalytic if and only if

$$\int_1^\infty \frac{\omega(t)}{t^2} dt = \infty$$

(e.g., by Corollary 5.8 and Theorem 5.14 below), and in this case we say that ω is quasianalytic.

5.4. **EXAMPLES.** For $s \in \mathbb{R}_{\geq 0}$ the weight function $\gamma^s(t) = t^{1/(1+s)}$ has all properties listed in 5.1 except (ω_8) and (ω_5) if $s = 0$; it is quasianalytic if and only if $s = 0$. The elements of $\mathcal{E}^{\{\gamma^0\}}(U)$ are exactly the real analytic functions $C^\omega(U)$, and the elements of $\mathcal{E}^{(\gamma^0)}(U)$ are exactly the restrictions of entire functions from $\mathcal{H}(\mathbb{C}^n)$. The class $\mathcal{E}^{\{\gamma^s\}}$ coincides with the Gevrey class \mathcal{G}^{1+s} .

5.5. **Associated sequences.** For $\omega \in \mathscr{W}$ and each $\rho > 0$ consider the sequence $\Omega^\rho \in \mathbb{R}_{>0}^{\mathbb{N}}$ defined by

$$\Omega_k^\rho := \frac{1}{k!} \exp\left(\frac{1}{\rho} \varphi^*(\rho k)\right).$$

By the properties of φ^* , each Ω^ρ is weakly log-convex, $(k! \Omega_k^\rho)^{1/k} \nearrow \infty$, and $\Omega^\rho \leq \Omega^\sigma$ if $\rho \leq \sigma$. By (5.2),

$$(5.6) \quad j! \Omega_j^\rho k! \Omega_k^\rho \leq (j+k)! \Omega_{j+k}^\rho \leq j! \Omega_j^{2\rho} k! \Omega_k^{2\rho}, \quad j, k \in \mathbb{N}.$$

In particular, $\Omega_{k+1}^\rho \leq C \Omega_k^{2\rho}$ for all k , where $C > 0$ is a constant depending on ρ .

With Ω^ρ we may associate the function $\omega_\rho := \log \circ T_{\Omega^\rho}$ (cf. [20, (3.1)]). Then

$$\omega_\rho(t) = \sup_{k \in \mathbb{N}} \left(k \log t - \frac{1}{\rho} \varphi^*(\rho k) \right) \leq \sup_{s \geq 0} \left(s \log t - \frac{1}{\rho} \varphi^*(\rho s) \right) = \frac{1}{\rho} \omega(t).$$

5.7. **LEMMA.** For $\omega \in \mathscr{W}$ we have $\omega \approx \omega_\rho$ for all $\rho > 0$.

Proof. It suffices to show that $\omega \approx \omega_1$; for arbitrary $\rho > 0$ replace ω by $\frac{1}{\rho} \omega$. By [27, 1.8.III],

$$\omega_1(t) = \sup_{k \in \mathbb{N}} (k \log t - \varphi^*(k)) = k_t \log t - \varphi^*(k_t),$$

where $k_t \in \mathbb{N}$ is such that $\varpi_{k_t} \leq t < \varpi_{k_t+1}$ and $\varpi_k := k \Omega_k^1 / \Omega_{k-1}^1 \nearrow \infty$. Consider the function $f_t : [0, \infty) \rightarrow \mathbb{R}$ given by $f_t(s) = s \log t - \varphi^*(s)$, which is concave (for $t \geq 1$) since φ^* is convex. Concavity of f_t shows that $\omega(t) = \sup_{s \geq 0} f_t(s) = f_t(s_t)$ for some point $s_t \in (k_t - 1, k_t + 1)$.

Assume that $s_t \in (k_t, k_t + 1)$. By concavity of f_t and since $f_t(0) = 0$, we find

$$f_t(s_t) \leq \frac{f_t(k_t)}{k_t} s_t \leq \frac{f_t(k_t)}{k_t} (k_t + 1) \leq 2f_t(k_t)$$

and hence $\omega(t) \leq 2\omega_1(t)$ for sufficiently large t . The case $s_t \in (k_t - 1, k_t)$ is similar. ■

5.8. **COROLLARY.** For $\omega \in \mathscr{W}$ we have:

- (1) ω is quasianalytic if and only if each (equivalently, some) Ω^ρ is quasianalytic.

(2) ω satisfies (ω_6) if and only if each (equivalently, some) Ω^ρ has moderate growth.

Proof. This follows from Lemma 5.7, [20, Lemma 4.1 and Prop. 3.6]. ■

5.9. LEMMA. For $\omega \in \mathscr{W}$ we have

$$(5.10) \quad \forall \sigma > 0 \exists H \geq 1 \forall \rho > 0 \exists C \geq 1 \forall k \in \mathbb{N} : \quad \sigma^k \Omega_k^\rho \leq C \Omega_k^{H\rho}.$$

Moreover, $\omega \in \mathscr{W}$ satisfies (ω_6) if and only if

$$(5.11) \quad \forall \rho > 0 \forall \tau > 0 : \quad \Omega^\rho \approx \Omega^\tau.$$

If $\omega \in \mathscr{W}$ satisfies (ω_8) then

$$(5.12) \quad \exists C > 1 \forall \rho > 0 : \quad \Omega^{\rho/C} \triangleleft \Omega^\rho.$$

It follows that (ω_8) is an obstruction for (ω_6) .

Proof. The following inequality is well-known (see e.g. [16, p. 404]):

$$(5.13) \quad \exists L \geq 1 \forall t \geq 0 \forall s \in \mathbb{N} : \quad L^s \varphi^*(t) + sL^s t \leq \varphi^*(L^s t) + \sum_{i=1}^s L^i.$$

For the reader's convenience we give a short proof. By (ω_1) , there exists $L_1 \geq 1$ such that $\omega(2t) \leq L_1 \omega(t) + L_1$ for all $t \geq 0$, and hence there exists $L \geq 1$ such that $\varphi(t+1) \leq L\varphi(t) + L$ for all $t \geq 0$. Thus, for $t \geq 0$,

$$\begin{aligned} \varphi^*(Lt) + L &= \sup_{s \geq 0} (Lts - (\varphi(s) - L)) \\ &\geq \sup_{s \geq 1} (Lts - L\varphi(s-1)) = L\varphi^*(t) + Lt, \end{aligned}$$

and (5.13) follows by iteration.

By choosing s such that $e^s \geq \sigma$ and by setting $t := \rho k$, $H := L^s$ and $C := \exp(\frac{1}{H\rho} \sum_{i=1}^s L^i)$, we see that (5.13) implies (5.10).

Let us prove that (ω_6) implies (5.11). By (ω_6) there exists a constant $H \geq 1$ such that $2\omega(t) \leq \omega(Ht) + H$ for all $t \geq 0$, and consequently, as $\omega|_{[0,1]} = 0$,

$$\begin{aligned} \varphi^*(t) &= \sup_{s \geq 0} (ts - \omega(e^s)) = \sup_{s \in \mathbb{R}} (ts - \omega(e^s)) = \sup_{u \geq 0} (t \log u - \omega(u)) \\ &\geq \sup_{u \geq 0} (t \log u - \frac{1}{2}\omega(Hu)) - \frac{1}{2}H = \frac{1}{2}\varphi^*(2t) - t \log H - \frac{1}{2}H. \end{aligned}$$

By setting $t := \rho k$, we may conclude that

$$\exists H \geq 1 \forall \rho > 0 \forall k \in \mathbb{N} : \quad \Omega_k^{2\rho} \leq e^{H/(2\rho)} H^k \Omega_k^\rho,$$

which implies $\Omega^{2\rho} \preceq \Omega^\rho$ for all $\rho > 0$. Iteration and the fact that $\Omega^\rho \preceq \Omega^{2\rho}$ yield $\Omega^{2^n \rho} \approx \Omega^\rho$ for all $\rho > 0$ and all $n \in \mathbb{N}$, and (5.11) follows.

Conversely, assume that (5.11) holds, which means that

$$\forall \rho > 0 \forall \tau > 0 \exists C > 0 \forall k \in \mathbb{N} : \quad \frac{1}{\rho} \varphi^*(\rho k) \leq Ck + \frac{1}{\tau} \varphi^*(\tau k).$$

By (5.2), we conclude that

$$\forall \rho > 0 \quad \forall \tau > 0 \quad \exists D > 0 \quad \forall t \geq 0 : \quad \frac{1}{\rho} \varphi^*(\rho t) \leq Dt + D + \frac{1}{2\tau} \varphi^*(2\tau t).$$

Thus

$$\begin{aligned} \frac{1}{2\tau} \varphi(t) &= \sup_{s \geq 0} \left(ts - \frac{1}{2\tau} \varphi^*(2\tau s) \right) \\ &\leq \sup_{s \geq 0} \left(ts + Ds - \frac{1}{\rho} \varphi^*(\rho s) \right) + D = \frac{1}{\rho} \varphi(t + D) + D, \end{aligned}$$

and hence

$$\frac{1}{2\tau} \omega(t) \leq \frac{1}{\rho} \omega(e^D t) + D.$$

Setting $\rho = 4$ and $\tau = 1$ implies (ω_6) .

Let us prove (5.12). By (ω_8) there exist constants $C, H > 0$ such that

$$\begin{aligned} C \varphi^* \left(\frac{2t}{C} \right) &= \sup_{u \geq 0} (2t \log u - C\omega(u)) = \sup_{u \geq 0} (2t \log u - C\omega(Hu)) + 2t \log H \\ &\leq \sup_{u \geq 0} (2t \log u - \omega(u^2)) + 2t \log H + C = \varphi^*(t) + 2t \log H + C. \end{aligned}$$

By setting $t := \rho k$, we find that for all $\rho > 0$ and all $k \in \mathbb{N}$,

$$(2k)! \Omega_{2k}^{\rho/C} \leq e^{C/\rho} H^{2k} k! \Omega_k^\rho.$$

Thus, the sequence $L = (L_k)$ defined by $k! L_k := (2k)! \Omega_{2k}^{\rho/C} \geq (k! \Omega_k^{\rho/C})^2$ satisfies $\Omega^{\rho/C} \triangleleft L \leq \Omega^\rho$, which implies (5.12). ■

5.14. THEOREM. *Let $\omega \in \mathscr{W}$, let $U \subseteq \mathbb{R}^n$ be open, and let $K \subseteq U$ be compact. Then:*

- (1) *For each $\rho > 0$ we have $\mathcal{E}^{\{\Omega^\rho\}}(U) \subseteq \mathcal{E}^{\{\omega\}}(U)$ and $\mathcal{E}^{(\omega)}(U) \subseteq \mathcal{E}^{(\Omega^\rho)}(U)$ with continuous inclusion.*
- (2) *We have, as locally convex spaces,*

$$\mathcal{E}^{(\omega)}(U) = \varprojlim_{\rho > 0} \mathcal{E}^{(\Omega^\rho)}(U) \quad \text{and} \quad \mathcal{E}^{\{\omega\}}(K) = \varinjlim_{\rho > 0} \mathcal{E}^{\{\Omega^\rho\}}(K).$$

- (3) *ω satisfies (ω_6) if and only if $\mathcal{E}^{[\Omega^\rho]}(U) = \mathcal{E}^{[\omega]}(U)$, for each $\rho > 0$, as locally convex spaces.*
- (4) *If ω satisfies (ω_8) , then also*

$$\mathcal{E}^{(\omega)}(U) = \varprojlim_{\rho > 0} \mathcal{E}^{(\Omega^\rho)}(U) = \varprojlim_{\rho > 0} \mathcal{E}^{\{\Omega^\rho\}}(U) \quad \text{and}$$

$$\mathcal{E}^{\{\omega\}}(K) = \varinjlim_{\rho > 0} \mathcal{E}^{\{\Omega^\rho\}}(K) = \varinjlim_{\rho > 0} \mathcal{E}^{(\Omega^\rho)}(K)$$

as locally convex spaces.

Proof. (1) Let $\rho > 0$ be fixed. If $f \in \mathcal{E}^{\{\Omega^\rho\}}(U)$ then for each compact $K \subseteq U$ there exists $\sigma > 0$ such that $\|f\|_{K,\sigma}^{\Omega^\rho} < \infty$. By (5.10), there exist constants $H, C \geq 1$ such that

$$\infty > C\|f\|_{K,\sigma}^{\Omega^\rho} \geq \|f\|_{K,1}^{\Omega^{H\rho}} = \|f\|_{K,H\rho}^\omega,$$

whence $f \in \mathcal{E}^{\{\omega\}}(U)$.

Assume that $f \in \mathcal{E}^{(\omega)}(U)$. Let $\rho, \sigma > 0$ be fixed. By (5.10), there exist constants $H, C \geq 1$ such that $\Omega_k^\rho \leq C\sigma^k\Omega_k^{H\rho}$ for all k . Since $f \in \mathcal{E}^{(\omega)}(U)$, for each compact $K \subseteq U$ we have $\|f\|_{K,\rho/H}^\omega < \infty$, and thus

$$\infty > C\|f\|_{K,\rho/H}^\omega = C\|f\|_{K,1}^{\Omega^{\rho/H}} \geq \|f\|_{K,\sigma}^{\Omega^\rho}.$$

Since $\sigma > 0$ was arbitrary, we conclude that $f \in \mathcal{E}^{(\Omega^\rho)}(U)$.

(2) follows from (1), since the inclusions $\mathcal{E}^{(\omega)}(U) \supseteq \varprojlim_{\rho>0} \mathcal{E}^{(\Omega^\rho)}(U)$ and $\mathcal{E}^{\{\omega\}}(K) \subseteq \varinjlim_{\rho>0} \mathcal{E}^{\{\Omega^\rho\}}(K)$ are clear and continuous by definition.

(3) follows from (2), (5.11), and Proposition 2.12(1).

(4) is a direct consequence of (2), (5.12), and Proposition 2.12(1). ■

5.15. COROLLARY. *Let $\omega \in \mathcal{W}$ and let $U \subseteq \mathbb{R}^n$ be open. Then $\mathcal{E}^{[\omega]}(U) = \mathcal{E}^{[\mathfrak{W}]}(U)$ as locally convex spaces, where the weight matrix $\mathfrak{W} := \{\Omega^\rho : \rho > 0\}$ satisfies*

- $(\mathfrak{M}_{(\text{mg})})$ and $(\mathfrak{M}_{\{\text{mg}\}})$,
- $(\mathfrak{M}_{(\text{alg})})$ and $(\mathfrak{M}_{\{\text{alg}\}})$,
- $(\mathfrak{M}_{(L)})$ and $(\mathfrak{M}_{\{L\}})$.

If ω satisfies (ω_4) , respectively (ω_5) , then \mathfrak{W} satisfies $(\mathfrak{M}_{\mathcal{H}})$, respectively $(\mathfrak{M}_{(C^\omega)})$. If ω satisfies (ω_8) , then \mathfrak{W} satisfies $(\mathfrak{M}_{(\text{BR})})$ and $(\mathfrak{M}_{\{\text{BR}\}})$.

Proof. This is an immediate consequence of Theorem 5.14, (5.6), and (5.10).

For $\omega(t) = \max\{0, t - 1\} \approx t$ we find $\varphi^*(t) = t \log t - t + 1$, for $t \geq 1$, $\varphi^*|_{[0,1]} = 0$, and it is easy to see that (ω_4) implies $(\mathfrak{M}_{\mathcal{H}})$ and (ω_5) implies $(\mathfrak{M}_{(C^\omega)})$, by Lemma 5.16. Finally, from (5.12) it follows that (ω_8) implies $(\mathfrak{M}_{(\text{BR})})$ and $(\mathfrak{M}_{\{\text{BR}\}})$. ■

5.16. LEMMA. *For $\omega, \sigma \in \mathcal{W}$ we have:*

- (1) *If $\omega \preceq \sigma$ then $\exists H \geq 1 \forall \rho > 0 \exists C > 0 : \Omega^\rho \leq C\Sigma^{H\rho}$.*
- (2) *If $\omega \triangleleft \sigma$ then $\forall H > 0 \forall \rho > 0 \exists C > 0 : \Omega^\rho \leq C\Sigma^{H\rho}$.*

Here Σ^ρ are the sequences associated with σ .

Proof. (1) If $\omega \preceq \sigma$ then there exists $H \geq 1$ such that $\sigma(t) \leq H\omega(t) + H$ for all $t \geq 0$, and thus also $\varphi_\sigma(t) \leq H\varphi_\omega(t) + H$ and finally $H\varphi_\omega^*(t) \leq \varphi_\sigma^*(Ht) + H$. Setting $t = \rho k$ gives the assertion.

(2) If $\omega \triangleleft \sigma$ then for all $H > 0$ there exists $D > 0$ such that $\sigma(t) \leq H\omega(t) + D$ for all $t \geq 0$, and thus $H\varphi_\omega^*(t) \leq \varphi_\sigma^*(Ht) + D$ as in (1). Setting $t = \rho k$ gives the assertion. ■

5.17. COROLLARY. For $\omega, \sigma \in \mathscr{W}$ we have:

- (1) $\omega \preceq \sigma \Rightarrow \mathcal{E}^{[\omega]} \subseteq \mathcal{E}^{[\sigma]}$ and $\mathcal{E}^{[\omega]}(\mathbb{R}) \subseteq \mathcal{E}^{[\sigma]}(\mathbb{R}) \Rightarrow \omega \preceq \sigma$.
- (2) $\omega \triangleleft \sigma \Rightarrow \mathcal{E}^{\{\omega\}} \subseteq \mathcal{E}^{(\sigma)}$ and $\mathcal{E}^{\{\omega\}}(\mathbb{R}) \subseteq \mathcal{E}^{(\sigma)}(\mathbb{R}) \Rightarrow \omega \triangleleft \sigma$.
- (3) There is no $\sigma \in \mathscr{W}$ such that $\mathcal{E}^{(\omega)}(\mathbb{R}) \subsetneq \mathcal{E}^{[\sigma]}(\mathbb{R}) \subsetneq \mathcal{E}^{\{\omega\}}(\mathbb{R})$.

Proof. (1) If $\mathfrak{W} := \{\Omega^\rho : \rho > 0\}$ and $\mathfrak{S} := \{\Sigma^\rho : \rho > 0\}$, where Σ^ρ are the sequences associated with σ , then in view of Proposition 4.6 and Corollary 5.15 it suffices to show

(1') $\omega \preceq \sigma$ if and only if $\mathfrak{W}[\preceq]\mathfrak{S}$.

If $\omega \preceq \sigma$ then Lemma 5.16 implies $\mathfrak{W}(\preceq)\mathfrak{S}$ as well as $\mathfrak{W}\{\preceq\}\mathfrak{S}$.

Conversely, assume $\mathfrak{W}\{\preceq\}\mathfrak{S}$, i.e., using (5.10),

$$\forall \rho > 0 \exists \tau > 0 \exists C > 0 \forall k \in \mathbb{N} : \frac{1}{\rho}\varphi_\omega^*(\rho k) \leq \frac{1}{\tau}\varphi_\sigma^*(\tau k) + C,$$

and, by (5.2),

$$\forall \rho > 0 \exists \tau > 0 \exists D > 0 \forall t \geq 0 : \frac{1}{\rho}\varphi_\omega^*(\rho t) \leq \frac{1}{2\tau}\varphi_\sigma^*(2\tau t) + D.$$

Thus

$$\frac{1}{2\tau}\varphi_\sigma(t) = \sup_{s \geq 0} \left(ts - \frac{1}{2\tau}\varphi_\sigma^*(2\tau s) \right) \leq \sup_{s \geq 0} \left(ts - \frac{1}{\rho}\varphi_\omega^*(\rho s) \right) + D = \frac{1}{\rho}\varphi_\omega(t) + D,$$

and hence

$$(5.18) \quad \frac{1}{2\tau}\sigma(t) \leq \frac{1}{\rho}\omega(t) + D,$$

which implies $\sigma(t) = O(\omega(t))$ as $t \rightarrow \infty$, i.e., $\omega \preceq \sigma$.

If $\mathfrak{W}(\preceq)\mathfrak{S}$, then the same arguments yield (5.18), but with swapped quantifiers:

$$\forall \tau > 0 \exists \rho > 0 \exists D > 0 \forall t \geq 0 : \frac{1}{2\tau}\sigma(t) \leq \frac{1}{\rho}\omega(t) + D.$$

Again this implies $\omega \preceq \sigma$.

(2) If $\omega \triangleleft \sigma$ then Lemma 5.16 implies $\mathcal{E}^{\{\omega\}} \subseteq \mathcal{E}^{(\sigma)}$. Conversely, if $\mathcal{E}^{\{\omega\}}(\mathbb{R}) \subseteq \mathcal{E}^{(\sigma)}(\mathbb{R})$, then $\mathcal{E}^{\{\Omega^\rho\}}$ admits a characteristic function and is contained in $\mathcal{E}^{(\sigma)}$, thus

$$\forall \rho > 0 \forall \tau > 0 \exists C > 0 \forall k \in \mathbb{N} : \frac{1}{\rho}\varphi_\omega^*(\rho k) \leq \frac{1}{\tau}\varphi_\sigma^*(\tau k) + C.$$

As in (1) we may derive that for all $\rho, \tau > 0$ there is $D > 0$ such that (5.18) for all $t \geq 0$, hence $\sigma(t) = o(\omega(t))$ as $t \rightarrow \infty$, i.e., $\omega \triangleleft \sigma$.

(3) If $\mathcal{E}^{(\omega)}(\mathbb{R}) \subseteq \mathcal{E}^{\{\sigma\}}(\mathbb{R})$, then $\mathfrak{M}(\preceq)\mathfrak{G}$, by Corollary 5.15 and Proposition 4.6. As in (1) we may derive that there exist $\rho, \tau > 0$ such that (5.18), and so $\omega \preceq \sigma$. This and (1) imply the assertion. ■

As $\mathcal{E}^{\{t\}}(U) = C^\omega(U)$ and $\mathcal{E}^{(t)}(U) = \mathcal{H}(\mathbb{C}^n)$ (via restriction), condition (ω_4) is equivalent to $C^\omega \subseteq \mathcal{E}^{\{\omega\}}$ and condition (ω_5) is equivalent to $C^\omega \subseteq \mathcal{E}^{(\omega)}$.

5.19. Intersection and union of all non-quasianalytic Gevrey classes. For the weight matrix $\mathfrak{G} = \{G^s : s > 0\}$ with $G^s = (G_k^s) = ((k!)^s)$

$$(5.20) \quad \mathcal{E}^{(\mathfrak{G})}(U) = \bigcap_{s>0} \mathcal{G}^{1+s}(U), \quad U \subseteq \mathbb{R}^n \text{ open,}$$

is the intersection and

$$(5.21) \quad \mathcal{E}^{\{\mathfrak{G}\}}(K) = \bigcup_{s>0} \mathcal{G}^{1+s}(K), \quad K \subseteq \mathbb{R}^n \text{ compact,}$$

is the union of all non-quasianalytic Gevrey classes $\mathcal{G}^{1+s} = \mathcal{E}^{\{G^s\}}$ (as locally convex spaces). Indeed, $G^s \triangleleft G^{s'}$ for all $s < s'$ (so \mathfrak{G} satisfies $(\mathfrak{M}_{\{\mathbb{R}\}})$ and $(\mathfrak{M}_{\{\mathbb{R}\}})$), and hence we get (5.20):

$$\mathcal{E}^{(\mathfrak{G})}(U) = \bigcap_{s>0} \mathcal{E}^{(G^s)}(U) = \bigcap_{s>0} \mathcal{E}^{\{G^s\}}(U) = \bigcap_{s>0} \mathcal{G}^{1+s}(U),$$

while (5.21) is evident by definition. Note that $\mathcal{E}^{(\mathfrak{G})}$, and hence also $\mathcal{E}^{\{\mathfrak{G}\}}$, is non-quasianalytic. In fact, the sequence $L = (L_k)$ defined by $k!L_k := k^k(\log(k+e))^{2k}$ is non-quasianalytic and satisfies $L \triangleleft G^s$ for all $s > 0$, and, as $(k!L_k)^{1/k}$ is increasing, $\mathcal{E}^{[L]}$ is non-quasianalytic, by the Denjoy–Carleman theorem.

The following theorem shows that there exist spaces $\mathcal{E}^{[\mathfrak{M}]}$ that are different from $\mathcal{E}^{[M]}$ as well as from $\mathcal{E}^{[\omega]}$.

5.22. THEOREM. *Neither $\mathcal{E}^{(\mathfrak{G})}(\mathbb{R})$ nor $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R})$ coincides (as vector space) with $\mathcal{E}^{(M)}(\mathbb{R})$, $\mathcal{E}^{\{M\}}(\mathbb{R})$, $\mathcal{E}^{(\omega)}(\mathbb{R})$, or $\mathcal{E}^{\{\omega\}}(\mathbb{R})$ for any weight sequence M or weight function ω .*

Proof. We show first that, given a weight matrix $\mathfrak{M} = \{M^\lambda : \lambda \in \Lambda\}$ with $M^\lambda \not\approx M^\mu$ for all $\lambda \neq \mu$, there cannot exist a weakly log-convex $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}) = \mathcal{E}^{[M]}(\mathbb{R})$. Indeed, if there is such an M , Proposition 4.6 implies $M \approx M^\lambda$ for some λ . Then, by Proposition 2.12(1),

$$\mathcal{E}^{(M)}(\mathbb{R}) = \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) = \bigcap_{\lambda} \mathcal{E}^{(M^\lambda)}(\mathbb{R}) \subsetneq \mathcal{E}^{(M)}(\mathbb{R}),$$

and, for compact $K \subseteq \mathbb{R}$,

$$\begin{aligned} \mathcal{E}^{\{M\}}(\mathbb{R}) &= \mathcal{E}^{\{\mathfrak{m}\}}(\mathbb{R}) = \bigcap_K \mathcal{E}^{\{\mathfrak{m}\}}(K) = \bigcap_K \bigcup_{\lambda} \mathcal{E}^{\{M^\lambda\}}(K) \\ &\supseteq \bigcup_{\lambda} \bigcap_K \mathcal{E}^{\{M^\lambda\}}(K) = \bigcup_{\lambda} \mathcal{E}^{\{M^\lambda\}}(\mathbb{R}) \not\supseteq \mathcal{E}^{\{M\}}(\mathbb{R}), \end{aligned}$$

which contradicts the assumption in both cases.

As $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R})$ contains $C^\omega(\mathbb{R})$ it cannot coincide with $\mathcal{E}^{\{M\}}(\mathbb{R})$ for any weight sequence M , by Theorem 2.15 and the first paragraph; neither can $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R})$ coincide with $\mathcal{E}^{\{M\}}(\mathbb{R})$.

If there exists $\omega \in \mathscr{W}$ such that $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R})$, then Proposition 4.6 implies that for each $\rho > 0$ there exist $s, \rho' > 0$ such that

$$(5.23) \quad \Omega^{\rho'} \preceq G^s \preceq \Omega^\rho,$$

and thus, by Proposition 2.12(1),

$$\mathcal{E}^{\{\Omega^{\rho'}\}} \subseteq \mathcal{G}^{1+s} \subseteq \mathcal{E}^{\{\Omega^\rho\}}.$$

Since $\mathcal{G}^{1+s} = \mathcal{E}^{\{\gamma\}}$ with $\gamma(t) = t^{1/(1+s)}$, using the fact that there exist characteristic $\mathcal{E}^{\{\Omega^\rho\}}$ - and $\mathcal{E}^{\{\Gamma^\tau\}}$ -functions (where Γ^τ are the sequences associated with γ), and by (5.10), we conclude that, for all k ,

$$\frac{1}{\rho'} \varphi_\omega^*(\rho'k) \leq \frac{1}{\tau} \varphi_\gamma^*(\tau k) + C \quad \text{and} \quad \frac{1}{\tau} \varphi_\gamma^*(\tau k) \leq \frac{1}{H\rho} \varphi_\omega^*(H\rho k) + D,$$

for suitable constants τ, C, D, H . As in the derivation of (5.18) this implies $\omega \approx \gamma$ and hence $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R}) = \mathcal{E}^{\{\gamma\}}(\mathbb{R}) = \mathcal{E}^{\{G^s\}}(\mathbb{R})$, a contradiction. Thus there is no $\omega \in \mathscr{W}$ with $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R})$.

If there exists $\omega \in \mathscr{W}$ such that $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R})$, then Proposition 4.6 implies that for each $\rho' > 0$ there exist $s, \rho > 0$ such that (5.23) holds. Then the same arguments show $\omega \approx \gamma$ and hence $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R}) = \mathcal{E}^{\{\gamma\}}(\mathbb{R}) = \mathcal{G}^{1+s}(\mathbb{R})$, a contradiction. Thus there is no $\omega \in \mathscr{W}$ with $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R})$.

For the remaining cases note that $\mathfrak{M}(\preceq)\mathfrak{N}\{\triangleleft\}\mathfrak{M}$ and $\mathfrak{M}\{\triangleleft\}\mathfrak{N}(\preceq)\mathfrak{M}$ is impossible for any two weight matrices $\mathfrak{M}, \mathfrak{N} \in \mathscr{M}$. This fact together with Proposition 4.6 (and Theorem 2.15) implies that there is no weight sequence M and no weight function ω so that $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{M\}}(\mathbb{R})$, $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R})$, $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{M\}}(\mathbb{R})$, or $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R})$. ■

5.24. COROLLARY. *Composition is continuous on the intersection of all non-quasianalytic Gevrey classes. More precisely, $\text{comp}^{\{\mathfrak{G}\}}$ is continuous, $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}^p, f)$, for $f \in \mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}^q, \mathbb{R}^r)$, is continuous, and $\text{comp}^{\{\mathfrak{G}\}}$ is sequentially continuous.*

Proof. This follows from Theorems 4.13 and 5.22. ■

We expect that $\text{comp}^{\{\mathfrak{G}\}}$ is even $\mathcal{E}^{\{\mathfrak{G}\}}$ (see Remark 4.23).

5.25. REMARK. More autonomous spaces $\mathcal{E}^{[\mathfrak{M}]}$ can be produced by choosing the weight matrix $\mathfrak{M} := \{M^\lambda : \lambda > 0\}$ such that each M^λ has moderate growth, satisfies $\underline{\lim} (M_k^\lambda)^{1/k} > 0$ and $\underline{\lim} \mu_{nk}^\lambda / \mu_k^\lambda > 1$ for some $n \in \mathbb{N}$ with $\mu_k^\lambda = kM_k^\lambda / M_{k-1}^\lambda$, and $M^\lambda \not\approx M^\mu$ for $\lambda \neq \mu$. Here we may use the comparison theorems of [8] and argue as above.

6. Stability under composition of $\mathcal{E}^{[\omega]}$. Stability under composition of $\mathcal{E}^{[\omega]}$ was characterized in [16] for non-quasianalytic weights ω . In this section we apply the characterization obtained by means of the associated weight matrix $\mathfrak{W} = \{\Omega^\rho : \rho > 0\}$ and relate it to the results of [16].

6.1. LEMMA. *If $\omega \in \mathcal{W}$ is subadditive, then $(\Omega^\rho)^\circ \preceq \Omega^{2\rho}$ for each $\rho > 0$.*

Then the weight matrix \mathfrak{W} satisfies $(\mathfrak{M}_{\{\text{FdB}\}})$ and $(\mathfrak{M}_{\{\text{FdB}\}})$.

Proof. Subadditivity of ω implies

$$(6.2) \quad \Omega_j^\rho \Omega_k^\rho \leq \Omega_{j+k}^\rho, \quad j, k \in \mathbb{N}$$

(cf. [17, Lemma 3.3]). Indeed, $\exp(\frac{1}{\rho}\varphi^*(\rho k)) = \sup_{s \geq 1} s^k \exp(-\frac{1}{\rho}\omega(s))$ and hence, using subadditivity of ω ,

$$\begin{aligned} \Omega_j^\rho \Omega_k^\rho &\leq \sup_{s,t \geq 1} \frac{s^j t^k}{j!k!} \exp\left(-\frac{1}{\rho}\omega(s+t)\right) \\ &\leq \sup_{s,t \geq 1} \frac{(s+t)^{j+k}}{(j+k)!} \exp\left(-\frac{1}{\rho}\omega(s+t)\right) \leq \Omega_{j+k}^\rho. \end{aligned}$$

By (5.6), (6.2) and since $\Omega^\rho \leq \Omega^{2\rho}$, we get, for $\alpha_i \in \mathbb{N}_{>0}$ with $\alpha_1 + \dots + \alpha_j = k$,

$$\Omega_j^\rho \Omega_{\alpha_1}^\rho \dots \Omega_{\alpha_j}^\rho \leq C^j \Omega_j^{2\rho} \Omega_{\alpha_1-1}^{2\rho} \dots \Omega_{\alpha_j-1}^{2\rho} \leq C^j \Omega_k^{2\rho},$$

which implies the assertion. ■

6.3. THEOREM. *For $\omega \in \mathcal{W}$ satisfying (ω_4) the following are equivalent:*

- (1) $\mathcal{E}^{\{\omega\}}$ is stable under composition.
- (2) For each $\rho > 0$ there is $\tau > 0$ such that $(\Omega^\rho)^\circ \preceq \Omega^\tau$, i.e., \mathfrak{W} satisfies $(\mathfrak{M}_{\{\text{FdB}\}})$.
- (3) There exists a subadditive $\tilde{\omega} \in \mathcal{W}$ such that $\omega \approx \tilde{\omega}$.
- (4) ω satisfies (ω_7) .

Proof. (1) \Leftrightarrow (2) follows from Theorem 4.9 and Corollary 5.15.

(3) \Leftrightarrow (4). See [32, Prop. 1.1] and [30, Lemma 1].

(3) \Rightarrow (2) follows from Lemma 6.1.

(2) \Rightarrow (3). The proof is inspired by [16, Prop. 2.3] which treats the non-quasianalytic case. We do not assume non-quasianalyticity (or quasianalyticity) and use Claim 4.14 to remedy the lack of $\mathcal{E}^{\{\omega\}}$ -functions of compact

support. If ω does not satisfy (ω_7) , then there exist increasing sequences $(k_n) \in \mathbb{N}^{\mathbb{N}}$ and $(t_n) \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that

$$(6.4) \quad \omega(k_n t_n) \geq n^2 k_n \omega(t_n).$$

Set $a_n := e^{-n\omega(t_n)}$ and $f_n(x) := a_n e^{it_n x}$, $x \in \mathbb{R}$. Then

$$\begin{aligned} \|f_n\|_{\mathbb{R}, \rho}^{\omega} &= a_n \sup_{j \in \mathbb{N}} t_n^j \exp\left(-\frac{1}{\rho} \varphi^*(\rho j)\right) = a_n \exp \sup_{j \in \mathbb{N}} \left(j \log t_n - \frac{1}{\rho} \varphi^*(\rho j)\right) \\ &= e^{-n\omega(t_n)} e^{\omega_{\rho}(t_n)} \leq e^{-(n-1/\rho)\omega(t_n)} \end{aligned}$$

and so $\{f_n : n \in \mathbb{N}\}$ is bounded in $\mathcal{E}^{\{\omega\}}(\mathbb{R}, \mathbb{C})$ (even in $\mathcal{E}^{(\omega)}(\mathbb{R}, \mathbb{C})$). The set $\{\mathbb{C} \ni z \mapsto z^k : k \in \mathbb{N}\}$ forms a bounded subset of $\mathcal{E}^{\{\omega\}}(\mathbb{D}, \mathbb{C})$, where $\mathbb{D} \subseteq \mathbb{C}$ is the unit disk and where we identify $\mathbb{C} \cong \mathbb{R}^2$). Indeed, for $|z| \leq r < 1$ choose $\rho > 0$ so that $r + 1/\rho < 1$, and thus

$$\sup_{j \in \mathbb{N}} \frac{|\partial_z^j z^k|}{\rho^j j!} \leq \sup_{j \leq k} \binom{k}{j} r^{k-j} \frac{1}{\rho^j} \leq \left(r + \frac{1}{\rho}\right)^k.$$

So $\{z \mapsto z^k : k \in \mathbb{N}\}$ is bounded in $C^{\omega}(\mathbb{D}, \mathbb{C})$ and, by (ω_4) , in $\mathcal{E}^{\{\omega\}}(\mathbb{D}, \mathbb{C})$. Since \mathfrak{W} satisfies $(\mathfrak{M}_{\{\text{FdB}\}})$ by assumption (2), we may conclude, from Claim 4.14, that the set $\{f_n^k : n, k \in \mathbb{N}\}$ is bounded in $\mathcal{E}^{\{\omega\}}(\mathbb{R}, \mathbb{C})$. Thus there exists $\rho > 0$ such that

$$\begin{aligned} \infty &> \sup_{n, k, j \in \mathbb{N}} |(f_n^k)^{(j)}(0)| \exp\left(-\frac{1}{\rho} \varphi^*(\rho j)\right) = \sup_{n, k, j \in \mathbb{N}} a_n^k (t_n k)^j \exp\left(-\frac{1}{\rho} \varphi^*(\rho j)\right) \\ &= \sup_{n, k \in \mathbb{N}} a_n^k e^{\omega_{\rho}(t_n k)} \geq D \sup_{n, k \in \mathbb{N}} a_n^k e^{C^{-1}\omega(t_n k)} = D \sup_{n, k \in \mathbb{N}} e^{-nk\omega(t_n) + C^{-1}\omega(t_n k)}, \end{aligned}$$

for constants $C, D > 0$, by Lemma 5.7, which contradicts (6.4). ■

6.5. THEOREM. *For $\omega \in \mathscr{W}$ satisfying (ω_4) the following are equivalent:*

- (1) $\mathcal{E}^{(\omega)}$ is stable under composition.
- (2) $\mathcal{E}^{(\omega)}$ is holomorphically closed.
- (3) For each $\rho > 0$ there exists $\tau > 0$ such that $(\Omega^{\tau})^{\circ} \preceq \Omega^{\rho}$, i.e., \mathfrak{W} satisfies $(\mathfrak{M}_{\{\text{FdB}\}})$.
- (4) There exists $H \geq 1$ such that for each $\rho > 0$ we have $(\Omega^{\rho})^{\circ} \preceq \Omega^{H\rho}$.
- (5) There exists a subadditive $\tilde{\omega} \in \mathscr{W}$ such that $\omega \approx \tilde{\omega}$.
- (6) ω satisfies (ω_7) .

Note that (ω_4) is needed only for (1) \Rightarrow (2).

Proof. (1) \Leftrightarrow (2) \Leftrightarrow (3) follows from Theorem 4.11 and Corollary 5.15.

(2) \Rightarrow (6) follows from an argument due to [10] (see [16, p. 405]).

(5) \Leftrightarrow (6). See [32, Prop. 1.1] and [30, Lemma 1].

(5) \Rightarrow (4) follows from Lemmas 6.1 and 5.16.

(4) \Rightarrow (3) is evident. ■

6.6. COROLLARY. For $\omega \in \mathcal{W}$ satisfying (ω_4) the following are equivalent:

- (1) For each $\rho > 0$ there exists $\tau > 0$ such that $(\Omega^\rho)^\circ \preceq \Omega^\tau$.
- (2) For each $\rho > 0$ there exists $\tau > 0$ such that $(\Omega^\tau)^\circ \preceq \Omega^\rho$.
- (3) There exists $H \geq 1$ such that $(\Omega^\rho)^\circ \preceq \Omega^{H\rho}$ for each $\rho > 0$.

Proof. Combine Theorems 6.3 and 6.5. ■

Special cases of Theorem 4.13 were proven in [16, 4.2 and 4.4]:

6.7. COROLLARY. Let $\omega \in \mathcal{W}$ satisfy (ω_7) . Then $\text{comp}^{(\omega)}$ is continuous, $\mathcal{E}^{\{\omega\}}(\mathbb{R}^p, f)$, for $f \in \mathcal{E}^{\{\omega\}}(\mathbb{R}^q, \mathbb{R}^r)$, is continuous, and $\text{comp}^{\{\omega\}}$ is sequentially continuous.

Proof. This is a special case of Theorem 4.13, by Corollary 5.15, Theorem 6.3, and Theorem 6.5. ■

We expect that the mapping $\text{comp}^{[\omega]}$ is even $\mathcal{E}^{[\omega]}$ (see Remark 4.23).

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