Unconditionally $p$-null sequences and unconditionally $p$-compact operators

by

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Abstract. We investigate sequences and operators via the unconditionally $p$-summable sequences. We characterize the unconditionally $p$-null sequences in terms of a certain tensor product and then prove that, for every $1 \leq p < \infty$, a subset of a Banach space is relatively unconditionally $p$-compact if and only if it is contained in the closed convex hull of an unconditionally $p$-null sequence.

1. Introduction and main results. Grothendieck [G] showed that a subset $K$ of a Banach space $X$ is relatively compact if and only if there exists a null sequence $(x_n)$ in $X$ such that

$$K \subset \left\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_1} \right\},$$

where we denote by $B_Z$ the unit ball of a Banach space $Z$. The notion of $p$-compactness of Sinha and Karn [SK] stems from this criterion. For $1 \leq p \leq \infty$, a subset $K$ of $X$ is called relatively $p$-compact if there exists $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ if $p = \infty$) such that

$$K \subset p\text{-co}(\{x_n\}) := \left\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_p^*} \right\},$$

where $1/p + 1/p^* = 1$ and $\ell_p(X)$ (resp. $c_0(X)$) is the Banach space with the norm $\| \cdot \|_p$ (resp. $\| \cdot \|_\infty$) of all $X$-valued absolutely $p$-summable (resp. null) sequences.

For $1 \leq p \leq \infty$, the closed subspace $\ell_p^u(X)$ of $\ell_p^w(X)$, the Banach space with the norm $\| \cdot \|_p^w$ of all $X$-valued weakly $p$-summable sequences, consists of sequences $(x_n)$ satisfying

$$\|(0, \ldots, 0, x_m, x_{m+1}, \ldots)\|_p^w \to 0$$

as $m \to \infty$. It is well known that $(x_n) \in \ell_1^u(X)$ if and only if $(x_n)$ is an 2010 Mathematics Subject Classification: 46B45, 46B50, 46B28, 47L20.

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unconditionally summable sequence (cf. [R, Example 3.4]). If \((x_n) \in \ell_p^u(X)\),
we call it an unconditionally p-summable sequence. We say that a subset \(K\)
of \(X\) is relatively unconditionally p-compact (u-p-compact) if there exists
\((x_n) \in \ell_p^u(X)\) such that \(K \subseteq \text{p-co}\{x_n\}\). Note that every u-p-compact set
is a compact set.

Piñeiro and Delgado [PD] introduced and studied p-null sequences. For
\(1 \leq p < \infty\), a sequence \((x_n)\) in a Banach space \(X\) is said to be p-null if for
every \(\varepsilon > 0\) there exist \(N \in \mathbb{N}\) and \((z_k) \in \ell_p(X)\) with \(\|(z_k)\|_p \leq \varepsilon\) such that
\(x_n \in \text{p-co}\{z_k\}\) for all \(n \geq N\). The collection of all p-null sequences in \(X\) is
denoted by \(c_0,up(X)\).

In this paper, a sequence is called unconditionally p-null (u-p-null) when
\(\ell_p(X)\) and \(\| \cdot \|_p\) are replaced by \(\ell_p^w(X)\) and \(\| \cdot \|_p^w\). We denote by \(c_0,wp(X)\) the
collection of all u-p-null sequences in \(X\). Note that for every \(1 \leq p < \infty\),
c\(c_0,wp(X) \subset c_0(X)\) and \(c_{0,up}(X) = c_0(X)\). As in [PD], we can analogously
define a norm on \(c_{0,up}(X)\) (see Section 3).

Fourie and Swart [FS2] studied the following norm on the tensor product
\(X \otimes Y\) of Banach spaces \(X\) and \(Y\). Let \(1 \leq p \leq \infty\). For \(u \in X \otimes Y\), define
\[w_p(u) = \inf \left\{ \| (x_j) \|_p^w \| (y_j) \|_p^w : u = \sum_{j=1}^{n} x_j \otimes y_j \right\}.\]
Then \((X \otimes Y, w_p)\) is a normed space and we denote by \(X \hat{\otimes} w_p Y\) its completion.
Recall that a norm on tensor products of Banach spaces is a tensor norm
if it is a finitely generated uniform crossnorm (cf. [R, Section 6.1]). It was
shown in [FS2] that \(w_p\) is a tensor norm. Oja [O] studied p-null sequences in
terms of the Chevet–Saphar tensor product. The following theorem is the

**Theorem 1.1.** Let \(1 \leq p \leq \infty\). The tensor product \(c_0 \hat{\otimes} w_p, X\) is
isometrically isomorphic to \(c_{0,up}(X)\) and for every \(u \in c_0 \hat{\otimes} w_p, X\) there exists
\((x_n) \in c_{0,up}(X)\) such that \(u = \sum_n e_n \otimes x_n\) in \(c_0 \hat{\otimes} w_p, X\).

Piñeiro and Delgado [PD, Proposition 2.6] showed that for \(1 \leq p < \infty\),
a sequence \((x_n)\) is in \(c_{0,p}(X)\) if and only if \((x_n) \in c_0(X)\) and the set
\(\{x_n\}\) is relatively p-compact under an assumption depending on \(p\), and
they asked whether the assumption could be deleted. Oja [O, Theorem 4.3]
gave an affirmative answer to that question. The following is the result of
[O, Theorem 4.3] adapted to u-p-null sequences.

**Theorem 1.2.** Let \((x_n)\) be a sequence in \(X\) and let \(1 \leq p < \infty\). Then
the following statements are equivalent:

(a) \((x_n) \in c_{0,up}(X)\).
(b) \((x_n)\) is null and the set \(\{x_n\}\) is relatively u-p-compact.
(c) \((x_n)\) is weakly null and the set \(\{x_n\}\) is relatively u-p-compact.
It was shown in [PD, Theorem 2.5] that for \(1 \leq p < \infty\), a subset of a Banach space \(X\) is relatively \(p\)-compact if and only if it is contained in the closed convex hull \(\overline{c_0}\{(x_n)\}\) of a \(p\)-null sequence \((x_n)\). For an alternative straightforward proof, see [AO]. For relatively \(u\)-\(p\)-compact sets we obtain the following result, where \(\overline{bco}(A)\) means the closed balanced convex hull of a set \(A\).

**Corollary 1.3.** Let \(K\) be a subset of \(X\) and let \(1 \leq p < \infty\). Then the following statements are equivalent:

(a) \(K\) is relatively \(u\)-\(p\)-compact.
(b) There exists \((x_n) \in c_{0,up}(X)\) such that \(K \subseteq \overline{c_0}\{(x_n)\}\).
(c) There exists \((x_n) \in c_{0,up}(X)\) such that \(K \subseteq \overline{bco}\{(x_n)\}\).

We prove Theorems 1.1 and 1.2 and Corollary 1.3 in Section 3 after studying operators via unconditionally \(p\)-summable sequences.

**2. Unconditionally \(p\)-compact and unconditionally (quasi) \(p\)-nuclear operators.** For \(1 \leq p \leq \infty\), following the definition of a \(p\)-compact operator in [SK], a linear map \(T : X \to Y\) is said to be \(u\)-\(p\)-compact if \(T(B_X)\) is a relatively \(u\)-\(p\)-compact subset of \(Y\). The collection of all \(u\)-\(p\)-compact operators from \(X\) to \(Y\) is denoted by \(K_{up}(X,Y)\) and we define a norm \(u_p\) on \(K_{up}(X,Y)\) by

\[
u_p(T) = \inf \{ \| (y_n) \|_p^w : (y_n) \in \ell_p^w(Y) \text{ and } T(B_X) \subseteq p-co(\{y_n\}) \}.
\]

From Grothendieck’s criterion of compactness, the ideal \([\mathcal{K}, \| \cdot \|]\) of compact operators equipped with the operator norm coincides with \([\mathcal{K}_{u\infty}, u_\infty]\) and we have:

**Theorem 2.1.** For every \(1 \leq p < \infty\), \([K_{up}, u_p]\) is a Banach operator ideal.

The proof of Theorem 2.1 is similar to the one of [PP, Lemma 4] and follows the scheme indicated by Delgado, Piñeiro and Serrano [DPS] for the ideal of \(p\)-compact operators.

Recall that a \(p\)-nuclear operator \(T \in N_p(X,Y)\) from \(X\) to \(Y\), for \(1 \leq p \leq \infty\), is represented as \(T = \sum_n x_n^* \otimes y_n\), where \((x_n^*) \in \ell_p(X^*)\) if \(p = \infty\) and \((y_n) \in \ell_p^w(Y)\), and the \(p\)-nuclear norm \(\nu_p(T)\) equals

\[
\inf \| (x_n^*) \|_p \| (y_n) \|_p^w,\text{ where the infimum is taken over all such representations of } T \text{ (cf. } [DJT, Proposition 5.23]).\]

When the spaces \(\ell_p(X^*)\) and \(\ell_p^w(Y)\) are replaced by \(\ell_p'(X^*)\) and \(\ell_p'(Y)\) respectively, the map is well known as a classical \(p\)-compact operator (cf. [P, Section 18.3] and [FS1, FS2]). To avoid confusion, in this paper, we call it an unconditionally \(p\)-nuclear (\(u\)-\(p\)-nuclear) operator. The collection of all \(u\)-\(p\)-nuclear operators from \(X\) to \(Y\) is denoted by \(N_{up}(X,Y)\) and the \(u\)-\(p\)-nuclear norm \(\nu_{up}\) is defined by
\[ \nu_{up}(T) = \inf \| (x_n^*)_p \|_p \| (y_n)_p \|_p, \] where the infimum is taken over all such representations of \( T \). It is well known that \([\mathcal{N}_{up}, \nu_{up}]\) is a Banach operator ideal (cf. [FSI Theorems 2.1 and 2.5]).

**Proposition 2.2.** \( \mathcal{K}_{u2} = \mathcal{N}_{u2} \).

**Proof.** Clearly \( \mathcal{N}_{u2} \subset \mathcal{K}_{u2} \) (in fact, \( \mathcal{N}_{up} \subset \mathcal{K}_{up} \) when \( 1 \leq p \leq \infty \)). To show the other inclusion, let \( T : X \to Y \) be a \( u \)-2-compact operator. Then there exists \( (y_n) \in \ell_2^u(Y) \) such that \( T(B_X) \subset \{ \sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_2} \} \). Define operators \( E_y : \ell_2 \to Y \) by \( E_y \alpha = \sum_n \alpha_n y_n \), and \( \hat{E}_y : \ell_2/\ker(E_y) \to Y \) by \( \hat{E}_y[\alpha] = E_y \alpha \). Then \( \hat{E}_y \) is a compact operator. In view of the factorization in [SK Section 3], \( T = \hat{E}_y T_y \), where \( T_y : X \to \ell_2/\ker(E_y) \) is a bounded operator. According to [FSI Theorem 2.3], \( T \) is a \( u \)-2-nuclear operator.

For \( 1 \leq p \leq \infty \), following the definition of a quasi \( p \)-nuclear operator in [PP], a linear map \( T : X \to Y \) is called quasi unconditionally \( p \)-nuclear (quasi \( u \)-\( p \)-nuclear) if there exists \( (x_n^*) \in \ell_p^u(X^*) \) such that \( \| Tx \| \leq \| (x_n^*(x)) \|_p \) for every \( x \in X \). We denote by \( \mathcal{N}_{up}^Q(X,Y) \) the collection of all quasi \( u \)-\( p \)-nuclear operators from \( X \) to \( Y \). For \( T \in \mathcal{N}_{up}^Q(X,Y) \), let \( \nu_{up}^Q(T) = \inf \| (x_n^*) \|_p^w \) where the infimum is taken over all such inequalities. Note that a quasi \( u \)-\( \infty \)-nuclear operator is just a compact operator (cf. [D Exercise II.6(ii)]).

We can also use the proof of [PP Lemma 4] to show that \([\mathcal{N}_{up}, \nu_{up}^Q] \) is a Banach operator ideal.

We now obtain the duality of \( u \)-\( p \)-compact operators, which is the analogue of the duality of \( p \)-compact operators from [DPS]. In fact, Theorem 2.3 and the “only if” part of Theorem 2.4 are essentially due to [DPS].

**Theorem 2.3.** Let \( 1 \leq p \leq \infty \) and let \( T : X \to Y \) be a linear map. Then \( T \in \mathcal{N}_{up}^Q(X,Y) \) if and only if \( T^* \in \mathcal{K}_{up}(Y^*,X^*) \). In this case, \( \nu_{up}^Q(T) = u_p(T^*) \).

**Proof.** This is immediate from [DPS Proposition 3.2].

**Theorem 2.4.** Let \( 1 \leq p \leq \infty \) and let \( T : X \to Y \) be a linear map. Then \( T \in \mathcal{K}_{up}(X,Y) \) if and only if \( T^* \in \mathcal{N}_{up}^Q(Y^*,X^*) \). In this case, \( \nu_{up}^Q(T^*) \leq u_p(T) \).

**Proof of the “only if” part of Theorem 2.4.** Let \( T \in \mathcal{K}_{up}(X,Y) \) and let \( (y_n) \in \ell_2^u(Y) \) be such that \( T(B_X) \subset p\co(\{y_n\}) \). Then by [DPS Proposition 3.1],

\[ \| T^* y^* \| \leq \| (i_Y(y_n)(y^*)) \|_p \]

for every \( y^* \in Y^* \). Note that \( (i_Y(y_n)) \in \ell_p^w(Y^{**}) \), where \( i_Y : Y \to Y^{**} \) is the natural isometry, and \( \| (i_Y(y_n)) \|_p^w = \| (y_n) \|_p^w \). Hence \( T^* \in \mathcal{N}_{up}^Q(Y^*,X^*) \) and \( \nu_{up}^Q(T^*) \leq u_p(T) \).
From Theorems 2.3 and 2.4, we have:

**Corollary 2.5.** Let $1 \leq p \leq \infty$ and let $T : X \to Y$ be a linear map. Then $T \in \mathcal{K}_{up}(X,Y)$ (resp. $\mathcal{N}_{up}^{\mathbb{Q}}(X,Y)$) if and only if $T^{**} \in \mathcal{K}_{up}(X^{**},Y^{**})$ (resp. $\mathcal{N}_{up}^{\mathbb{Q}}(X^{**},Y^{**})$). In this case, $u_p(T^{**}) \leq u_p(T)$ (resp. $\nu_{up}^{\mathbb{Q}}(T^{**}) \leq \nu_{up}^{\mathbb{Q}}(T)$).

In order to prove the “if” part of Theorem 2.4, we also use an argument from [DPS]. By definition we see that $\mathcal{N}_{up}(X,Y) \subseteq \mathcal{N}_{up}^{\mathbb{Q}}(X,Y)$ and $\nu_{up}^{\mathbb{Q}}(T) \leq \nu_{up}(T)$ for every $T \in \mathcal{N}_{up}(X,Y)$, and we have:

**Lemma 2.6.** Let $1 \leq p \leq \infty$. Suppose that $Y$ is injective. If $T \in \mathcal{N}_{up}^{\mathbb{Q}}(X,Y)$, then $T \in \mathcal{N}_{up}(X,Y)$ and $\nu_{up}^{\mathbb{Q}}(T) = \nu_{up}(T)$.

**Proof.** This proof is essentially due to [PP]. Let $T \in \mathcal{N}_{up}^{\mathbb{Q}}(X,Y)$. Let $\epsilon > 0$ be given. Then there exists $(x_n^*) \in \ell^u_p(X^*)$ such that for every $x \in X$, $\|Tx\| \leq \|(x_n^*)(x)\|_p$ and $\|(x_n^*)\|_p^{\ell^w_p} \leq \nu_{up}^{\mathbb{Q}}(T) + \epsilon$. Consider the linear subspace $Z = \{(x_n^*)(x) : x \in X\}$ of $\ell^p$ (or of $c_0$ if $p = \infty$) and the map $J : Z \to Y$, $(x_n^*)(x) \mapsto Tx$. Then $J$ is well defined and linear, and $\|J\| \leq 1$. Since $Y$ is injective, there exists an extension $\hat{J} : \ell^p \to Y$ of $J$ with $\|\hat{J}\| = \|J\|$. Define a map $U : X \to \ell^p$ by $Ux = (x_n^*)(x))$. Then $U$ is a compact operator and the following diagram is commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{U} & & \downarrow{\hat{J}} \\
\ell^p & & \\
\end{array}
$$

Hence by [FS1] Theorem 2.5], $T \in \mathcal{N}_{up}(X,Y)$ and $\nu_{up}(T) \leq \|U\| \|\hat{J}\| \leq \|(x_n^*)\|_p^w \leq \nu_{up}^{\mathbb{Q}}(T) + \epsilon$, and so $\nu_{up}^{\mathbb{Q}}(T) = \nu_{up}(T)$. ■

Let $K$ be a bounded subset of $X$. In [DPS], the authors defined the operators $u_K : \ell^1(K) \to X$ and $j_K : X^* \to \ell^\infty(K)$, respectively, by $u_K(\xi_x)_{x \in K} = \sum_{x \in K} \xi_x x$ and $j_K x^* = (x^*(x))_{x \in K}$. We see that $u_K^* = j_K$.

We now obtain the versions for $u$-p-compactness of [DPS] Proposition 3.5, Corollary 3.6, Remark 3.7].

**Proposition 2.7.** Let $1 \leq p \leq \infty$ and let $K$ be a bounded subset of $X$. Then the following statements are equivalent:

(a) $K$ is relatively $u$-$p$-compact.

(b) $u_K$ is $u$-$p$-compact.

(c) $j_K$ is $u$-$p$-nuclear.

**Proof.** (a)⇒(b). Let $(x_n) \in \ell^u_p(X)$ be such that $K \subseteq p$-$\text{co}(\{x_n\})$. Then $u_K(B_{\ell^1(K)}) \subseteq \text{bco}(K) \subseteq p$-$\text{co}(\{x_n\})$. Hence $u_K$ is $u$-$p$-compact.
(b) $\Rightarrow$ (a). Since $K \subset u_K(B_{\ell_1(K)})$, this is clear.

(b) $\Rightarrow$ (c). If $u_K$ is $u$-p-compact, then by Theorem 2.4($\Rightarrow$), $u_K^*$ is quasi $u$-p-nuclear. Since $\ell_\infty(K)$ is injective, (c) follows from Lemma 2.6.

(c) $\Rightarrow$ (b). If $u_K^* = j_K$ is $u$-p-nuclear, then by [DFS Proposition 1.5.7], $u_K$ is $u$-p$^*$-nuclear. Hence $u_K$ is $u$-p-compact, because, clearly, $N_{u_K} \subset K_{u_K}$. ■

The following is a duality version of Proposition 2.7.

**Proposition 2.8.** Let $1 \leq p \leq \infty$ and let $C$ be a bounded subset of $X^*$. Then the following statements are equivalent:

(a) $C$ is relatively $u$-p-compact.

(b) The map $u_C : \ell_1(C) \to X^*$ defined by $u_C((\xi_{x^*})_{x^*} \in C) = \sum_{x^*} \in C \xi_{x^*} x^*$ is a $u$-p-compact operator.

(c) The map $j_C : X \to \ell_\infty(C)$ defined by $j_C x = (x^*(x))_{x^*} \in C$ is a $u$-p-nuclear operator.

*Proof.* Use the duality relationships $u_C^* i_X = j_C$ and $j_C^* i_{\ell_1(C)} = u_C$, and Theorems 2.3 and 2.4($\Rightarrow$). ■

**Corollary 2.9.** Let $1 \leq p \leq \infty$ and let $K$ be a subset of $X$. If $i_X(K)$ is a relatively $u$-p-compact subset of $X^{**}$, then $K$ is a relatively $u$-p-compact subset of $X$.

*Proof.* If $i_X(K)$ is a relatively $u$-p-compact subset of $X^{**}$, then by Proposition 2.8, the operator $j_K : X^* \to \ell_\infty(i_X(K))$, which is actually the operator $j_K : X^* \to \ell_\infty(K)$ in Proposition 2.7, is $u$-p-nuclear. Hence $K$ is relatively $u$-p-compact. ■

We now complete the proof of Theorem 2.4.

*Proof of the “if” part of Theorem 2.4.* If $T^* \in N_{u_K}^Q(Y^*, X^*)$, then by Theorem 2.3, $T^{**} \in K_{u_K}(X^{**}, Y^{**})$. Thus $i_Y T(B_X) = T^{**} i_X(B_X)$ is a relatively $u$-p-compact subset of $Y^{**}$. Hence by Corollary 2.9, $T(B_X)$ is a relatively $u$-p-compact subset of $Y$ and so $T \in K_{u_K}(X, Y)$. ■

### 3. Proofs of main results.

For a bounded sequence $\hat{x} := (x_n)$ in $X$, define an operator $u_{\hat{x}} : \ell_1 \to X$ by $u_{\hat{x}}(\alpha_n) = \sum_n \alpha_n x_n$. Then for $1 \leq p \leq \infty$, by Proposition 2.7, the set $\{x_n\}$ is relatively $u$-p-compact if and only if the operator $u_{\hat{x}}$ is $u$-p-compact. As indicated for $p$-null sequences [PD Remark 2.2], a simple verification shows that a sequence $(x_n)$ is $u$-p-null if and only if $\{x_n\}$ is relatively $u$-p-compact and $u_p(u_{\hat{x}}(n) - u_{\hat{x}}) \to 0$ as $n \to \infty$, where $\hat{x}(n) := (x_1, \ldots, x_n, 0, \ldots)$.

Let $1 \leq p \leq \infty$. We define a norm on $c_{0,u_p}(X)$ by $\|(x_n)\|_{u_p}^P = u_p(u_{\hat{x}})$ for $(x_n) \in c_{0,u_p}(X)$. Then $(c_{0,u_p}(X), \|\cdot\|_{u_p}^P) = (c_0(X), \|\cdot\|_{\infty})$ and we have the following result whose proof is straightforward.
PROPOSITION 3.1. Let $1 \leq p < \infty$. Then $(c_{0,up}(X), \| \cdot \|_{up}^0)$ is a Banach space.

We need the following result to prove Theorem 1.1. Its prototype is [DPS Proposition 3.11].

LEMMA 3.2. Let $1 \leq p \leq \infty$ and let $T : Y \to X$ be a linear map. Then $T \in K_{up}(Y,X)$ if and only if $Tu_{B_Y} \in N_{up}^Q(\ell_1(B_Y), X)$. In this case, $u_p(T) = \nu_{up}^*(Tu_{B_Y})$.

Proof. If $T \in K_{up}(Y,X)$, then by Theorem 2.4, $T^* \in N_{up}^Q(X^*, Y^*)$ and $\nu_{up}^*(T^*) \leq u_p(T)$. It follows from Lemma 2.6 that

$$(Tu_{B_Y})^* = j_{B_Y}T^* \in N_{up}(X^*, \ell_\infty(B_Y)).$$

Hence by [DFS Proposition 1.5.7], $Tu_{B_Y} \in N_{up}(\ell_1(B_Y), X)$ and $\nu_{up}^*(Tu_{B_Y}) = \nu_{up}(j_{B_Y}T^*) = \nu_{up}^Q(j_{B_Y}T^*) \leq u_p(T)$.

To show the converse, let $Tu_{B_Y} = \sum_n \langle \zeta^n_y \rangle y_n \otimes x_n \in N_{up}(\ell_1(B_Y), X)$. Then $T(Y) = \{ \sum_n \zeta^n_y x_n : y \in B_Y \}$ is a representation, where $((\zeta^n_y)_y) \in \ell^u_p(\ell_\infty(B_Y))$ and $(x_n) \in \ell^u_p(X)$. Then

$$T(B_Y) = \{ \sum_n \zeta^n_y x_n : y \in B_Y \} \subseteq p_{co}(\{ \|((\zeta^n_y)_y)_k\|_p^w, x_n\}, \|x_n\|_p^w).$$

Hence $T \in K_{up}(Y,X)$ and $u_p(T) \leq \|((\zeta^n_y)_y)_k\|_p^w, \|x_n\|_p^w$. Since the representation was arbitrary, $u_p(T) \leq \nu_{up}^*(Tu_{B_Y})$. 

Since for every operator $T : \ell_1 \to X$, $T$ coincides with $Tu_{B_{\ell_1}}i$, where the map $i : \ell_1 \to \ell_1(B_{\ell_1})$ is the canonical isometry, by Lemma 3.2 we have:

COROLLARY 3.3. Let $1 \leq p \leq \infty$ and let $T : \ell_1 \to X$ be a linear map. Then $T \in K_{up}(\ell_1, X)$ if and only if $T \in N_{up}(\ell_1, X)$. In this case, $u_p(T) = \nu_{up}^*(T)$. 

We also need a result of Fourie and Swart [FS2] to prove Theorem 1.1.

LEMMA 3.4 ([FS2 Proposition 3.2]). Let $1 \leq p \leq \infty$. If $(x_n) \in \ell^u_p(X)$ and $(y_n) \in \ell^u_p(Y)$, then $\sum_n x_n \otimes y_n$ converges in $X \hat{\otimes}_{wp} Y$. Conversely, if $u \in X \hat{\otimes}_{wp} Y$, then there exist $(x_n) \in \ell^u_p(X)$ and $(y_n) \in \ell^u_p(Y)$ such that $\sum_n x_n \otimes y_n$ converges to $u$. Moreover,

$$w_p(u) = \inf \{ \|x_n\|_p^w, \|y_n\|_p^w : u = \sum_{n=1}^\infty x_n \otimes y_n, (x_n) \in \ell^u_p(X), (y_n) \in \ell^u_p(Y) \}.$$ 

Proof of Theorem 1.1. In order to show the first part, consider the linear map $J : (c_{0} \otimes X, w^*) \to c_{0,up}(X)$ defined by

$$J\left( \sum_{j=1}^n (\lambda_i^j) \otimes x_j \right) = \left( \sum_{j=1}^n \lambda_i^j x_j \right)_i.$$
First, one may check that \( J(c_0 \otimes X) \subset c_{0,up}(X) \) using elementary tensors. Since, for every \((x_n) \in c_{0,up}(X)\) and \(m \in \mathbb{N}\), \( \hat{x}(m) = J(\sum_{j \leq m} e_j \otimes x_j) \) and 
\[
\lim_{m \to \infty} \|\hat{x}(m) - \hat{x}\|_{up}^0 = \lim_{m \to \infty} u_p(u(\hat{x}(m)) - u(\hat{x})) = 0,
\]
\( J(c_0 \otimes X) \) is dense in \( c_{0,up}(X) \).

To show that the map \( J : (c_0 \otimes X, w_p^*) \to c_{0,up}(X) \) is an isometry, let 
\[
T = \sum_{j \leq n} (\lambda^j_i)_i \otimes x_j \in c_0 \otimes X \text{ and let } (z_i) := (\sum_{j \leq n} \lambda^j_i x_j)_i.
\]
Then \( T = u\hat{z} \). From [DFS, Proposition 1.5.5] and Corollary 3.3, it follows that 
\[
\| (z_i) \|_{up}^0 = u_p(u(\hat{x})) = u_p(T) = \nu_{up}^*(T) = w_p^* \left( \sum_{j \leq n} (\lambda^j_i)_i \otimes x_j \right).
\]
Since \( c_{0,up}(X) \) is a Banach space, the extension \( \check{J} : c_0 \hat{\otimes}_{w_p^*} X \to c_{0,up}(X) \) of \( J \) is a surjective linear isometry.

In order to show the second part, let \( u \in c_0 \hat{\otimes}_{w_p^*} X \). Then by Lemma 3.4 there exists \((\lambda^j_i)_i_j \in \ell^w_p(c_0)\) and \((z_j) \in \ell^w_p(X)\) such that \( u = \sum_{i=1}^{\infty} (\lambda^j_i)_i \otimes z_j \) in \( c_0 \hat{\otimes}_{w_p^*} X \). For every \( i \), let 
\[
x_i := \sum_{j=1}^{\infty} \lambda^j_i z_j.
\]
We show that \((x_i)\) is the desired sequence. Let \( \varepsilon > 0 \) be given. Since \((\lambda^j_i)_i_j \in \ell^w_p(c_0)\), it is easily seen that there exists an \( N \in \mathbb{N} \) such that 
\[
\sup_{i \geq N} \| (\lambda^j_i)_j \|_{p^*} \| (z_j) \|_p^w \leq \varepsilon.
\]
Let \( C := \sup_{i \geq N} \| (\lambda^j_i)_j \|_{p^*} \). We may assume that \( C \neq 0 \). Then \( i \geq N \) implies that 
\[
x_i = \sum_{j=1}^{\infty} \lambda^j_i \frac{C}{C} C z_j \subset p-co(\{Cz_j\}).
\]
Since \( \| (Cz_j) \|_p^w \leq \varepsilon, (x_i) \in c_{0,up}(X) \). Recall the isometry \( \check{J} : c_0 \hat{\otimes}_{w_p^*} X \to c_{0,up}(X) \). Let \( \check{J}(u) := (u_i) \). Since \( \lim_{n \to \infty} (\sum_{j \leq n} \lambda^j_i z_j)_i = \check{J}(u) \) in \( c_{0,up}(X) \), \( \lim_{n \to \infty} \sum_{j \leq n} \lambda^j_i z_j = u_i \) in \( X \) for every \( i \). Hence \( (x_i) = (u_i) = \check{J}(u) \) and so 
\[
u = \check{J}^{-1}((x_i)) = \lim_{m \to \infty} \check{J}^{-1}(\hat{x}(m)) = \lim_{m \to \infty} \sum_{i \leq m} e_i \otimes x_i. \]
We need the main theorem in [O] to prove Theorem 1.2.

**Lemma 3.5 ([O] Theorem 2.4).** Let \( \alpha \) be a tensor norm. Assume that \( X^{***} \) or \( Y \) has the approximation property. If \( T \in N_\alpha(X^*, Y) \) and \( T^*(Y^*) \subset i_X(X) \), then \( T \in X \otimes Y \in N_\alpha(X^*, Y) \).
Corollary 3.6. Let $1 \leq p \leq \infty$. Assume that $X^{**}$ or $Y$ has the approximation property. If $T \in \mathcal{N}_{up}(X^*, Y)$ and $T^*(Y^*) \subset i_X(X)$, then $T = \sum_n x_n \otimes y_n$ in $\mathcal{N}_{up}(X^*, Y)$, where $(x_n) \in \ell^u_p(X)$ and $(y_n) \in \ell^u_p(Y)$.

Proof. From Lemma 3.5, [DFS, Corollary 1.4.9 and Proposition 1.5.5], it follows that
$$T \in \overline{X \otimes Y}^{\mathcal{N}_{up}(X^*, Y)} = \overline{X \otimes Y}^{X^{**} \hat{\otimes} w_p Y} = X \hat{\otimes} w_p Y.$$ Hence from Lemma 3.4 we obtain the conclusion. ■

Proof of Theorem 1.2. (a)⇒(b) and (b)⇒(c) are obvious.

(c)⇒(a). This proof is essentially due to [O, Theorem 4.3]. If $(x_n)$ is relatively $u$-$p$-compact, then $u_x \in K_{up}(\ell_1, X)$. By Corollary 3.3, $u_x \in \mathcal{N}_{up^*}(\ell_1, X)$ and $u_p(u_x) = \nu_{up^*}(u_x)$.

Since $(x_n)$ is weakly null, we see that $u_x^*(X^*) \subset c_0$. Since $c_0^{**}$ has the approximation property, it follows from Corollary 3.6 that $u_x \in c_0 \hat{\otimes} w_{p^*}, X$. Hence by Theorem 1.1 there exists $(z_n) \subset c_0, u_{p^*}(X)$ such that $u_x = \sum_n e_n \otimes z_n$ in $c_0 \hat{\otimes} w_{p^*}, X$ and so $z_k = x_k$ for every $k$, which completes the proof. ■

Remark 3.7. We can also prove Theorem 1.2 using the argument of Lassalle and Turco [LT] based on Carl–Stephani theory [CS].

Proof of Corollary 1.3. (b)⇒(c) is trivial.

(c)⇒(a). If $(x_n) \subset c_{0, up}(X)$, then the set $\{x_n\}$ is relatively $u$-$p$-compact. Thus there exists $(z_n) \subset \ell^u_p(X)$ such that $\{x_n\} \subset p$-$\text{co}(\{z_n\})$. By (c) we have $K \subset p$-$\text{co}(\{z_n\})$, hence the assertion (a) follows.

(a)⇒(b). Since $K$ is relatively $u$-$p$-compact, there exists $(x_n) \subset \ell^u_p(X)$ such that $K \subset p$-$\text{co}(\{x_n\})$. By a standard argument we can find a sequence $(\beta_n)$ of positive numbers such that $\beta_n \to 0$ and $(x_n/\beta_n) \subset \ell^u_p(X)$. Recall the operators $E_x, E_{x/\beta} : \ell_{p^*} \to X$ defined in the proof of Proposition 2.2, and the diagonal operator $D_\beta : \ell_{p^*} \to \ell_{p^*}$ via $(\beta_n)$. We see that $D_\beta$ is a compact operator. Then there exists a null sequence $(z_n) \subset \ell_{p^*}$, such that $D_\beta(B_{\ell_{p^*}}) \subset \overline{\mathfrak{c}}(\{z_n\})$. Hence we have
$$K \subset p$-$\text{co}(\{x_n\}) = E_x(B_{\ell_{p^*}}) = E_{x/\beta}D_\beta(B_{\ell_{p^*}}) \subset \overline{\mathfrak{c}}(\{E_{x/\beta}z_n\})$$ and, by Theorem 1.2, $(E_{x/\beta}z_n) \subset c_{0, up}(X)$ because $(E_{x/\beta}z_n)$ is a null sequence in $X$ and the set $\{E_{x/\beta}z_n\}$ is relatively $u$-$p$-compact. ■

We can also prove [PD, Theorem 2.5] using [O, Theorem 4.3].

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