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Hessian determinants as elements of dual Sobolev spaces

by

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Abstract. In this short note we present new integral formulas for the Hessian determinant. We use them for new definitions of Hessian under minimal regularity assumptions. The Hessian becomes a continuous linear functional on a Sobolev space.

1. Introduction. To every differential operator, linear or nonlinear, there corresponds a natural domain of definition; usually a very specific Sobolev space of functions in question.

In recent years there has been increasing interest in Jacobian determinants of Sobolev mappings, Hessians of scalar functions, and more general null Lagrangians [BFS], [CLMS], [GI], [GIM], [I1], [I2], [IL], [IO], [IV], [IS], [Mo], [Mu1], [Mu2], [Mu3]. A common feature of such nonlinear differential expressions is that they possess an important cancellation property. These properties lead, through integration by parts, to a study of null Lagrangians beyond their natural domain of definition. In particular, the very useful concepts of weak Jacobians and distributional Hessian have been introduced and developed in [I1], [CLMS], [Mu2], [Mu3]. They are understood as Schwartz distributions. In this short note we define new classes of weak Hessians which are elements of dual Sobolev spaces.

For a smooth function $u \in C_0^\infty(\Omega)$ in a domain $\Omega \subset \mathbb{R}^n$ we have the pointwise Hessian

$$\mathcal{H}u = \det\left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]_{i,j=1,\ldots,n}.$$

There are many more equivalent formulas for $\mathcal{H}u$. The reader is referred to [I1] for a complete list of such formulas and the associated definitions of the distributional Hessian. The novelty of our approach lies in letting the test functions be part of Sobolev mappings, whose Jacobian determinants lie in the Hardy space $H^1(\Omega)$. The integrals, which diverge under the usual

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regularity conditions, become well defined via H^1-BMO pairing. In this way the very weak Hessian $\mathcal{H}[u]$ becomes an element of the space $[W_0^{1,p}(\Omega)]^*$ dual to $W_0^{1,p}(\Omega)$. Furthermore, we gain new estimates. Let us take a quick look at particular situations of this kind.

2. Preliminaries. We shall work with the Hardy space of functions defined on a Lipschitz domain $\Omega \subset \mathbb{R}^n$ (see, for example [BIJZ], [M]). Following [BIJZ] we fix a nonnegative $\Phi \in C_0^\infty(B)$ supported in the unit ball $B = \{x \in \mathbb{R}^n;\ |x| < 1\}$ and having integral equal to 1. The one-parameter family of mollifiers (called an *approximation of unity*)

$$\Phi_\varepsilon(x) = \varepsilon^{-n}\Phi\left(\frac{x}{\varepsilon}\right), \qquad \varepsilon > 0,$$

gives rise to a maximal operator for Schwartz distributions $\mathcal{D}'(\Omega)$. Let $f \in \mathcal{D}'(\Omega)$. We consider smooth functions defined on the level sets $\Omega_\varepsilon = \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ by

$$f_\varepsilon(x) = (f * \Phi_\varepsilon)(x) = f[\Phi_\varepsilon(x - \cdot)].$$

This latter notation stands for the action of f on the test function $y \mapsto \Phi_\varepsilon(x-y) \in C_0^\infty(B(x,\varepsilon))$ in the y-variable. Thus $f_\varepsilon \to f$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0$.

For $f \in L^1_{\mathrm{loc}}(\Omega)$ the above convolution formula takes the usual integral form,

$$f_\varepsilon(x) = \int_\Omega f(y)\Phi_\varepsilon(x-y)\,dy \to f(x) \qquad \text{as } \varepsilon \to 0$$

whenever $x \in \Omega$ is a Lebesgue point of f. Now the maximal operator $\mathcal{M}$ on $\mathcal{D}'(\Omega)$ is defined by

$$\mathcal{M}f(x) = \sup\{|f_\varepsilon(x)|;\ 0 < \varepsilon < \operatorname{dist}(x, \partial\Omega)\}.$$

Definition 2.1. A distribution $f \in \mathcal{D}'(\Omega)$ is said to belong to the Hardy space $H^1(\Omega)$ if $\mathcal{M}f \in L^1(\Omega)$.

$H^1(\Omega)$ is a Banach space with respect to the norm

$$\|f\|_{H^1(\Omega)} = \int_\Omega \mathcal{M}f(x)\,dx.$$

Actually, distributions in $H^1(\Omega)$ are represented by integrable functions, so we have the inclusion $H^1(\Omega) \subset L^1(\Omega)$. Similarly, the BMO-norm in a domain $\Omega \subset \mathbb{R}^n$ (see [BIJZ], [J]) is given by

$$\|b\|_{\mathrm{BMO}(\Omega)} = \sup\left\{ \fint |b - b_Q|;\ Q \text{ is a cube in } \Omega\right\}, \text{ where } b_Q = \frac{1}{|Q|}\int_Q b = \fint_Q b.$$

The following result of [CLMS], adapted to domains in $\mathbb{R}^n$ as Theorem 8.3 in [IV], will be useful:

THEOREM 2.2. *Let $\Omega \subset \mathbb{R}^n$ and $f^i \in W^{1,p_i}(\Omega)$, $1 < p_i < \infty$, $1/p_1 + \cdots + 1/p_n = 1$. Then the Jacobian determinant of the mapping $f = (f^1, \ldots, f^n) : \Omega \to \mathbb{R}^n$ lies in the Hardy space $H^1(\Omega)$. Moreover, we have the estimate*

$$\|J_f = J(f^1, \ldots, f^n)\|_{H^1(\Omega)} \le C\|\nabla f^1\|_{L^{p_1}(\Omega)} \times \cdots \times \|\nabla f^n\|_{L^{p_n}(\Omega)}. \tag{2.1}$$

We shall view functions $b \in \mathrm{BMO}(\Omega)$ as continuous linear functionals on $H^1(\Omega)$ (see [FS] and [BIJZ]). The action of b on $h \in H^1(\Omega)$ will be designated by the symbol $\int_\Omega^* bh$.

LEMMA 2.3. *The* BMO-H^1 *pairing is represented by the inequality*

$$\Big|\int_\Omega^* bh\Big| \le C\|b\|_{\mathrm{BMO}(\Omega)}\|h\|_{H^1(\Omega)}. \tag{2.2}$$

3. Weak extensions of the Hessian. To simplify the writing, we use subscripts for partial derivatives; for instance, $u_x = \frac{\partial u}{\partial x}$, $u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$. The two-dimensional Hessian

$$(\mathcal{H}u)(x,y) = \begin{vmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{vmatrix} = u_{xx}u_{yy} - u_{xy}^2,$$

for smooth functions $u \in C^\infty(\Omega)$, can be written in three different ways (see [I1]):

$$(\mathcal{H}u)(x,y) = (u_x u_{yy})_x - (u_x u_{xy})_y = (u_y u_{xx})_y - (u_y u_{xy})_x \tag{3.1}$$

$$= \frac{1}{2}(u u_{xx})_{yy} + \frac{1}{2}(u u_{yy})_{xx} - (u u_{xy})_{xy} \tag{3.2}$$

$$= (u_x u_y)_{xy} - \frac{1}{2}(u_x u_x)_{yy} - \frac{1}{2}(u_y u_y)_{xx}. \tag{3.3}$$

One advantage of having partial derivatives outside the round parentheses is that they can be understood in the sense of distributions. Indeed, if we multiply by a test function $\varphi \in C_0^\infty(\Omega)$ and integrate by parts, formulas (3.1)–(3.3) lead us to three different concepts of distributional Hessian:

$$\mathcal{H}[u](\varphi) = \int_\Omega (u_x u_{yy} \varphi_x - u_x u_{xy} \varphi_y), \tag{3.4}$$

$$\mathcal{H}[u](\varphi) = \frac{1}{2}\int_\Omega (u_{xx}\varphi_{yy} - u_{xy}\varphi_{xy})u + \frac{1}{2}\int_\Omega (u_{yy}\varphi_{xx} - u_{xy}\varphi_{xy})u, \tag{3.5}$$

$$\mathcal{H}[u](\varphi) = \frac{1}{2}\int_\Omega (2u_x u_y \varphi_{xy} - u_x u_x \varphi_{yy} - u_y u_y \varphi_{xx}). \tag{3.6}$$

Thus we have three well defined nonlinear differential operators:

$$\mathcal{H}_1 : W^{2,4/3}_{\mathrm{loc}}(\Omega) \to \mathcal{D}'_1(\Omega), \tag{3.7}$$

$$\mathcal{H}_2 : W^{2,1}_{\mathrm{loc}}(\Omega) \to \mathcal{D}'_2(\Omega), \tag{3.8}$$

$$\mathcal{H}^*_2 : W^{1,2}_{\mathrm{loc}}(\Omega) \to \mathcal{D}'_2(\Omega). \tag{3.9}$$

The purpose of the present note is to extend the above domains of definition of $\mathcal{H}[u]$ to other classes of Sobolev functions. We shall take advantage of the H^1-regularity of the Jacobians. $\mathcal{H}u$ will be considered as a continuous linear functional on $W^{1,p}_0(\Omega)$ rather than on $C^\infty_0(\Omega)$. We also provide improved estimates.

First consider expression (3.4) to observe that:

Proposition 3.1. *Suppose* $u_x \in \mathrm{BMO}(\Omega)$ *and* $u_y \in W^{1,q}(\Omega)$, $1<q<\infty$. *For* $\varphi \in W^{1,p}_0(\Omega)$, *where* $1/p + 1/q = 1$, *define*

$$\mathcal{H}[u](\varphi) = \int^*_\Omega u_x J_f, \tag{3.10}$$

where $f = (\varphi, u_y) : \Omega \to \mathbb{R}^2$ *and* $J_f = J(\varphi, u_y)$. *This is consistent with* (3.4) *in the smooth case. In other words,* $\mathcal{H}[u] \in [W^{1,p}_0(\Omega)]^*$. *The last integral is understood as* H^1-BMO *pairing in which the Jacobian determinant* J_f *belongs to the Hardy space* $H^1(\Omega)$. *By* (2.2) *and* (2.1) *we gain the estimate*

$$|\mathcal{H}[u](\varphi)| \le C\|u_x\|_{\mathrm{BMO}(\Omega)}\|\nabla\varphi\|_{L^p(\Omega)}\|\nabla u_y\|_{L^q(\Omega)}.$$

In this context it is worth recalling the classical definition of the space dual to $W^{1,p}_0(\Omega)$:

$$[W^{1,p}_0(\Omega)]^* = W^{-1,q}(\Omega) = \{\operatorname{div} F;\ F \in L^q(\Omega, \mathbb{R}^2)\}$$

where the divergence of a vector field $F : \Omega \to \mathbb{R}^2$ acts on a test function $\varphi \in C^\infty_0(\Omega)$ by

$$(\operatorname{div} F)[\varphi] = -\int_\Omega \langle F, \nabla\varphi\rangle.$$

In (3.4), however, the corresponding vector field $F = (-u_x u_{yy}, u_x u_{xy})$ cannot be used to give a meaning to $\mathcal{H}[u]$ as an element of $W^{-1,q}(\Omega)$, because $F \notin L^q(\Omega, \mathbb{R}^2)$, unless one assumes that $u_x \in L^\infty(\Omega)$. Proposition 3.1 gains in interest because we only need to assume $u_x \in \mathrm{BMO}(\Omega)$, due to H^1-regularity of the Jacobian of f.

Example 3.2. The observant reader may also notice that in Proposition 3.1, $\mathcal{H}[u]$ is well defined under less restrictive conditions on u than in (3.7). Indeed, for (3.7) it is required that $u \in W^{2,4/3}_{\mathrm{loc}}(\Omega)$, whereas in (3.10) one may consider $u(x, y) = x \log x + y^\alpha$ in $\Omega = (0, 1) \times (0, 1)$ where $\alpha > 1 + 1/p$. We have $u_x = 1 + \log x \in \mathrm{BMO}(\Omega)$, $u_{xx} = 1/x \notin L^{4/3}(\Omega)$ and $u_{yy} =$

$\alpha(\alpha-1)y^{\alpha-2} \in L^q(\Omega)$. Thus this function satisfies hypotheses of Proposition 3.1 but $u \notin W^{2,4/3}_{\mathrm{loc}}(\Omega)$.

Next we reexamine expression (3.5).

PROPOSITION 3.3. *For $u \in W^{2,q}(\Omega)$, $1 < q < 2$, and $\varphi \in W^{2,p}_0(\Omega)$, where $1/p + 1/q = 1$, define*

$$\mathcal{H}[u](\varphi) = \frac{1}{2}\int^*_\Omega [J(u_x, \varphi_y) + J(\varphi_x, u_y)] \cdot u.$$

This agrees with (3.5) *in the smooth case. In the more general setting, $\mathcal{H}[u]$ becomes an element of the dual to $W^{2,p}_0(\Omega)$,*

$$\mathcal{H} : W^{2,q}(\Omega) \to [W^{2,p}_0(\Omega)]^*. \tag{3.11}$$

Let us put in evidence the differences between the two notions of Hessian; in (3.8) and in (3.11). By the definition in (3.8), it follows from (3.5) that

$$|\mathcal{H}[u](\varphi)| \le C\|\nabla^2\varphi\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)}\|\nabla^2 u\|_{L^1(\Omega)}. \tag{3.12}$$

In the definition (3.11), on the other hand, Lemma 2.3 and Theorem 2.2 yield

$$\begin{aligned}\mathcal{H}[u](\varphi) &\le C\|u\|_{\mathrm{BMO}(\Omega)}\|J(u_x, \varphi_y) - J(u_y, \varphi_x)\|_{H^1(\Omega)}\\ &\le C\|u\|_{\mathrm{BMO}(\Omega)}\|\nabla^2 u\|_{L^q(\Omega)}\|\nabla^2\varphi\|_{L^p(\Omega)},\end{aligned}$$

which is slightly better than (3.12).

Concerning expression (3.6) (see also Example VII.2 in [CLMS]), we regard φ_{xx}, φ_{xy} and φ_{yy} as entries of Jacobian determinants to gain a new estimate:

PROPOSITION 3.4. *Suppose $u \in W^{1,q}(\Omega)$ and $\varphi \in W^{2,p}_0(\Omega)$, where $1 < p, q < \infty$, $1/p + 1/q = 1$. Define*

$$\mathcal{H}[u](\varphi) = \frac{1}{2}\int^*_\Omega [u_x J(\varphi_y, u) + u_y J(u, \varphi_x)], \tag{3.13}$$

which is the same as (3.6) *in the smooth case.*

Again, $\mathcal{H}[u]$ becomes an element of the dual of $W^{2,p}_0(\Omega)$, so we obtain a nonlinear operator

$$\mathcal{H} : W^{1,q}(\Omega) \to [W^{2,p}_0(\Omega)]^*. \tag{3.14}$$

Using (3.9) it follows from (3.6) that

$$|\mathcal{H}[u](\varphi)| \le C\|\nabla^2\varphi\|_{L^\infty(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}.$$

On the other hand, using (3.13) we can apply Lemma 2.3 and Theorem 2.2

to obtain

$$\begin{aligned}|\mathcal{H}[u](\varphi)| &\le C\|\nabla u\|_{\mathrm{BMO}(\Omega)}\|J(\varphi_y,u)-J(\varphi_x,u)\|_{H^1(\Omega)}\\ &\le C\|\nabla u\|_{\mathrm{BMO}(\Omega)}\|\nabla^2\varphi\|_{L^p(\Omega)}\|\nabla u\|_{L^q(\Omega)}.\end{aligned}$$

4. Three-dimensional Hessian. There are many more weak formulations of null Lagrangians in higher dimensions [I1]. Their domain of definition can be improved in the spirit demonstrated above. Let us conclude this note with one such example.

Recall from [I1] the formula

$$\begin{aligned}(4.1)\quad & 3\mathcal{H}[u](\varphi)\\ &= \int_\Omega \left(\varphi_{xx}\left[\begin{vmatrix}u_{yy} & u_{yz}\\ u_{zy} & u_{zz}\end{vmatrix} u\right] + \varphi_{yy}\left[\begin{vmatrix}u_{xx} & u_{xz}\\ u_{zx} & u_{zz}\end{vmatrix} u\right] + \varphi_{zz}\left[\begin{vmatrix}u_{xx} & u_{xy}\\ u_{yx} & u_{yy}\end{vmatrix} u\right]\right)\\ &\quad - 2\int_\Omega \left(\varphi_{xy}\left[\begin{vmatrix}u_{xy} & u_{xz}\\ u_{zy} & u_{zz}\end{vmatrix} u\right] + \varphi_{yz}\left[\begin{vmatrix}u_{yz} & u_{yx}\\ u_{xz} & u_{xx}\end{vmatrix} u\right] + \varphi_{zx}\left[\begin{vmatrix}u_{zx} & u_{zy}\\ u_{yx} & u_{yy}\end{vmatrix} u\right]\right).\end{aligned}$$

Naturally, it defines the distributional Hessian

$$\mathcal{H} : W^{2,2}_{\mathrm{loc}}(\Omega) \to \mathcal{D}'_2(\Omega).$$

As in Proposition 3.4, we will gain slightly by placing φ_x, φ_y and φ_z into Jacobian determinants.

Proposition 4.1. *Suppose $u \in W^{2,s}(\Omega) \subset BMO(\Omega)$ and $\varphi \in W^{2,p}_0(\Omega)$, where $1/p + 2/s = 1$, $p > 1$, $s > 2$. Write (4.1) in the form*

$$\mathcal{H}[u](\varphi) = \frac{1}{3}\int^*_\Omega [J(\varphi_x,u_y,u_z) + J(u_x,\varphi_y,u_z) + J(u_x,u_y,\varphi_z)]\cdot u.$$

This integral defines a nonlinear differential operator

$$\mathcal{H} : W^{2,s}(\Omega) \to [W^{2,p}_0(\Omega)]^*.$$

We obtain new estimates which are not direct consequences of (4.1):

$$\begin{aligned}|\mathcal{H}[u](\varphi)| &\le C\|u\|_{\mathrm{BMO}(\Omega)}\|J(\varphi_x,u_y,u_z) + J(u_x,\varphi_y,u_z) + J(u_x,u_y,\varphi_z)\|_{H^1(\Omega)}\\ &\le C\|u\|_{\mathrm{BMO}(\Omega)}\|\nabla^2\varphi\|_{L^p(\Omega)}\|\nabla^2 u\|^2_{L^s(\Omega)}.\end{aligned}$$

Historically, any improvement, no matter how little, in the definitions and regularity properties of the weak Jacobians [Mu1], [Mu2], [Mu3], Hessians [I1], [CLMS] and null Lagrangians [I2] turned out to be extremely useful in the study of minima of variational integrals. Indeed, having such improvements in hand we gain compactness, which in turn yields the existence of the solutions. It is in this context that our quick look at the Hessians is certainly worth noting.

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