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Sharp endpoint results for imaginary powers and Riesz transforms on certain noncompact manifolds

by

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Abstract. We consider a complete connected noncompact Riemannian manifold M with bounded geometry and spectral gap. We prove that the imaginary powers of the Laplacian and the Riesz transform are bounded from the Hardy space $X^1(M)$, introduced in previous work of the authors, to $L^1(M)$.

1. Introduction. Denote by M a complete connected noncompact Riemannian manifold of dimension n with Ricci curvature bounded from below, positive injectivity radius and spectral gap. Denote by $\mathscr{L}$ (minus) the Laplace–Beltrami operator on M and by $X^k(M)$ the Hardy-type spaces introduced in [23, 24] (see Definitions 2.1 and 2.2 below). These spaces play for harmonic analysis on M much the same role as the classical Hardy space $H^1(\mathbb{R}^n)$ plays for harmonic analysis on $\mathbb{R}^n$ [23, 24] (see also [25] for the theory of the duals of these spaces). In particular, in [23, 24] we proved that the operators $\mathscr{L}^{iu}$ and $\nabla\mathscr{L}^{-1/2}$ are bounded from $X^k(M)$ to $L^1(M)$ for an integer k large enough and depending on n.

The purpose of this paper is to prove the following result.

Theorem 1.1. *For every u in $\mathbb{R}$ the operators $\mathscr{L}^{iu}$ and $\nabla\mathscr{L}^{-1/2}$ are bounded from $X^1(M)$ to $L^1(M)$.*

Clearly Theorem 1.1 is an improvement of the aforementioned results. We believe that its main interest lies not only in the fact that all these operators are bounded from the same space $X^1(M)$ to $L^1(M)$, but also in the method of proof, which appears to be quite adaptable to the geometry of manifolds and could pave the way to obtaining similar results for more general manifolds and different operators. In particular, we wish to empha-

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sise the role of Takeda's inequality in the proof of the boundedness of the imaginary powers of $\mathscr{L}$, and of parabolic Caccioppoli and Harnack type estimates in the proof of the boundedness of the Riesz transform. Furthermore, an important role in the proof of our main result is played by a local Faber–Krahn type estimate.

The imaginary powers of $\mathscr{L}$ and the Riesz transforms on Riemannian manifolds have been investigated in a number of papers [1–4, 6–9, 18, 20–24, 26–28]. For a discussion of these papers and their relations to our results we refer the reader to the introductions of [23, 24].

We also mention that there is a large literature on Hardy spaces associated with particular classes of operators (see for instance [10–14, 19] and the bibliography in [18]).

We now give a brief outline of the paper. In Section 2 we recall the definition and the basic properties of the atomic Hardy space $X^1(M)$. In Section 3 we estimate the L^2 norm of the resolvent of the Laplacian $\mathscr{L}$ on atoms. In Section 4 we prove the boundedness of the imaginary powers of $\mathscr{L}$ and in Section 5 that of the Riesz transform $\nabla\mathscr{L}^{-1/2}$.

We shall use the "variable constant convention", and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards.

2. Background on Hardy-type spaces. Let M denote a connected, complete n-dimensional Riemannian manifold of infinite volume with Riemannian measure μ. Denote by Ric the Ricci tensor, by $-\mathscr{L}$ the Laplace–Beltrami operator on M, by b the bottom of the $L^2(M)$ spectrum of $\mathscr{L}$, and set $\beta = \limsup_{r\to\infty}[\log \mu(B(o,r))]/(2r)$, where o is any reference point of M. By a result of Brooks [5], $b \le \beta^2$.

We denote by $\mathscr{B}$ the family of all geodesic balls on M. For each B in $\mathscr{B}$ we denote by c_B and r_B the centre and the radius of B respectively. Furthermore, we denote by cB the ball with centre c_B and radius cr_B. For each scale parameter s in $\mathbb{R}^+$, we denote by $\mathscr{B}_s$ the family of all balls B in $\mathscr{B}$ such that $r_B \le s$.

Standing Assumptions. We assume that the injectivity radius of M is positive, that the Ricci tensor is bounded from below and that M has spectral gap, to wit $b > 0$.

It is well known that for manifolds satisfying the assumptions above there are positive constants α, β and C such that

$$\mu(B) \le C r_B^{\alpha} e^{2\beta r_B} \quad \forall B \in \mathscr{B} \text{ such that } r_B \ge 1. \tag{2.1}$$

Moreover, the measure μ is *locally doubling*, i.e. for every $s > 0$ there exists a constant D_s such that

$$\mu(2B) \leq D_s \mu(B) \quad \forall B \in \mathscr{B}_s.$$

Furthermore (see [24, Remark 2.3]) there exists a positive constant C such that

$$C^{-1} r_B^n \leq \mu(B) \leq C r_B^n \quad \forall B \in \mathscr{B}_1. \tag{2.2}$$

In this section we gather some known facts about the Hardy-type space $X^1(M)$, introduced in [23] and studied in [24, 25]. For each open ball B, we denote by

(i) $h^2(B)$ the space of all $\mathscr{L}$-harmonic functions in $L^2(B)$;
(ii) $q^2(B)$ the space all functions $u \in L^2(B)$ such that $\mathscr{L}u$ is constant on B.

We say that a function u is in $h^2(\overline{B})$ (respectively $q^2(\overline{B})$) if u is the restriction to $\overline{B}$ of a function in $h^2(B')$ (respectively $q^2(B')$) for some open ball B' containing B.

We shall refer to $h^2(B)$ as the *harmonic Bergman space* on B, while functions in $q^2(\overline{B})$ are referred to as *quasi-harmonic functions on* $\overline{B}$. Often we think of $q^2(\overline{B})$ as a subspace of $L^2(B)$. When we do, $q^2(\overline{B})^\perp$ will denote the orthogonal complement of $q^2(\overline{B})$ in $L^2(B)$. Clearly $q^2(B)^\perp$ is a subspace of $q^2(\overline{B})^\perp$ and of $h^2(B)^\perp$.

Definition 2.1. An X^1-*atom* associated to the geodesic ball B is a function A in $L^2(M)$, supported in B, such that

(i) $\int Av \, d\mu = 0$ for all $v \in q^2(\overline{B})$;
(ii) $\|A\|_2 \leq \mu(B)^{-1/2}$.

Note that condition (i) implies that $\int_M A \, d\mu = 0$, because $\mathbf{1}_{2B}$ is in $q^2(\overline{B})$. Given a positive "scale parameter" s, we say that an X^k-atom is *at scale* s if it is supported in a ball B of $\mathscr{B}_s$.

Definition 2.2. Choose a "scale parameter" $s > 0$. The *Hardy-type space* $X^1(M)$ is the space of all functions F that admit a decomposition of the form $F = \sum_j c_j A_j$, where $\{c_j\}$ is a sequence in ℓ^1 and $\{A_j\}$ is a sequence of X^1-atoms at scale s. We endow $X^1(M)$ with the natural "atomic norm"

$$\|F\|_{X^1} := \Big\{ \sum_{j=1}^{\infty} |c_j| : F = \sum_{j=1}^{\infty} c_j A_j, \; A_j \; X^1\text{-atoms at scale } s \Big\}.$$

Remark 2.3. It is known [23, 24] that all these atomic norms are equivalent and it becomes a matter of convenience to choose one or another. In our situation any value $< \operatorname{Inj}(M)$ of the scale parameter s would be a convenient choice for the following reasons. Balls of radius $< \operatorname{Inj}(M)$ have no holes and their boundaries are smooth, so that various results concerning Sobolev spaces on balls hold. We shall, implicitly or explicitly, make use of them in what follows. Another advantage of choosing $s < \operatorname{Inj}(M)$ is that we can make use of the fact that the cancellation condition (i) in Definition 2.1 may then be equivalently formulated by requiring that A be in $q^2(B)^\perp$ [25, Proposition 3.5 and the comments after Theorem 4.12]. This will be used below without any further comment. In the following, we shall choose $s_0 = \frac{1}{2}\operatorname{Inj}(M)$ and we shall call atoms at scale s_0 *admissible*.

Another result that plays a crucial role in the proof of the $X^1(M)$-$L^1(M)$-boundedness of imaginary powers and Riesz transforms is the following theorem proved in [25, Corollary 6.2 and Proposition 6.3].

THEOREM 2.4. *If $\mathscr{T}$ is a bounded linear operator on $L^2(M)$ such that*

$$\sup\{\|\mathscr{T}A\|_1 : A \text{ an admissible } X^1\text{-atom}\} < \infty,$$

then $\mathscr{T}$ extends to a bounded linear operator from $X^1(M)$ to $L^1(M)$ that agrees with $\mathscr{T}$ on $X^1(M) \cap L^2(M)$.

3. Atoms and the Laplace–Beltrami operator. Henceforth we denote by $\mathscr{L}$ the unique self-adjoint extension of minus the Laplace–Beltrami operator on $L^2(M)$. We recall that the domain of $\mathscr{L}$ is the space of all functions in $L^2(M)$ such that the distribution $\mathscr{L}u$ is in $L^2(M)$. For a geodesic ball B we denote by $\mathscr{L}_B$ the restriction of $\mathscr{L}$ to the subspace

$$\operatorname{Dom}(\mathscr{L}_B) = \{f \in \operatorname{Dom}(\mathscr{L}) : \operatorname{supp}(f) \subset \overline{B}\}.$$

Even though the operator $\mathscr{L}_B$ is defined on $L^2(M)$, in the following we shall often consider it as acting on $L^2(B)$. In addition to $\mathscr{L}_B$, we also consider the Dirichlet Laplacian $\mathscr{L}_{B,\mathrm{Dir}}$ on the ball B, i.e. the Friedrichs extension of the restriction of $\mathscr{L}$ to $C_c^\infty(B)$. We recall that the domain of $\mathscr{L}_{B,\mathrm{Dir}}$ is

$$\operatorname{Dom}(\mathscr{L}_{B,\mathrm{Dir}}) = \{u \in W_0^{1,2}(B) : \mathscr{L}u \in L^2(B)\},$$

where $\mathscr{L}u$ is interpreted in the sense of distributions on B and $W_0^{1,2}(B)$ denotes the closure of $C_c^\infty(B)$ in the Sobolev space

$$W^{1,2}(B) = \{u \in L^2(B) : |\nabla u| \in L^2(B)\}.$$

We shall restrict our attention to balls B which are the interior of their closure and ∂B is smooth. Observe that any ball B of radius $< \operatorname{Inj}(M)$ is the interior of its closure and has smooth boundary. The following proposition will be useful later.

PROPOSITION 3.1. *Assume that B is a ball in M with smooth boundary. The following hold:*

(i) $\mathscr{L}_{B,\mathrm{Dir}}$ *is an extension of* $\mathscr{L}_B$;
(ii) $\operatorname{Ran}(\mathscr{L}_B) = h^2(B)^\perp$ *and* $\mathscr{L}_B$ *is an isomorphism between its domain, endowed with the graph norm, and its range;*

(iii)
$$\|\mathscr{L}^{-1}f\|_2 \leq \frac{1}{\lambda_1(B)}\|f\|_{L^2(B)} \qquad \forall f \in h^2(B)^\perp,$$

where $\lambda_1(B)$ is the first eigenvalue of the Dirichlet Laplacian $\mathscr{L}_{B,\mathrm{Dir}}$.

Proof. If $u \in \mathrm{Dom}(\mathscr{L}_B)$ then $\mathscr{L}u \in L^2(M)$ and $\mathrm{supp}(u) \subset \overline{B}$. Hence, by elliptic regularity, $u, |\nabla u| \in L^2_{\mathrm{loc}}(M)$. Thus $u \in W^{1,2}(B)$. Since $u = 0$ on the complement of $\overline{B}$ and the boundary of B is smooth, the trace of u on the boundary of B is zero. Hence $u \in W^{1,2}_0(B)$ by a classical result. This proves that $\mathrm{Dom}(\mathscr{L}_B) \subset \mathrm{Dom}(\mathscr{L}_{B,\mathrm{Dir}})$. Thus $\mathscr{L}_B \subset \mathscr{L}_{B,\mathrm{Dir}}$ because both operators are defined in the sense of distributions on their domains.

Next we prove (ii). First we observe that, since functions in $\mathrm{Ran}(\mathscr{L}_B)$ are supported in $\overline{B}$, we may identify isometrically $\mathrm{Ran}(\mathscr{L}_B)$ with the subspace of $L^2(B)$ obtained by restricting functions to B. Thus $\mathrm{Ran}(\mathscr{L}_B)$ is closed in $L^2(B)$, since it is closed in $L^2(M)$, because $\mathscr{L}$ is strictly positive and closed. Thus, to prove the inclusion $h^2(B)^\perp \subseteq \mathrm{Ran}(\mathscr{L}_B)$, it suffices to show that $\mathrm{Ran}(\mathscr{L}_B)^\perp \subseteq h^2(B)$. Now, if $g \in L^2(B)$ is orthogonal to $\mathrm{Ran}(\mathscr{L}_B)$, then

$$0 = \int_B \mathscr{L}\psi\, \overline{g}\, d\mu = \langle \psi, \mathscr{L}g\rangle \qquad \forall \psi \in C^\infty_c(B),$$

where $\mathscr{L}g$ is in the sense of distributions on B. Therefore $\mathscr{L}g = 0$ in B, i.e., g is harmonic in B and belongs to $L^2(B)$, i.e., $g \in h^2(B)$.

To prove the opposite inclusion, we observe that by [25, Prop. 3.5],

$$h^2(B) = \overline{h^2(\overline{B})}.$$

Thus, to prove the inclusion $\mathrm{Ran}(\mathscr{L}_B) \subseteq h^2(B)^\perp$ it suffices to show that $\mathrm{Ran}(\mathscr{L}_B)$ is orthogonal to $h^2(\overline{B})$, i.e. $\int_B \mathscr{L}_B f\, \overline{g}\, d\mu = 0$ for all f in $\mathrm{Dom}(\mathscr{L}_B)$ and all g in $h^2(\overline{B})$. Pick $f \in \mathrm{Dom}(\mathscr{L}_B)$, $g \in h^2(\overline{B})$ and denote by $\hat{g}$ an extension of g to all of M, which is in $\mathrm{Dom}(\mathscr{L})$. Since $\mathscr{L}_B f = \mathscr{L} f$ and $\mathrm{supp}(\mathscr{L}f) \subset \overline{B}$,

$$\int_B \mathscr{L}_B f\, \overline{g}\, d\mu = \int_M \mathscr{L} f\, \overline{\hat{g}}\, d\mu = \int_M f\, \overline{\mathscr{L}\hat{g}}\, d\mu = 0,$$

because $\mathrm{supp}(f) \subseteq \overline{B}$ and $\mathscr{L}\hat{g}$ vanishes in a neighbourhood of $\overline{B}$. This concludes the proof that $\mathrm{Ran}(\mathscr{L}_B) = h^2(B)^\perp$.

Next, we observe that the operator $\mathscr{L}_B$ is injective and continuous from its domain, endowed with the graph norm, to its range, since it is the restriction of $\mathscr{L}$ which is injective and closed. Thus the fact that $\mathscr{L}_B$ is an isomorphism between its domain and its range follows from the Open Mapping Theorem, since the range $h^2(B)^\perp$ is closed.

Finally, to prove (iii), we observe that by (ii) if $f \in h^2(B)^\perp$ then there exists $u \in \mathrm{Dom}(\mathscr{L}_B)$ such that $f = \mathscr{L}_B u = \mathscr{L}u$. Thus $\mathscr{L}^{-1}f = u =$

$\mathscr{L}_B^{-1} f = \mathscr{L}_{B,\mathrm{Dir}}^{-1} f$, since $\mathscr{L}_{B,\mathrm{Dir}}^{-1}$ is an extension of $\mathscr{L}_B^{-1}$, by (i). Hence

$$\|\mathscr{L}^{-1}u\|_2 = \|\mathscr{L}_{B,\mathrm{Dir}}^{-1} f\|_2 \le \frac{1}{\lambda_1(B)}\|f\|_2. \ \blacksquare$$

Remark 3.2. Note that if A is an X^1-atom supported in B, then the function $\mathscr{L}^{-1}A$ has support contained in $\overline{B}$ [24, Remark 3.5].

A straightforward consequence of Proposition 3.1 is the following.

Corollary 3.3. *If A is an X^1-atom with support contained in $\overline{B}$ and $r_B < \mathrm{Inj}(M)$ then the support of $\mathscr{L}^{-1}A$ is contained in $\overline{B}$ and*

$$\|\mathscr{L}^{-1}A\|_2 \le \frac{1}{\lambda_1(B)\mu(B)^{1/2}}. \tag{3.1}$$

Proof. The proof of Proposition 3.1 (or Remark 3.2 above) shows that the support of $\mathscr{L}^{-1}A$ is contained in $\overline{B}$. The estimate (3.1) is a direct consequence of the size estimate in the definition of an atom and of the norm estimate for $\mathscr{L}^{-1}$ in Proposition 3.1(iii). $\blacksquare$

This result sheds light on the definition of $(1,2,M)$-atom in [18]. In fact, a direct consequence of (3.1) is that if A is an X^1-atom and $\lambda_1(B) \asymp r_B^{-2}$, then A is an $(1,2,M)$-atom for every positive integer M. A similar observation applies to X^k-atoms for $k \ge 2$. This suggests that the normalisation of $(1,2,M)$-atoms introduced in [18] may be profitably modified on manifolds whenever the geometry of M determines a somewhat different behaviour of $\lambda_1(B)$.

4. Boundedness of imaginary powers. In this section we analyse the boundedness of $\mathscr{L}^{iu}$ from $X^1(M)$ to $L^1(M)$ in the case where M satisfies our standing assumptions. In this case the (minimal) heat kernel h_t of M satisfies the following pointwise estimate:

$$h_t(x,y) \le \frac{C}{\min(1,t^{n/2})} e^{-bt-d(x,y)^2/(2Dt)} \qquad \forall x,y \in M\ \forall t>0 \tag{4.1}$$

(see, for instance, [16]). In particular under our standing assumptions, M satisfies the Faber–Krahn inequality

$$\lambda_1(\Omega) \ge a\mu(\Omega)^{-2/n}, \tag{4.2}$$

where a is a positive constant and Ω is any precompact region in M.

We recall the following special case of *Takeda's inequality*, which holds on all connected, complete, noncompact Riemannian manifolds (see, for instance, [17, Theorem 12.9]). Suppose that B is a ball in M and denote by $\mathscr{H}_t$ the heat semigroup. Then

$$\int_B (\mathscr{H}_t \mathbf{1}_{(2B)^c})^2\,d\mu \le e\mu((2B)\setminus B)\|\mathscr{H}_t\mathbf{1}_{(2B)^c}\|_\infty^2 \max\left(\frac{r_B^2}{2t}, \frac{2t}{r_B^2}\right) e^{-r_B^2/(2t)} \tag{4.3}$$

for all $t > 0$. Observe that $\mathscr{H}_t$ is submarkovian, so that

$$\|\mathscr{H}_t \mathbf{1}_{(2B)^c}\|_\infty \leq 1 \quad \forall t > 0.$$

Under our standing assumptions on M, for each $s > 0$ there exist constants C_1 and C_2 such that

$$C_1 \mu(B) \leq \mu((2B) \setminus B) \leq C_2 \mu(B) \quad \forall B \in \mathscr{B}_s.$$

Then, by Takeda's inequality and the estimate above, there exist positive constants c and C such that

$$\frac{1}{\mu(B)} \int_B (\mathscr{H}_t \mathbf{1}_{(2B)^c})^2 \, d\mu \leq C e^{-cr_B^2/t} \quad \forall t \in (0, r_B^2] \; \forall B \in \mathscr{B}_s. \tag{4.4}$$

THEOREM 4.1. *Suppose that M is a Riemannian manifold satisfying our standing assumptions. Then for every u in $\mathbb{R} \setminus \{0\}$ the imaginary powers $\mathscr{L}^{iu}$ are bounded from $X^1(M)$ to $L^1(M)$.*

Proof. In view of Theorem 2.4 it suffices to prove that

$$\sup\{\|\mathscr{L}^{iu} A\|_1 : A \text{ an admissible } X^1\text{-atom}\} < \infty.$$

Recall that admissible X^1-atoms are supported in balls of radius at most $s_0 = \frac{1}{2} \operatorname{Inj}(M)$. Suppose that A is such an atom, with support contained in B. Observe that

$$\|\mathscr{L}^{iu} A\|_1 = \|\mathbf{1}_{2B} \, \mathscr{L}^{iu} A\|_1 + \|\mathbf{1}_{(2B)^c} \, \mathscr{L}^{iu} A\|_1.$$

We estimate the two summands on the right hand side separately. To estimate the first, simply observe that, by Schwarz's inequality, the size condition for A, and the spectral theorem,

$$\|\mathbf{1}_{2B} \, \mathscr{L}^{iu} A\|_1 \leq \mu(2B)^{1/2} \, |\!|\!| \mathscr{L}^{iu} |\!|\!|_2 \, \|A\|_2 \leq \left(\frac{\mu(2B)}{\mu(B)} \right)^{1/2}.$$

The right hand side is bounded independently of B, because μ is locally doubling.

To estimate the second summand, we denote by $k_{\mathscr{L}^{iu+1}}(x, y)$ the kernel of the operator $\mathscr{L}^{iu+1}$. Then, by Schwarz's inequality and (3.1), we obtain

$$\begin{aligned} \|\mathbf{1}_{(2B)^c} \mathscr{L}^{iu} A\|_1 &\leq \|\mathscr{L}^{-1} A\|_2 \Big[\int_B d\mu(y) \Big(\int_{(2B)^c} |k_{\mathscr{L}^{iu+1}}(x, y)| \, d\mu(x) \Big)^2 \Big]^{1/2} \\ &\leq \frac{C}{\lambda_1(B)} \Big[\frac{1}{\mu(B)} \int_B d\mu(y) \Big(\int_{(2B)^c} |k_{\mathscr{L}^{iu+1}}(x, y)| \, d\mu(x) \Big)^2 \Big]^{1/2}. \end{aligned}$$

It remains to show that

$$\Big[\frac{1}{\mu(B)} \int_B d\mu(y) \Big(\int_{(2B)^c} |k_{\mathscr{L}^{iu+1}}(x, y)| \, d\mu(x) \Big)^2 \Big]^{1/2} \leq C \lambda_1(B), \tag{4.5}$$

where C is independent of B in $\mathscr{B}_{s_0}$. Observe that off the diagonal the following formula for the kernel of $\mathscr{L}^{iu+1}$ holds:

$$k_{\mathscr{L}^{iu+1}}(x,y) = c_u \int_0^\infty t^{-iu-1} h_t(x,y)\,\frac{dt}{t}.$$

We write the integral on the right hand side as the sum of integrals over $(0, r_B^2]$ and (r_B^2, ∞). Note that

$$\int_{(2B)^c} \Big| \int_{r_B^2}^\infty t^{-iu-1} h_t(x,y)\,\frac{dt}{t}\Big|\,d\mu(x) \le \int_{r_B^2}^\infty \frac{dt}{t^2} \int_{(2B)^c} h_t(x,y)\,d\mu(x) \le r_B^{-2},$$

because the heat semigroup is contractive on $L^\infty(M)$. Hence

$$\text{(4.6)}\qquad \bigg[\frac{1}{\mu(B)}\int_B d\mu(y)\Big(\int_{(2B)^c}\Big|\int_{r_B^2}^\infty t^{-iu-1} h_t(x,y)\,\frac{dt}{t}\Big|\,d\mu(x)\Big)^2\bigg]^{1/2} \le C\lambda_1(B),$$

for $r_B^{-2} \le C\lambda_1(B)$ (just take $\Omega = B$ in formula (4.2) above).

We now prove that there is a constant C, independent of B, such that

$$\text{(4.7)}\qquad \bigg[\frac{1}{\mu(B)}\int_B d\mu(y)\Big(\int_{(2B)^c}\Big|\int_0^{r_B^2} t^{-iu-1} h_t(x,y)\,\frac{dt}{t}\Big|\,d\mu(x)\Big)^2\bigg]^{1/2} \le C\lambda_1(B).$$

By the generalised Minkowski inequality, the left hand side in (4.7) is majorised by

$$\int_0^{r_B^2} \frac{dt}{t^2}\bigg[\frac{1}{\mu(B)}\int_B d\mu(y)\Big(\int_{(2B)^c} h_t(x,y)\,d\mu(x)\Big)^2\bigg]^{1/2},$$

which, by (4.4), is in turn bounded above by

$$\int_0^{r_B^2} e^{-cr_B^2/(2t)}\,\frac{dt}{t^2} = \frac{1}{r_B^2}\int_0^1 e^{-c/(2v)}\,\frac{dv}{v^2} \le C r_B^{-2}.$$

Finally, note that $r_B^{-2} \le C\lambda_1(B)$, and (4.7) is proved. Then (4.6) and (4.7) prove (4.5), as required to conclude the proof of the theorem. ∎

5. Boundedness of the Riesz transform. In this section we prove that the Riesz transform is bounded from $X^1(M)$ to $L^1(M)$. As a preliminary step, we prove the following:

LEMMA 5.1. *For every η in $(0,1)$ and every $s > 0$ there exist positive constants c and C such that for every B in $\mathscr{B}_s$,*

$$\text{(5.1)}\qquad \int_{(4B)^c} e^{-d(x,y)^2/(Dt)}\,d\mu(x) \le C(t^{n/2} e^{-\eta r_B^2/Dt} + e^{-c/t})$$

for every t in $(0, r_B^2]$ and every y in B.

Proof. For simplicity we prove the lemma for $s = 1$. The general case requires only minor modifications. Since $y \in B$ and $x \notin 4B$,

$$\begin{aligned} d(x,y) &\geq d(x,c_B) - d(y,c_B) \\ &\geq d(x,c_B) - r_B \geq \frac{1}{2} d(x,c_B). \end{aligned}$$

Hence

$$\int_{(4B)^c} e^{-d(x,y)^2/(Dt)}\, d\mu(x) \leq \int_{(4B)^c} e^{-d(x,c_B)^2/(4Dt)}\, d\mu(x).$$

Thus, it suffices to estimate the last integral. We split the set $(4B)^c$ into annuli. If r_B is in $(1/4, 1]$, then we simply write

$$(4B)^c = \bigcup_{k=1}^{\infty} A(4kr_B, 4(k+1)r_B),$$

where $A(u,v)$ denotes the annulus $\{x \in M : u \leq d(x,c_B) \leq v\}$. If, instead, $r_B < 1/4$, then we write

$$(4B)^c = \Big[\bigcup_{j=0}^{J-1} A(2^j 4r_B, 2^{j+1} 4r_B)\Big] \cup \Big[\bigcup_{k=1}^{\infty} A(2^J 4kr_B, 2^J 4(k+1)r_B)\Big],$$

where J is chosen so that $R := 2^J 4r_B$ is in $(1/2, 1]$, i.e.,

$$\log_2(1/r_B) - 3 \leq J \leq \log_2(1/r_B) - 2.$$

We give details in the case where $r_B < 1/4$. The case where r_B is in $(1/4, 1]$ is simpler and we omit the details. By (2.2),

$$\begin{aligned} \int_{A(2^j 4r_B, 2^{j+1} 4r_B)} e^{-d(x,c_B)^2/(4Dt)}\, d\mu(x) &\leq C(2^{j+1} 4r_B)^n e^{-2^{2j+2} r_B^2/(Dt)} \\ &= C' t^{n/2} \left(\frac{2^{2j+2} r_B^2}{Dt}\right)^{n/2} e^{-2^{2j+2} r_B^2/(Dt)} \\ &\leq C_\eta t^{n/2} e^{-\eta 2^{2j+2} r_B^2/(Dt)}. \end{aligned}$$

We have used the fact that $t \leq r_B^2$ in the last inequality. By summing over j between 0 and $J-1$, we obtain

$$\begin{aligned} \int_{(2^J 4B)\setminus(4B)} e^{-d(x,c_B)^2/(4Dt)}\, d\mu(x) &\leq C_\eta t^{n/2} \sum_{j=0}^{\infty} [e^{-4\eta r_B^2/(Dt)}]^{2^{2j}} \\ &\leq C_\eta t^{n/2} e^{-4\eta r_B^2/(Dt)}. \end{aligned} \tag{5.2}$$

By (2.1) and the estimate $(Rk)^\alpha e^{2\beta R(k+1)} \leq C_\varepsilon e^{(2\beta+\varepsilon)Rk}$, which holds for every k,

(5.3)
$$\int_{A(2^J 4kr_B, 2^J 4(k+1)r_B)} e^{-d(x,c_B)^2/(4Dt)}\, d\mu(x) \le C(Rk)^\alpha e^{2\beta R(k+1)-R^2k^2/(4Dt)} \le C_\varepsilon e^{(2\beta+\varepsilon)Rk-R^2k^2/(4Dt)}.$$

By completing the square, and using the fact that $t \le r_B^2$, we see that

$$(2\beta+\varepsilon)Rk - \frac{R^2k^2}{4Dt} = \left(\beta+\frac{\varepsilon}{2}\right)^2 4Dt - \left[\frac{Rk}{2\sqrt{Dt}} - 2\left(\beta+\frac{\varepsilon}{2}\right)\sqrt{Dt}\right]^2 \le \left(\beta+\frac{\varepsilon}{2}\right)^2 4Dr_B^2 - \left[\frac{Rk}{2\sqrt{Dt}} - 2\left(\beta+\frac{\varepsilon}{2}\right)\sqrt{Dt}\right]^2.$$

Now observe that if $Rk \ge 4D(2\beta+\varepsilon)r_B^2$, then $Rk - (2\beta+\varepsilon)2Dt \ge Rk/2$, so that

$$(2\beta+\varepsilon)Rk - \frac{R^2k^2}{4Dt} \le C - \frac{R^2k^2}{16Dt}, \tag{5.4}$$

where $C = (\beta+\varepsilon/2)^2 4D$. Choose $K := [\![4D(2\beta+\varepsilon)r_B^2/R]\!] + 1$. Now,

$$\int_{M\setminus(2^J 4B)} e^{-d(x,c_B)^2/(4Dt)}\, d\mu(x) = \sum_{k=1}^{\infty} \int_{A(2^J 4kr_B, 2^J 4(k+1)r_B)} e^{-d(x,c_B)^2/(4Dt)}\, d\mu(x).$$

Note that $K \le D(\beta+\varepsilon/2)$, so K has an upper bound that does not depend on r_B. We estimate each of the terms of the series up to the $(K-1)$th as in (5.3), so that the sum for k from 1 to $K-1$ may be estimated by

$$C_\varepsilon K e^{(2\beta+\varepsilon)D} e^{-R^2/(4Dt)} \le Ce^{-1/(8Dt)}.$$

The series for k from K to ∞ may be estimated as

$$C\sum_{k=K}^{\infty} e^{-R^2k^2/(16Dt)} \le Ce^{-c/t}$$

for some positive c. By combining the estimates above, we obtain

$$\int_{M\setminus(RB)} e^{-d(x,c_B)^2/(4Dt)}\, d\mu(x) \le Ce^{-c/t}, \tag{5.5}$$

which, together with (5.2), gives the required estimate. ∎

Lemma 5.2. *Suppose that M is a Riemannian manifold satisfying our standing assumptions. Fix a scale parameter $s < \mathrm{Inj}(M)$. Then there exists a constant C such that for every ball B in $\mathscr{B}_s$,*

$$\|\nabla \mathscr{L}^{1/2} f\|_{L^1((4B)^c)} \le Cr_B^{-2}\|f\|_{L^1(B)} \quad \forall f \in L^1(B).$$

Proof. Step I: *Reduction of the problem and conclusion.* A straightforward argument shows that

$$\nabla \mathscr{L}^{1/2} f(x) = \int_M k_{\nabla \mathscr{L}^{1/2}}(x,y) f(y)\, d\mu(y) \quad \forall f \in C_c(M)\ \forall x \notin \operatorname{supp}(f),$$

where

$$k_{\nabla \mathscr{L}^{1/2}}(x,y) = \frac{1}{\Gamma(-1/2)} \int_0^\infty \nabla_x h_t(x,y) \frac{dt}{t^{3/2}} \tag{5.6}$$

for all (x,y) off the diagonal in $M \times M$. Here h_t denotes the heat kernel (with respect to the Riemannian measure μ). Define

$$\mathscr{I}^B(y) := \int_0^{r_B^2} \frac{dt}{t^{3/2}} \int_{(4B)^c} |\nabla_x h_t(x,y)|\, d\mu(x),$$

$$\mathscr{I}_B(y) := \int_{r_B^2}^\infty \frac{dt}{t^{3/2}} \int_{(4B)^c} |\nabla_x h_t(x,y)|\, d\mu(x).$$

Note that, by (5.6) and Tonelli's theorem,

$$\begin{aligned} \|\nabla \mathscr{L}^{1/2} f\|_{L^1((4B)^c)} &\le \int_{(4B)^c} \int_0^\infty \int_B |\nabla_x h_t(x,y)|\, |f(y)|\, d\mu(x)\, d\mu(y) \frac{dt}{t^{3/2}} \\ &= \int_B [\mathscr{I}^B(y) + \mathscr{I}_B(y)] |f(y)|\, d\mu(y). \end{aligned} \tag{5.7}$$

We *claim* that there exists a constant C such that

$$\mathscr{I}^B(y) \le C r_B^{-2} \quad \text{and} \quad \mathscr{I}_B(y) \le C r_B^{-2}. \tag{5.8}$$

These estimates will be proved in Steps II and III, respectively. Assuming the claim, we may deduce from (5.7) and (5.8) that

$$\|\nabla \mathscr{L}^{1/2} f\|_{L^1((4B)^c)} \le \int_B [\mathscr{I}^B(y) + \mathscr{I}_B(y)] |f(y)|\, d\mu(y) \le C r_B^{-2} \|f\|_{L^1(B)},$$

as required to conclude the proof of the lemma.

Step II: *Estimate of $\mathscr{I}^B(y)$.* We shall use Grigor'yan's integral estimates for the gradient of the heat kernel [15]. It will be convenient to introduce more notation. We fix $D > 4$, and set, for every y in M and $t > 0$,

$$E_0(y,t) := \int_M h_t(x,y)^2 e^{d(x,y)^2/(Dt)}\, d\mu(x), \tag{5.9}$$

$$E_1(y,t) := \int_M |\nabla_x h_t(x,y)|^2 e^{d(x,y)^2/(Dt)}\, d\mu(x). \tag{5.10}$$

Recall that, under our standing assumptions on M, the Faber–Krahn type inequality (4.2) holds on M. Furthermore, the constant a in (4.2) is uniformly bounded from below as long as $r_B \leq s$ (because M has bounded geometry). Therefore [17, Theorem 15.8, p. 400]

$$E_0(y,t) \leq Ct^{-n/2} \qquad \forall t \in (0, r_B^2]\ \forall y \in M.$$

Hence [15, Theorem 1.1]

$$E_1(y,t) \leq Ct^{-n/2-1} \qquad \forall t \in (0, r_B^2]\ \forall y \in M.$$

By using Schwarz's inequality, the estimate above and Lemma 5.1, we obtain

$$\begin{aligned} \mathscr{I}^B(y) &\leq C \int_0^{r_B^2} (t^{n/2} e^{-\eta r_B^2/(Dt)} + e^{-c/t})^{1/2} E_1(y,t)^{1/2} \frac{dt}{t^{3/2}} \\ &\leq C \int_0^{r_B^2} t^{-1} e^{-\eta r_B^2/(2Dt)} \frac{dt}{t} + C \int_0^{r_B^2} e^{-c/(2t)} \frac{dt}{t^{n/4+2}} \\ &\leq C(r_B^{-2} + 1) \qquad \forall y \in M, \end{aligned} \tag{5.11}$$

as required to prove the first statement in (5.8).

STEP III: *Estimate of* $\mathscr{I}_B(y)$. The main idea is to combine Caccioppoli's inequality with Harnack's inequality for balls of small radius. We denote by $\{\varphi_j\}$ a smooth partition of unity associated to a locally finite covering $\{B'_j\}$ of $(4B)^c$ by balls of radius r_B. We set

$$\mathscr{I}_{B;j,k}(y) := \int_{(k-1)r_B^2}^{kr_B^2} \frac{dt}{t^{3/2}} \int_{B'_j} |\nabla_x h_t(x,y)| \varphi_j(x)\, d\mu(x). \tag{5.12}$$

Clearly

$$\begin{aligned} \mathscr{I}_B(y) &\leq \sum_j \int_{r_B^2}^{\infty} \frac{dt}{t^{3/2}} \int_{B'_j} |\nabla_x h_t(x,y)| \varphi_j(x)\, d\mu(x) \\ &= \sum_j \sum_{k=2}^{\infty} \mathscr{I}_{B;j,k}(y). \end{aligned} \tag{5.13}$$

We now introduce the parabolic cylinder $\mathscr{C}_{j,k}$, defined as follows:

$$\mathscr{C}_{j,k} := B'_j \times ((k-1)r_B^2, kr_B^2].$$

Clearly $(\mu \times \lambda)(\mathscr{C}_{j,k}) = \mu(B'_j) r_B^2$, where λ is the Lebesgue measure on the real line. Recall the following version of the parabolic Caccioppoli inequality:

$$\int_{\mathscr{C}_{j,k}} |\nabla_x h_t(x,y)|^2\, d\mu(x)\, dt \leq \frac{C}{r_B^2} \int_{2\mathscr{C}_{j,k}} |h_t(x,y)|^2\, d\mu(x)\, dt, \tag{5.14}$$

where

$$2\mathscr{C}_{j,k} := 2B'_j \times ((k-2)r_B^2, (k+1)r_B^2].$$

This inequality is a straightforward consequence of [17, Lemmas 15.2 and 15.3]. Observe that

$$\mathscr{I}_{B;j,k}(y) \asymp \frac{1}{(kr_B^2)^{3/2}} \int_{\mathscr{C}_{j,k}} |\nabla_x h_t(x,y)| \, d\mu(x) \, dt.$$

Therefore, by Schwarz's inequality and Caccioppoli's inequality,

$$\begin{aligned} \mathscr{I}_{B;j,k}(y) &\leq \frac{(\mu \times \lambda)(\mathscr{C}_{j,k})}{(kr_B^2)^{3/2}} \left[\frac{1}{(\mu \times \lambda)(\mathscr{C}_{j,k})} \int_{\mathscr{C}_{j,k}} |\nabla_x h_t(x,y)|^2 \, d\mu(x) \, dt \right]^{1/2} \\ &\leq \frac{(\mu \times \lambda)(\mathscr{C}_{j,k})}{(kr_B^2)^{3/2}} \frac{1}{r_B} \left[\frac{1}{(\mu \times \lambda)(2\mathscr{C}_{j,k})} \int_{2\mathscr{C}_{j,k}} h_t(x,y)^2 \, d\mu(x) \, dt \right]^{1/2}. \end{aligned}$$

We now apply the parabolic Harnack inequality to the parabolic cylinder $2\mathscr{C}_{j,k}$ to conclude that

$$\begin{aligned} &\left[\frac{1}{(\mu \times \lambda)(2\mathscr{C}_{j,k})} \int_{2\mathscr{C}_{j,k}} h_t(x,y)^2 \, d\mu(x) \, dt \right]^{1/2} \\ &\qquad \leq C \inf_{(z,t) \in 2\mathscr{C}_{j,k+2}} h_t(z,y) \\ &\qquad \leq C \frac{1}{(\mu \times \lambda)(2\mathscr{C}_{j,k})} \int_{2\mathscr{C}_{j,k+2}} h_t(x,y) \, d\mu(x) \, dt. \end{aligned} \tag{5.15}$$

By combining the last two estimates, we obtain

$$\begin{aligned} \mathscr{I}_{B;j,k}(y) &\leq \frac{C}{(kr_B^2)^{3/2}} \frac{1}{r_B} \int_{2\mathscr{C}_{j,k+2}} h_t(x,y) \, d\mu(x) \, dt \\ &\leq \frac{C}{r_B} \int_{kr_B^2}^{(k+3)r_B^2} \frac{dt}{t^{3/2}} \int_{2B'_j} h_t(x,y) \, d\mu(x). \end{aligned} \tag{5.16}$$

We now sum over j and k, and then use the facts that the covering $\{B'_j\}$ is uniformly locally finite and that $\|h_t(\cdot, y)\|_1 \leq 1$ for every y in M, to obtain

$$\begin{aligned} \mathscr{I}_B(y) &\leq \frac{C}{r_B} \int_{r_B^2}^{\infty} \frac{dt}{t^{3/2}} \int_{(2B)^c} h_t(x,y) \, d\mu(x) \\ &\leq \frac{C}{r_B} \int_{r_B^2}^{\infty} \frac{dt}{t^{3/2}} \leq \frac{C}{r_B^2}, \end{aligned} \tag{5.17}$$

as required to prove the second estimate in (5.8), and to conclude the proof of the claim. ∎

THEOREM 5.3. *Suppose that M is a Riemannian manifold satisfying our standing assumptions. The Riesz transform $\nabla\mathscr{L}^{-1/2}$ is bounded from $X^1(M)$ to $L^1(M)$.*

Proof. In view of Theorem 2.4 it suffices to prove that

$$\sup\{\|\nabla\mathscr{L}^{-1/2}A\|_1 : A \text{ an admissible } X^1\text{-atom}\} < \infty. \tag{5.18}$$

Fix such an atom A, and denote by B the ball associated to A. Recall that $r_B \le s_0$. Observe that

$$\|\nabla\mathscr{L}^{-1/2}A\|_1 = \|\nabla\mathscr{L}^{-1/2}A\|_{L^1(4B)} + \|\nabla\mathscr{L}^{-1/2}A\|_{L^1((4B)^c)}.$$

We shall estimate the two summands on the right hand side separately. Clearly

$$\begin{aligned}\|\nabla\mathscr{L}^{-1/2}A\|_{L^1(4B)} &\le \mu(4B)^{1/2}\|\nabla\mathscr{L}^{-1/2}A\|_{L^2(4B)}\\ &\le \left(\frac{\mu(4B)}{\mu(B)}\right)^{1/2} \le C.\end{aligned}$$

In the second inequality above we have used the fact that

$$\|\nabla\mathscr{L}^{-1/2}A\|_{L^2(4B)} \le \|A\|_2 \le \mu(B)^{-1/2},$$

which follows from the L^2-boundedness of the Riesz transform and the size property of A. In the last inequality we have used the fact that the measure μ is locally doubling. Therefore

$$\sup\|\nabla\mathscr{L}^{-1/2}A\|_{L^1(4B)} < \infty, \tag{5.19}$$

where the supremum is taken over all admissible X^1-atoms A.

Thus, to conclude the proof of the theorem it suffices to show that

$$\sup\|\nabla\mathscr{L}^{-1/2}A\|_{L^1((4B)^c)} < \infty, \tag{5.20}$$

where the supremum is taken over all admissible X^1-atoms A. Observe that

$$\nabla\mathscr{L}^{-1/2}A = \nabla\mathscr{L}^{-1/2}\mathscr{L}\mathscr{L}^{-1}A = \nabla\mathscr{L}^{1/2}(\mathscr{L}^{-1}A).$$

Recall that by Corollary 3.3,

$$\|\mathscr{L}^{-1}A\|_{L^2(B)} \le \frac{1}{\lambda_1(B)}\mu(B)^{-1/2},$$

so that

$$\|\mathscr{L}^{-1}A\|_{L^1(B)} \le \mu(B)^{1/2}\|\mathscr{L}^{-1}A\|_{L^2(B)} \le \frac{1}{\lambda_1(B)}. \tag{5.21}$$

Therefore

$$\begin{aligned}\|\nabla\mathscr{L}^{-1/2}A\|_{L^1((4B)^c)} &= \|\nabla\mathscr{L}^{1/2}(\mathscr{L}^{-1}A)\|_{L^1((4B)^c)}\\ &\le Cr_B^{-2}\|\mathscr{L}^{-1}A\|_{L^1(B)}\\ &\le Cr_B^{-2}\lambda_1(B)^{-1} \le C;\end{aligned}$$

the first inequality follows from Lemma 5.2 and the support property of A combined with Remark 3.2, the second from (5.21), and the last from (4.2) and (2.2). ∎

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