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Unconditionally p-null sequences and unconditionally p-compact operators

by

Ju Myung Kim (Seoul)

Abstract. We investigate sequences and operators via the unconditionally p-summable sequences. We characterize the unconditionally p-null sequences in terms of a certain tensor product and then prove that, for every $1 \le p < \infty$, a subset of a Banach space is relatively unconditionally p-compact if and only if it is contained in the closed convex hull of an unconditionally p-null sequence.

1. Introduction and main results. Grothendieck [G] showed that a subset K of a Banach space X is relatively compact if and only if there exists a null sequence (x_n) in X such that

$$K \subset \Big\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_1} \Big\},$$

where we denote by B_Z the unit ball of a Banach space Z. The notion of p-compactness of Sinha and Karn [SK] stems from this criterion. For $1 \le p \le \infty$, a subset K of X is called relatively *p-compact* if there exists $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ if $p = \infty$) such that

$$K \subset p\text{-co}(\{x_n\}) := \Big\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_{p^*}} \Big\},$$

where $1/p + 1/p^* = 1$ and $\ell_p(X)$ (resp. $c_0(X)$) is the Banach space with the norm $\|\cdot\|_p$ (resp. $\|\cdot\|_\infty$) of all X-valued absolutely p-summable (resp. null) sequences.

For $1 \le p \le \infty$, the closed subspace $\ell_p^u(X)$ of $\ell_p^w(X)$, the Banach space with the norm $\|\cdot\|_p^w$ of all X-valued weakly p-summable sequences, consists of sequences (x_n) satisfying

$$\|(0, \dots, 0, x_m, x_{m+1}, \dots)\|_p^w \to 0$$

as $m \to \infty$. It is well known that $(x_n) \in \ell_1^u(X)$ if and only if (x_n) is an

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unconditionally summable sequence (cf. [R, Example 3.4]). If $(x_n) \in \ell_p^u(X)$, we call it an *unconditionally p-summable sequence*. We say that a subset K of X is relatively *unconditionally p-compact* (u-p-compact) if there exists $(x_n) \in \ell_p^u(X)$ such that $K \subset p\text{-co}(\{x_n\})$. Note that every u-p-compact set is a compact set.

Piñeiro and Delgado [PD] introduced and studied p-null sequences. For $1 \le p < \infty$, a sequence (x_n) in a Banach space X is said to be *p-null* if for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $(z_k) \in \ell_p(X)$ with $\|(z_k)\|_p \le \varepsilon$ such that $x_n \in p\text{-co}(\{z_k\})$ for all $n \ge N$. The collection of all p-null sequences in X is denoted by $c_{0,p}(X)$.

In this paper, a sequence is called *unconditionally p-null* (u-p-null) when $\ell_p(X)$ and $\|\cdot\|_p$ are replaced by $\ell_p^u(X)$ and $\|\cdot\|_p^w$. We denote by $c_{0,up}(X)$ the collection of all u-p-null sequences in X. Note that for every $1 \le p < \infty$, $c_{0,up}(X) \subset c_0(X)$ and $c_{0,u\infty}(X) = c_0(X)$. As in [PD], we can analogously define a norm on $c_{0,up}(X)$ (see Section 3).

Fourie and Swart [FS2] studied the following norm on the tensor product $X \otimes Y$ of Banach spaces X and Y. Let $1 \le p \le \infty$. For $u \in X \otimes Y$, define

$$w_p(u) = \inf\Big\{\|(x_j)\|_p^w \|(y_j)\|_{p^*}^w : u = \sum_{j=1}^n x_j \otimes y_j\Big\}.$$

Then $(X \otimes Y, w_p)$ is a normed space and we denote by $X \hat{\otimes}_{w_p} Y$ its completion. Recall that a norm on tensor products of Banach spaces is a *tensor norm* if it is a finitely generated uniform crossnorm (cf. [R, Section 6.1]). It was shown in [FS2] that w_p is a tensor norm. Oja [O] studied p-null sequences in terms of the Chevet–Saphar tensor product. The following theorem is the analogue of [O, Theorem 4.1] for u-p-null sequences.

THEOREM 1.1. *Let $1 \le p \le \infty$. The tensor product $c_0 \hat{\otimes}_{w_{p^*}} X$ is isometrically isomorphic to $c_{0,up}(X)$ and for every $u \in c_0 \hat{\otimes}_{w_{p^*}} X$ there exists $(x_n) \in c_{0,up}(X)$ such that $u = \sum_n e_n \otimes x_n$ in $c_0 \hat{\otimes}_{w_{p^*}} X$.*

Piñeiro and Delgado [PD, Proposition 2.6] showed that for $1 \le p < \infty$, a sequence (x_n) is in $c_{0,p}(X)$ if and only if $(x_n) \in c_0(X)$ and the set $\{x_n\}$ is relatively p-compact under an assumption depending on p, and they asked whether the assumption could be deleted. Oja [O, Theorem 4.3] gave an affirmative answer to that question. The following is the result of [O, Theorem 4.3] adapted to u-p-null sequences.

THEOREM 1.2. *Let (x_n) be a sequence in X and let $1 \le p < \infty$. Then the following statements are equivalent:*

(a) $(x_n) \in c_{0,up}(X)$.
(b) *(x_n) is null and the set $\{x_n\}$ is relatively u-p-compact.*
(c) *(x_n) is weakly null and the set $\{x_n\}$ is relatively u-p-compact.*

It was shown in [PD, Theorem 2.5] that for $1 \leq p < \infty$, a subset of a Banach space X is relatively p-compact if and only if it is contained in the closed convex hull $\overline{\mathrm{co}}(\{x_n\})$ of a p-null sequence (x_n). For an alternative straightforward proof, see [AO]. For relatively u-p-compact sets we obtain the following result, where $\overline{\mathrm{bco}}(A)$ means the closed balanced convex hull of a set A.

COROLLARY 1.3. *Let K be a subset of X and let $1 \leq p < \infty$. Then the following statements are equivalent:*

(a) *K is relatively u-p-compact.*
(b) *There exists $(x_n) \in c_{0,up}(X)$ such that $K \subset \overline{\mathrm{co}}(\{x_n\})$.*
(c) *There exists $(x_n) \in c_{0,up}(X)$ such that $K \subset \overline{\mathrm{bco}}(\{x_n\})$.*

We prove Theorems 1.1 and 1.2 and Corollary 1.3 in Section 3 after studying operators via unconditionally p-summable sequences.

2. Unconditionally p-compact and unconditionally (quasi) p-nuclear operators. For $1 \leq p \leq \infty$, following the definition of a p-compact operator in [SK], a linear map $T : X \to Y$ is said to be *u-p-compact* if $T(B_X)$ is a relatively u-p-compact subset of Y. The collection of all u-p-compact operators from X to Y is denoted by $\mathcal{K}_{up}(X,Y)$ and we define a norm u_p on $\mathcal{K}_{up}(X,Y)$ by

$$u_p(T) = \inf\big\{\|(y_n)\|_p^w : (y_n) \in \ell_p^u(Y) \text{ and } T(B_X) \subset p\text{-co}(\{y_n\})\big\}.$$

From Grothendieck's criterion of compactness, the ideal $[\mathcal{K}, \|\cdot\|]$ of compact operators equipped with the operator norm coincides with $[\mathcal{K}_{u\infty}, u_\infty]$ and we have:

THEOREM 2.1. *For every $1 \leq p < \infty$, $[\mathcal{K}_{up}, u_p]$ is a Banach operator ideal.*

The proof of Theorem 2.1 is similar to the one of [PP, Lemma 4] and follows the scheme indicated by Delgado, Piñeiro and Serrano [DPS] for the ideal of p-compact operators.

Recall that a *p-nuclear operator* $T \in \mathcal{N}_p(X,Y)$ from X to Y, for $1 \leq p \leq \infty$, is represented as $T = \sum_n x_n^* \otimes y_n$, where $(x_n^*) \in \ell_p(X^*)$ ($(x_n) \in c_0(X^*)$ if $p = \infty$) and $(y_n) \in \ell_{p^*}^w(Y)$, and the p-nuclear norm $\nu_p(T)$ equals $\inf \|(x_n^*)\|_p \|(y_n)\|_{p^*}^w$, where the infimum is taken over all such representations of T (cf. [DJT, Proposition 5.23]). When the spaces $\ell_p(X^*)$ and $\ell_{p^*}^w(Y)$ are replaced by $\ell_p^u(X^*)$ and $\ell_{p^*}^u(Y)$ respectively, the map is well known as a classical p-compact operator (cf. [P, Section 18.3] and [FS1, FS2]). To avoid confusion, in this paper, we call it an *unconditionally p-nuclear* (u-p-nuclear) operator. The collection of all u-p-nuclear operators from X to Y is denoted by $\mathcal{N}_{up}(X,Y)$ and the u-p-nuclear norm ν_{up} is defined by

$\nu_{up}(T) = \inf \|(x_n^*)\|_p^w \|(y_n)\|_{p^*}^w$, where the infimum is taken over all such representations of T. It is well known that $[\mathcal{N}_{up}, \nu_{up}]$ is a Banach operator ideal (cf. [FS1, Theorems 2.1 and 2.5]).

Proposition 2.2. $\mathcal{K}_{u2} = \mathcal{N}_{u2}$.

Proof. Clearly $\mathcal{N}_{u2} \subset \mathcal{K}_{u2}$ (in fact, $\mathcal{N}_{up^*} \subset \mathcal{K}_{up}$ when $1 \le p \le \infty$). To show the other inclusion, let $T : X \to Y$ be a u-2-compact operator. Then there exists $(y_n) \in \ell_2^u(Y)$ such that $T(B_X) \subset \{\sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_2}\}$. Define operators $E_y : \ell_2 \to Y$ by $E_y\alpha = \sum_n \alpha_n y_n$, and $\hat{E}_y : \ell_2/\ker(E_y) \to Y$ by $\hat{E}_y[\alpha] = E_y\alpha$. Then $\hat{E}_y$ is a compact operator. In view of the factorization in [SK, Section 3], $T = \hat{E}_y T_y$, where $T_y : X \to \ell_2/\ker(E_y)$ is a bounded operator. According to [FS1, Theorem 2.3], T is a u-2-nuclear operator. ∎

For $1 \le p \le \infty$, following the definition of a quasi p-nuclear operator in [PP], a linear map $T : X \to Y$ is called *quasi unconditionally p-nuclear* (quasi u-p-nuclear) if there exists $(x_n^*) \in \ell_p^u(X^*)$ such that $\|Tx\| \le \|(x_n^*(x))\|_p$ for every $x \in X$. We denote by $\mathcal{N}_{up}^Q(X,Y)$ the collection of all quasi u-p-nuclear operators from X to Y. For $T \in \mathcal{N}_{up}^Q(X,Y)$, let $\nu_{up}^Q(T) = \inf \|(x_n^*)\|_p^w$, where the infimum is taken over all such inequalities. Note that a quasi u-∞-nuclear operator is just a compact operator (cf. [D, Exercise II.6(ii)]). We can also use the proof of [PP, Lemma 4] to show that $[\mathcal{N}_{up}^Q, \nu_{up}^Q]$ is a Banach operator ideal.

We now obtain the duality of u-p-compact operators, which is the analogue of the duality of p-compact operators from [DPS]. In fact, Theorem 2.3 and the "only if" part of Theorem 2.4 are essentially due to [DPS].

Theorem 2.3. *Let $1 \le p \le \infty$ and let $T : X \to Y$ be a linear map. Then $T \in \mathcal{N}_{up}^Q(X,Y)$ if and only if $T^* \in \mathcal{K}_{up}(Y^*, X^*)$. In this case, $\nu_{up}^Q(T) = u_p(T^*)$.*

Proof. This is immediate from [DPS, Proposition 3.2]. ∎

Theorem 2.4. *Let $1 \le p \le \infty$ and let $T : X \to Y$ be a linear map. Then $T \in \mathcal{K}_{up}(X,Y)$ if and only if $T^* \in \mathcal{N}_{up}^Q(Y^*, X^*)$. In this case, $\nu_{up}^Q(T^*) \le u_p(T)$.*

Proof of the "only if" part of Theorem 2.4. Let $T \in \mathcal{K}_{up}(X,Y)$ and let $(y_n) \in \ell_p^u(Y)$ be such that $T(B_X) \subset p\text{-co}(\{y_n\})$. Then by [DPS, Proposition 3.1],

$$\|T^*y^*\| \le \|(i_Y(y_n)(y^*))\|_p$$

for every $y^* \in Y^*$. Note that $(i_Y(y_n)) \in \ell_p^u(Y^{**})$, where $i_Y : Y \to Y^{**}$ is the natural isometry, and $\|(i_Y(y_n))\|_p^w = \|(y_n)\|_p^w$. Hence $T^* \in \mathcal{N}_{up}^Q(Y^*, X^*)$ and $\nu_{up}^Q(T^*) \le u_p(T)$. ∎

From Theorems 2.3 and 2.4, we have:

COROLLARY 2.5. *Let $1 \le p \le \infty$ and let $T : X \to Y$ be a linear map. Then $T \in \mathcal{K}_{up}(X,Y)$ (resp. $\mathcal{N}^Q_{up}(X,Y)$) if and only if $T^{**} \in \mathcal{K}_{up}(X^{**},Y^{**})$ (resp. $\mathcal{N}^Q_{up}(X^{**},Y^{**})$). In this case, $u_p(T^{**}) \le u_p(T)$ (resp. $\nu^Q_{up}(T^{**}) \le \nu^Q_{up}(T)$).*

In order to prove the "if" part of Theorem 2.4, we also use an argument from [DPS]. By definition we see that $\mathcal{N}_{up}(X,Y) \subset \mathcal{N}^Q_{up}(X,Y)$ and $\nu^Q_{up}(T) \le \nu_{up}(T)$ for every $T \in \mathcal{N}_{up}(X,Y)$, and we have:

LEMMA 2.6. *Let $1 \le p \le \infty$. Suppose that Y is injective. If $T \in \mathcal{N}^Q_{up}(X,Y)$, then $T \in \mathcal{N}_{up}(X,Y)$ and $\nu^Q_{up}(T) = \nu_{up}(T)$.*

Proof. This proof is essentially due to [PP]. Let $T \in \mathcal{N}^Q_{up}(X,Y)$. Let $\varepsilon > 0$ be given. Then there exists $(x^*_n) \in \ell^u_p(X^*)$ such that for every $x \in X$, $\|Tx\| \le \|(x^*_n(x))\|_p$ and $\|(x^*_n)\|^w_p \le \nu^Q_{up}(T) + \varepsilon$. Consider the linear subspace $Z = \{(x^*_n(x)) : x \in X\}$ of ℓ_p (or of c_0 if $p = \infty$) and the map $J : Z \to Y$, $(x^*_n(x)) \mapsto Tx$. Then J is well defined and linear, and $\|J\| \le 1$. Since Y is injective, there exists an extension $\hat{J} : \ell_p \to Y$ of J with $\|\hat{J}\| = \|J\|$. Define a map $U : X \to \ell_p$ by $Ux = (x^*_n(x))$. Then U is a compact operator and the following diagram is commutative:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "U"'] & & Y \\ & \ell_p \arrow[ur, "\hat{J}"'] & \end{tikzcd}$$

Hence by [FS1, Theorem 2.5], $T \in \mathcal{N}_{up}(X,Y)$ and $\nu_{up}(T) \le \|U\|\,\|\hat{J}\| \le \|(x^*_n)\|^w_p \le \nu^Q_{up}(T) + \varepsilon$, and so $\nu^Q_{up}(T) = \nu_{up}(T)$. ∎

Let K be a bounded subset of X. In [DPS], the authors defined the operators $u_K : \ell_1(K) \to X$ and $j_K : X^* \to \ell_\infty(K)$, respectively, by $u_K(\xi_x)_{x\in K} = \sum_{x\in K} \xi_x x$ and $j_K x^* = (x^*(x))_{x\in K}$. We see that $u^*_K = j_K$.

We now obtain the versions for u-p-compactness of [DPS, Proposition 3.5, Corollary 3.6, Remark 3.7].

PROPOSITION 2.7. *Let $1 \le p \le \infty$ and let K be a bounded subset of X. Then the following statements are equivalent:*

(a) *K is relatively u-p-compact.*
(b) *u_K is u-p-compact.*
(c) *j_K is u-p-nuclear.*

Proof. (a)⇒(b). Let $(x_n) \in \ell^u_p(X)$ be such that $K \subset p\text{-co}(\{x_n\})$. Then $u_K(B_{\ell_1(K)}) \subset \overline{\mathrm{bco}}(K) \subset p\text{-co}(\{x_n\})$. Hence u_K is u-p-compact.

(b)⇒(a). Since $K \subset u_K(B_{\ell_1(K)})$, this is clear.

(b)⇒(c). If u_K is u-p-compact, then by Theorem 2.4(⇒), $u_K^* = j_K$ is quasi u-p-nuclear. Since $\ell_\infty(K)$ is injective, (c) follows from Lemma 2.6.

(c)⇒(b). If $u_K^* = j_K$ is u-p-nuclear, then by [DFS, Proposition 1.5.7], u_K is u-p^*-nuclear. Hence u_K is u-p-compact, because, clearly, $\mathcal{N}_{up^*} \subset \mathcal{K}_{up}$. ∎

The following is a duality version of Proposition 2.7.

PROPOSITION 2.8. *Let $1 \le p \le \infty$ and let C be a bounded subset of X^*. Then the following statements are equivalent:*

(a) *C is relatively u-p-compact.*
(b) *The map $u_C : \ell_1(C) \to X^*$ defined by $u_C((\xi_{x^*})_{x^* \in C}) = \sum_{x^* \in C} \xi_{x^*} x^*$ is a u-p-compact operator.*
(c) *The map $j_C : X \to \ell_\infty(C)$ defined by $j_C x = (x^*(x))_{x^* \in C}$ is a u-p-nuclear operator.*

Proof. Use the duality relationships $u_C^* i_X = j_C$ and $j_C^* i_{l_1(C)} = u_C$, and Theorems 2.3 and 2.4(⇒). ∎

COROLLARY 2.9. *Let $1 \le p \le \infty$ and let K be a subset of X. If $i_X(K)$ is a relatively u-p-compact subset of X^{**}, then K is a relatively u-p-compact subset of X.*

Proof. If $i_X(K)$ is a relatively u-p-compact subset of X^{**}, then by Proposition 2.8, the operator $j_{i_X(K)} : X^* \to \ell_\infty(i_X(K))$, which is actually the operator $j_K : X^* \to \ell_\infty(K)$ in Proposition 2.7, is u-p-nuclear. Hence K is relatively u-p-compact. ∎

We now complete the proof of Theorem 2.4.

Proof of the "if" part of Theorem 2.4. If $T^* \in \mathcal{N}_{up}^Q(Y^*, X^*)$, then by Theorem 2.3, $T^{**} \in \mathcal{K}_{up}(X^{**}, Y^{**})$. Thus $i_Y T(B_X) = T^{**} i_X(B_X)$ is a relatively u-p-compact subset of Y^{**}. Hence by Corollary 2.9, $T(B_X)$ is a relatively u-p-compact subset of Y and so $T \in \mathcal{K}_{up}(X, Y)$. ∎

3. Proofs of main results. For a bounded sequence $\hat{x} := (x_n)$ in X, define an operator $u_{\hat{x}} : \ell_1 \to X$ by $u_{\hat{x}}(\alpha_n) = \sum_n \alpha_n x_n$. Then for $1 \le p \le \infty$, by Proposition 2.7, the set $\{x_n\}$ is relatively u-p-compact if and only if the operator $u_{\hat{x}}$ is u-p-compact. As indicated for p-null sequences [PD, Remark 2.2], a simple verification shows that a sequence (x_n) is u-p-null if and only if $\{x_n\}$ is relatively u-p-compact and $u_p(u_{\hat{x}(n)} - u_{\hat{x}}) \to 0$ as $n \to \infty$, where $\hat{x}(n) := (x_1, \ldots, x_n, 0, \ldots)$.

Let $1 \le p \le \infty$. We define a norm on $c_{0,up}(X)$ by $\|(x_n)\|_{up}^0 = u_p(u_{\hat{x}})$ for $(x_n) \in c_{0,up}(X)$. Then $(c_{0,u\infty}(X), \|\cdot\|_{u\infty}^0) = (c_0(X), \|\cdot\|_\infty)$ and we have the following result whose proof is straightforward.

PROPOSITION 3.1. *Let $1 \le p < \infty$. Then $(c_{0,up}(X), \|\cdot\|_{up}^0)$ is a Banach space.*

We need the following result to prove Theorem 1.1. Its prototype is [DPS, Proposition 3.11].

LEMMA 3.2. *Let $1 \le p \le \infty$ and let $T : Y \to X$ be a linear map. Then $T \in \mathcal{K}_{up}(Y, X)$ if and only if $Tu_{B_Y} \in \mathcal{N}_{up^*}(\ell_1(B_Y), X)$. In this case, $u_p(T) = \nu_{up^*}(Tu_{B_Y})$.*

Proof. If $T \in \mathcal{K}_{up}(Y, X)$, then by Theorem 2.4, $T^* \in \mathcal{N}_{up}^Q(X^*, Y^*)$ and $\nu_{up}^Q(T^*) \le u_p(T)$. It follows from Lemma 2.6 that

$$(Tu_{B_Y})^* = j_{B_Y} T^* \in \mathcal{N}_{up}(X^*, \ell_\infty(B_Y)).$$

Hence by [DFS, Proposition 1.5.7], $Tu_{B_Y} \in \mathcal{N}_{up^*}(\ell_1(B_Y), X)$ and

$$\nu_{up^*}(Tu_{B_Y}) = \nu_{up}(j_{B_Y} T^*) = \nu_{up}^Q(j_{B_Y} T^*) \le u_p(T).$$

To show the converse, let $Tu_{B_Y} = \sum_n (\zeta_y^n)_y \otimes x_n \in \mathcal{N}_{up^*}(\ell_1(B_Y), X)$ be a representation, where $((\zeta_y^n)_y) \in \ell_{p^*}^u(\ell_\infty(B_Y))$ and $(x_n) \in \ell_p^u(X)$. Then

$$T(B_Y) = \Big\{ \sum_n \zeta_y^n x_n : y \in B_Y \Big\} \subset p\text{-co}\big(\{\|((\zeta_y^k)_y)_k\|_{p^*}^w x_n\}_n\big).$$

Hence $T \in \mathcal{K}_{up}(Y, X)$ and $u_p(T) \le \|((\zeta_y^k)_y)_k\|_{p^*}^w \|(x_n)\|_p^w$. Since the representation was arbitrary, $u_p(T) \le \nu_{up^*}(Tu_{B_Y})$. ∎

Since for every operator $T : \ell_1 \to X$, T coincides with $Tu_{B_{\ell_1}} i$, where the map $i : \ell_1 \to \ell_1(B_{\ell_1})$ is the canonical isometry, by Lemma 3.2 we have:

COROLLARY 3.3. *Let $1 \le p \le \infty$ and let $T : \ell_1 \to X$ be a linear map. Then $T \in \mathcal{K}_{up}(\ell_1, X)$ if and only if $T \in \mathcal{N}_{up^*}(\ell_1, X)$. In this case, $u_p(T) = \nu_{up^*}(T)$.*

We also need a result of Fourie and Swart [FS2] to prove Theorem 1.1.

LEMMA 3.4 ([FS2, Proposition 3.2]). *Let $1 \le p \le \infty$. If $(x_n) \in \ell_p^u(X)$ and $(y_n) \in \ell_{p^*}^u(Y)$, then $\sum_n x_n \otimes y_n$ converges in $X \hat{\otimes}_{w_p} Y$. Conversely, if $u \in X \hat{\otimes}_{w_p} Y$, then there exist $(x_n) \in \ell_p^u(X)$ and $(y_n) \in \ell_{p^*}^u(Y)$ such that $\sum_n x_n \otimes y_n$ converges to u. Moreover,*

$$w_p(u) = \inf \Big\{ \|(x_n)\|_p^w \|(y_n)\|_{p^*}^w : u = \sum_{n=1}^\infty x_n \otimes y_n, (x_n) \in \ell_p^u(X), (y_n) \in \ell_{p^*}^u(Y) \Big\}.$$

Proof of Theorem 1.1. In order to show the first part, consider the linear map $J : (c_0 \otimes X, w_{p^*}) \to c_{0,up}(X)$ defined by

$$J\Big(\sum_{j \le n} (\lambda_i^j)_i \otimes x_j\Big) = \Big(\sum_{j \le n} \lambda_i^j x_j\Big)_i.$$

First, one may check that $J(c_0 \otimes X) \subset c_{0,up}(X)$ using elementary tensors. Since, for every $(x_n) \in c_{0,up}(X)$ and $m \in \mathbb{N}$, $\hat{x}(m) = J(\sum_{j\le m} e_j \otimes x_j)$ and

$$\lim_{m\to\infty} \|\hat{x}(m) - \hat{x}\|_{up}^0 = \lim_{m\to\infty} u_p(u_{\hat{x}(m)} - u_{\hat{x}}) = 0,$$

$J(c_0 \otimes X)$ is dense in $c_{0,up}(X)$.

To show that the map $J : (c_0 \otimes X, w_{p^*}) \to c_{0,up}(X)$ is an isometry, let $T = \sum_{j\le n} (\lambda_i^j)_i \otimes x_j \in c_0 \otimes X$ and let $(z_i) := (\sum_{j\le n} \lambda_i^j x_j)_i$. Then $T = u_{\hat{z}}$. From [DFS, Proposition 1.5.5] and Corollary 3.3, it follows that

$$\|(z_i)\|_{up}^0 = u_p(u_{\hat{z}}) = u_p(T) = \nu_{up^*}(T) = w_{p^*}\Big(\sum_{j\le n} (\lambda_i^j)_i \otimes x_j\Big).$$

Since $c_{0,up}(X)$ is a Banach space, the extension $\hat{J} : c_0 \hat{\otimes}_{w_{p^*}} X \to c_{0,up}(X)$ of J is a surjective linear isometry.

In order to show the second part, let $u \in c_0 \hat{\otimes}_{w_{p^*}} X$. Then by Lemma 3.4 there exist $((\lambda_i^j)_i)_j \in \ell_{p^*}^u(c_0)$ and $(z_j) \in \ell_p^u(X)$ such that $u = \sum_{j=1}^\infty (\lambda_i^j)_i \otimes z_j$ in $c_0 \hat{\otimes}_{w_{p^*}} X$. For every i, let

$$x_i := \sum_{j=1}^\infty \lambda_i^j z_j.$$

We show that (x_i) is the desired sequence. Let $\varepsilon > 0$ be given. Since $((\lambda_i^j)_i)_j \in \ell_{p^*}^u(c_0)$, it is easily seen that there exists an $N \in \mathbb{N}$ such that

$$\sup_{i\ge N} \|(\lambda_i^j)_j\|_{p^*} \|(z_j)\|_p^w \le \varepsilon.$$

Let $C := \sup_{i\ge N} \|(\lambda_i^j)_j\|_{p^*}$. We may assume that $C \ne 0$. Then $i \ge N$ implies that

$$x_i = \sum_{j=1}^\infty \frac{\lambda_i^j}{C} C z_j \subset p\text{-co}(\{C z_j\}).$$

Since $\|(C z_j)\|_p^w \le \varepsilon$, $(x_i) \in c_{0,up}(X)$. Recall the isometry $\hat{J} : c_0 \hat{\otimes}_{w_{p^*}} X \to c_{0,up}(X)$. Let $\hat{J}(u) := (u_i)$. Since $\lim_{n\to\infty} (\sum_{j\le n} \lambda_i^j z_j)_i = \hat{J}(u)$ in $c_{0,up}(X)$, $\lim_{n\to\infty} \sum_{j\le n} \lambda_i^j z_j = u_i$ in X for every i. Hence $(x_i) = (u_i) = \hat{J}(u)$ and so

$$u = \hat{J}^{-1}((x_i)) = \lim_{m\to\infty} \hat{J}^{-1}(\hat{x}(m)) = \lim_{m\to\infty} \sum_{i\le m} e_i \otimes x_i. \blacksquare$$

We need the main theorem in [O] to prove Theorem 1.2.

Lemma 3.5 ([O, Theorem 2.4]). *Let α be a tensor norm. Assume that X^{***} or Y has the approximation property. If $T \in \mathcal{N}_\alpha(X^*, Y)$ and $T^*(Y^*) \subset i_X(X)$, then $T \in \overline{X \otimes Y}$ in $\mathcal{N}_\alpha(X^*, Y)$.*

COROLLARY 3.6. *Let $1 \le p \le \infty$. Assume that X^{***} or Y has the approximation property. If $T \in \mathcal{N}_{up}(X^*, Y)$ and $T^*(Y^*) \subset i_X(X)$, then $T = \sum_n x_n \otimes y_n$ in $\mathcal{N}_{up}(X^*, Y)$, where $(x_n) \in \ell_p^u(X)$ and $(y_n) \in \ell_{p^*}^u(Y)$.*

Proof. From Lemma 3.5, [DFS, Corollary 1.4.9 and Proposition 1.5.5], it follows that

$$T \in \overline{X \otimes Y}^{\mathcal{N}_{up}(X^*,Y)} = \overline{X \otimes Y}^{X^{**} \hat{\otimes}_{w_p} Y} = X \hat{\otimes}_{w_p} Y.$$

Hence from Lemma 3.4 we obtain the conclusion. ∎

Proof of Theorem 1.2. (a)⇒(b) and (b)⇒(c) are obvious.

(c)⇒(a). This proof is essentially due to [O, Theorem 4.3]. If $\{x_n\}$ is relatively u-p-compact, then $u_{\hat{x}} \in \mathcal{K}_{up}(\ell_1, X)$. By Corollary 3.3, $u_{\hat{x}} \in \mathcal{N}_{up^*}(\ell_1, X)$ and $u_p(u_{\hat{x}}) = \nu_{up^*}(u_{\hat{x}})$.

Since (x_n) is weakly null, we see that $u_{\hat{x}}^*(X^*) \subset c_0$. Since c_0^{***} has the approximation property, it follows from Corollary 3.6 that $u_{\hat{x}} \in c_0 \hat{\otimes}_{w_{p^*}} X$. Hence by Theorem 1.1 there exists $(z_n) \in c_{0,up}(X)$ such that $u_{\hat{x}} = \sum_n e_n \otimes z_n$ in $c_0 \hat{\otimes}_{w_{p^*}} X$ and so $z_k = x_k$ for every k, which completes the proof. ∎

REMARK 3.7. We can also prove Theorem 1.2 using the argument of Lassalle and Turco [LT] based on Carl–Stephani theory [CS].

Proof of Corollary 1.3. (b)⇒(c) is trivial.

(c)⇒(a). If $(x_n) \in c_{0,up}(X)$, then the set $\{x_n\}$ is relatively u-p-compact. Thus there exists $(z_n) \in \ell_p^u(X)$ such that $\{x_n\} \subset p\text{-co}(\{z_n\})$. By (c) we have $K \subset p\text{-co}(\{z_n\})$, hence the assertion (a) follows.

(a)⇒(b). Since K is relatively u-p-compact, there exists $(x_n) \in \ell_p^u(X)$ such that $K \subset p\text{-co}(\{x_n\})$. By a standard argument we can find a sequence (β_n) of positive numbers such that $\beta_n \to 0$ and $(x_n/\beta_n) \in \ell_p^u(X)$. Recall the operators $E_x, E_{x/\beta} : \ell_{p^*} \to X$ defined in the proof of Proposition 2.2, and the diagonal operator $D_\beta : \ell_{p^*} \to \ell_{p^*}$ via (β_n). We see that D_β is a compact operator. Then there exists a null sequence (z_n) in ℓ_{p^*} such that $D_\beta(B_{\ell_{p^*}}) \subset \overline{\text{co}}(\{z_n\})$. Hence we have

$$K \subset p\text{-co}(\{x_n\}) = E_x(B_{\ell_{p^*}}) = E_{x/\beta} D_\beta(B_{\ell_{p^*}}) \subset \overline{\text{co}}(\{E_{x/\beta} z_n\})$$

and, by Theorem 1.2, $(E_{x/\beta} z_n) \in c_{0,up}(X)$ because $(E_{x/\beta} z_n)$ is a null sequence in X and the set $\{E_{x/\beta} z_n\}$ is relatively u-p-compact. ∎

We can also prove [PD, Theorem 2.5] using [O, Theorem 4.3].

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Ju Myung Kim
Department of Mathematical Sciences
Seoul National University
Seoul, 151-747, Korea
E-mail: kjm21@kaist.ac.kr

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