STUDIA MATHEMATICA 194 (1) (2009)

Uniformly ergodic A-contractions on Hilbert spaces

by

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Abstract. We study the concept of uniform (quasi-) A-ergodicity for A-contractions on a Hilbert space, where A is a positive operator. More precisely, we investigate the role of closedness of certain ranges in the uniformly ergodic behavior of A-contractions. We use some known results of M. Lin, M. Mbekhta and J. Zemánek, and S. Grabiner and J. Zemánek, concerning the uniform convergence of the Cesàro means of an operator, to obtain similar versions for A-contractions. Thus, we continue the study of A-ergodic operators developed earlier by the author.

1. Notations and preliminaries. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$, where the identity operator is denoted by $I = I_{\mathcal{H}}$. For $T \in \mathcal{B}(\mathcal{H})$, T^* means the adjoint operator, while $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote the range and the null-space of T, respectively.

An operator $T \in \mathcal{B}(\mathcal{H})$ is *mean ergodic* on $\mathcal{H}$ if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j h = Qh \tag{1.1}$$

exists for any $h \in \mathcal{H}$, where Q is the projection onto $\mathcal{N}(I - T)$ along $\overline{\mathcal{R}(I - T)}$. In this case, one has the (direct sum) decomposition

$$\mathcal{H} = \overline{\mathcal{R}(I - T)} \dotplus \mathcal{N}(I - T). \tag{1.2}$$

If the projection Q is orthogonal, or equivalently $\mathcal{N}(I-T) = \mathcal{N}(I-T^*)$, we say that T is *orthogonally mean ergodic*. Similarly, if the limit (1.1) is uniform on the unit ball and Q is an orthogonal projection, we say that T is *orthogonally uniformly ergodic* on $\mathcal{H}$. This is a stronger concept than the one studied by N. Dunford [3], E. Ed-Dari [4], S. Grabiner and J. Zemánek [6], M. Lin [8], M. Mbekhta and J. Zemánek [9], Y. Tomilov and J. Zemánek [15], J. Zemánek [16] and other authors, in the context of Banach spaces.

2000 *Mathematics Subject Classification*: Primary 47A35, 47B65; Secondary 47B20.
Key words and phrases: mean ergodic operator, uniformly ergodic operator, A-contraction, A-ergodicity, uniform A-ergodicity.

DOI: 10.4064/sm194-1-1

Clearly, any contraction on $\mathcal{H}$, that is, an operator $T \in \mathcal{B}(\mathcal{H})$ satisfying $T^*T \le I$, is orthogonally mean ergodic, but in general it is not orthogonally uniformly ergodic. At the same time, there exist mean ergodic operators without bounded powers. In [12–14] we have obtained such operators belonging to a class of generalized contractions which are defined below.

Let $A \in \mathcal{B}(\mathcal{H})$ be a fixed positive operator, $A \neq 0$. Then $T \in \mathcal{B}(\mathcal{H})$ is called an A-*contraction* on $\mathcal{H}$ if it satisfies the inequality

$$T^*AT \le A. \tag{1.3}$$

Clearly, we can, and will, always assume $\|A\| = 1$.

If there is equality in (1.3), we say that T is an A-*isometry* on $\mathcal{H}$. It is known ([2], [7]) that if T is a contraction such that the strong limit $S_T = s\text{-}\lim_n T^{*n}T^n$ is not the zero operator then T is an S_T-isometry. Moreover, S_T is an orthogonal projection if and only if S_T commutes with T.

An A-contraction T induces the orthogonal decomposition

$$\mathcal{H} = \overline{\mathcal{R}(A - AT)} \oplus \mathcal{N}(A - AT), \tag{1.4}$$

where

$$\mathcal{N} := \mathcal{N}(A - AT) = \mathcal{N}(A^{1/2} - A^{1/2}T) = \mathcal{N}(A - T^*A)$$

and it is an invariant subspace for T (see [11]), $A^{1/2}$ being the square root of A. But the subspace $\mathcal{N}$ is not invariant for A and T^*, and in general we have

$$\mathcal{N} \neq \mathcal{N}_* := \mathcal{N}(A^{1/2} - T^*A^{1/2}).$$

We recall from [12, 14] that $\mathcal{N} = \mathcal{N}_*$ if and only if $\mathcal{N}_*$ is invariant for A, which also means that $\mathcal{H}$ admits the orthogonal decomposition

$$\mathcal{H} = \overline{\mathcal{R}(A^{1/2} - A^{1/2}T)} \oplus \mathcal{N}. \tag{1.5}$$

When $\mathcal{N} = \mathcal{N}_*$ one has $\mathcal{N} = \mathcal{N}(A) \oplus \overline{A^{1/2}\mathcal{N}}$ and $\mathcal{N} \ominus \mathcal{N}(A) = \mathcal{N}(I - \widehat{T})$, where $\widehat{T}$ is the contraction on $\overline{\mathcal{R}(A)}$ defined (in virtue of (1.3)) by

$$\widehat{T}A^{1/2}h = A^{1/2}Th \quad (h \in \mathcal{H}). \tag{1.6}$$

Since $\widehat{T}$ is orthogonally mean ergodic on $\overline{\mathcal{R}(A)}$ it follows (see [14]) that the sequence $A_n(T) := n^{-1}\sum_{j=0}^{n-1} A^{1/2}T^j$ $(n \ge 1)$ strictly converges in $\mathcal{B}(\mathcal{H})$ (that is, $A_n(T)$ and $A_n(T)^*$ strongly converge in $\mathcal{B}(\mathcal{H})$), and we have

$$\lim_{n\to\infty} A_n(T)h = P_*A^{1/2}h, \quad \lim_{n\to\infty} A_n(T)^*h = A^{1/2}P_*h \tag{1.7}$$

for any $h \in \mathcal{H}$, where $P_* \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection onto $\mathcal{N}_*$. Clearly, the two limits in (1.7) coincide if and only if $\mathcal{N} = \mathcal{N}_*$, which, according to the terminology from [12, 14], means that the A-contraction T is A-*ergodic*. In this case, we briefly say that the operator T is A-*ergodic*.

An important class of A-ergodic operators is the class of *regular* A-*contractions* T, that is, satisfying the condition $AT = A^{1/2}TA^{1/2}$. For a regular A-contraction T we have $A - T^*A = A^{1/2}(A^{1/2} - T^*A^{1/2})$, whence it follows that $\mathcal{N}_* \subset \mathcal{N}$. This implies $A^{1/2}\mathcal{N}_* \subset A^{1/2}\mathcal{N} \subset \mathcal{N}_*$, hence T is A-ergodic on $\mathcal{H}$. Ergodic properties of regular A-contractions were studied in [13]. But those facts refer only to the strong convergence of the corresponding Cesàro means.

In the present paper we introduce and study the concept of uniform (quasi-) A-ergodicity for A-contractions, an analogous notion in this setting to that of orthogonally uniform ergodicity (defined above). We obtain similar versions for A-contractions of some results of Lin [8], Mbekhta–Zemánek [9] and Grabiner–Zemánek [6]. Especially, we want to stress the role of some closed operator ranges in the uniformly ergodic behavior of A-contractions.

2. Ranges and null-spaces for A-contractions. In this section we study certain properties of ranges and null-spaces of operators which play an important role in the ergodic behavior of A-contractions. We begin with facts concerning null-spaces.

Proposition 2.1. *If T is an A-contraction on $\mathcal{H}$ then*

$$\mathcal{N} = \mathcal{N}(A^{1/2}(I-T)^m) = \mathcal{N}(A(I-T)^m) = \mathcal{N}((I-T^*)^m A) \tag{2.1}$$

and

$$\mathcal{N}_* = \mathcal{N}((I-T^*)^m A^{1/2}), \tag{2.2}$$

for any integer $m \geq 2$.

Proof. Let T be an A-contraction and let $h \in \mathcal{N}(A^{1/2}(I-T)^2)$. Then $k = (I-T)h \in \mathcal{N}$, and since $\mathcal{N}$ is invariant for T we have $A^{1/2}T^j k = A^{1/2}k$ for $j \geq 1$. Thus for $n \geq 1$ we get

$$A^{1/2}k = \frac{1}{n}\sum_{j=0}^{n-1} A^{1/2}T^j(I-T)h = \frac{1}{n}\,(A^{1/2}h - A^{1/2}T^n h),$$

which implies by (1.3) that $\|A^{1/2}k\| \leq (2/n)\|A^{1/2}h\| \to 0$ $(n \to \infty)$. Hence $A^{1/2}k = 0$, which means $h \in \mathcal{N}$, and so we have proved the inclusion $\mathcal{N}(A^{1/2}(I-T)^2) \subset \mathcal{N}$. The converse inclusion is true because $\mathcal{N}$ is invariant for T. Thus, we have the first equality in (2.1) for $m = 2$. Also, the second equality in (2.1) for any m is based on the fact that the operator $A^{1/2}$ is injective on its range.

Now, we show (2.2) for $m = 2$. Let $h \in \mathcal{N}((I-T^*)^2A^{1/2})$, so that $k = (I-T^*)A^{1/2}h \in \mathcal{N}(I-T^*)$. Then $T^{*j}k = k$ for $j \geq 1$, hence for $n \geq 1$,

$$k = \frac{1}{n}\sum_{j=0}^{n-1} T^{*j}(I-T^*)A^{1/2}h = \frac{1}{n}\,(A^{1/2}h - T^{*n}A^{1/2}h).$$

Since (1.3) implies $T^{*n}AT^n \le A$ we have $\|T^{*n}A^{1/2}\| \le 1$ (as $\|A\| = 1$), and obtain

$$\|k\| \le \frac{1}{n}\,(\|A^{1/2}h\| + \|T^{*n}A^{1/2}\|\,\|h\|) \le \frac{2}{n}\,\|h\|$$

for any $n \ge 1$, and consequently $k = 0$. Thus, $h \in \mathcal{N}_*$ and we have proved the inclusion $\mathcal{N}((I-T^*)^2A^{1/2}) \subset \mathcal{N}_*$. As the converse inclusion is trivial, the equality (2.2) is true for $m = 2$.

Considering now the last equality in (2.1), let $h \in \mathcal{N}((I-T^*)^2A)$. Then by (2.2) for $m = 2$ we have $A^{1/2}h \in \mathcal{N}((I-T^*)^2A^{1/2}) = \mathcal{N}_*$, therefore $h \in \mathcal{N}((I-T^*)A) = \mathcal{N}$. Conversely, if $h \in \mathcal{N}$ then as above $A^{1/2}h \in \mathcal{N}_*$ and by (2.2), the case $m = 2$, one has $(I-T^*)^2Ah = 0$, that is, $h \in \mathcal{N}((I-T^*)^2A)$. Hence, the last equality in (2.1) also holds for $m = 2$.

Finally, all equalities of (2.1) and (2.2) for $m > 2$ can be obtained by recurrence. ■

Corollary 2.2. *If T is A-ergodic then*

$$\begin{aligned}\mathcal{N} &= \mathcal{N}(A^{1/2}(I-T)^m) = \mathcal{N}(A(I-T)^m)\\ &= \mathcal{N}((I-T^*)^mA) = \mathcal{N}((I-T^*)^mA^{1/2})\end{aligned}$$

and the orthogonal decomposition

$$\mathcal{H} = \overline{\mathcal{R}(A^{1/2}(I-T)^m)} \oplus \mathcal{N}$$

holds for any integer $m \ge 2$.

Clearly, the case $m = 1$ holds trivially in this corollary. Now, we can infer certain relations between the orthogonal complements of the subspaces from (2.1) and (2.2), that is, between the corresponding operator ranges. But in ergodic theory it is important to know when some of these ranges are closed. Concerning this, we have the following.

Theorem 2.3. *Let T be a regular A-contraction on $\mathcal{H}$.*

(i) *$\mathcal{R}(A(I-T)^p)$ is closed for some (equivalently, all) $p \ge 1$ if and only if $\mathcal{R}(A^{1/2}(I-T)^q)$ is closed for some (equivalently, all) $q \ge 1$. In this case*

$$\mathcal{R}(A(I-T)^p) = \mathcal{R}(A^{1/2}(I-T)^q), \quad p, q \ge 1. \tag{2.3}$$

(ii) *$\mathcal{R}((I-T^*)^pA)$ is closed for some (equivalently, all) $p \ge 2$ if and only if $\mathcal{R}((I-T^*)^qA^{1/2})$ is closed for some (equivalently, all) $q \ge 1$. In this case*

$$\mathcal{R}((I-T^*)^pA) = \mathcal{R}((I-T^*)^qA^{1/2}), \quad p \ge 2,\ q \ge 1. \tag{2.4}$$

If furthermore $\mathcal{R}((I-T^*)A)$ is closed, then the relations (2.4) hold true for $p, q \ge 1$.

Proof. (i) Suppose that $\mathcal{R}(A(I-T)^m)$ is closed for some integer $m \geq 2$. Then from (2.1) and (2.2) we infer for $1 \leq p \leq m$,

$$\begin{aligned}\overline{\mathcal{R}(A(I-T)^p)} = \overline{\mathcal{R}(A^{1/2}(I-T)^p)} = \overline{\mathcal{R}(A(I-T)^m)} &= \mathcal{R}(A(I-T)^m)\\ &= \mathcal{R}(A^{1/2}(I-T)^p A^{1/2}(I-T)^{m-p})\\ &\subset \mathcal{R}(A^{1/2}(I-T)^p) \cap \mathcal{R}(A(I-T)^p),\end{aligned}$$

where we have used the fact that T is a regular A-contraction. This implies

$$\mathcal{R}(A(I-T)^p) = \mathcal{R}(A^{1/2}(I-T)^p) = \mathcal{R}(A(I-T)^m) =: \mathcal{R}$$

for $1 \leq p \leq m$, and that $\mathcal{R}$ is a closed invariant subspace for $A^{1/2}(I-T)$. This operator is surjective on the subspace $\mathcal{R} = \mathcal{R}(A^{1/2}(I-T)^{m-1}) = \mathcal{R}(A(I-T)^m)$, and since T is a regular A-contraction, we also have

$$\mathcal{R}(A(I-T)^{m+1}) = A^{1/2}(I-T)\mathcal{R}(A^{1/2}(I-T)^m) = A^{1/2}(I-T)\mathcal{R} = \mathcal{R}.$$

In particular, $\mathcal{R}(A(I-T)^{m+1})$ is closed. Using the above argument, one shows by recurrence that all subspaces $\mathcal{R}(A(I-T)^p)$ and $\mathcal{R}(A^{1/2}(I-T)^q)$ $(p, q \geq 1)$ coincide, that is, the relations (2.3) hold, if $\mathcal{R}(A(I-T)^m)$ is closed for some $m \geq 2$.

Now assume that $\mathcal{R}(A(I-T))$ is closed. Since T is a regular A-contraction one has

$$\overline{\mathcal{R}(A^{1/2}(I-T))} = \overline{\mathcal{R}(A(I-T))} = \mathcal{R}(A(I-T)) \subset \mathcal{R}(A^{1/2}(I-T)),$$

and it follows that $\mathcal{R}(A^{1/2}(I-T)) = \mathcal{R}(A(I-T))$. Then, using the decomposition (1.5) we deduce that

$$\begin{aligned}\mathcal{R}(A(I-T)^2) &= A^{1/2}(I-T)\mathcal{R}(A^{1/2}(I-T))\\ &= A^{1/2}(I-T)\mathcal{H} = \mathcal{R}(A^{1/2}(I-T)),\end{aligned}$$

and in particular $\mathcal{R}(A(I-T)^2)$ is closed. This implies by the above remark that the relations (2.3) hold true for any $p, q \geq 1$.

Conversely, if $\mathcal{R}(A^{1/2}(I-T)^m)$ is closed for some integer $m \geq 1$, then by Proposition 2.1 we have the decomposition

$$\mathcal{H} = \mathcal{R}(A^{1/2}(I-T)^m) \oplus \mathcal{N}(A^{1/2}(I-T)^m),$$

whence as above $\mathcal{R}(A(I-T)^{2m}) = \mathcal{R}(A^{1/2}(I-T)^m)$. Hence $\mathcal{R}(A(I-T)^{2m})$ is closed if $\mathcal{R}(A^{1/2}(I-T)^m)$ is closed.

(ii) Suppose that $\mathcal{R}((I-T^*)^m A)$ is closed for some integer $m \geq 2$. Since T is a regular A-contraction, from (2.1) and (2.2) we have, for $1 \leq q \leq m$,

$$\begin{aligned}\overline{\mathcal{R}((I-T^*)^q A^{1/2})} = \overline{\mathcal{R}((I-T^*)^m A)} &= \mathcal{R}((I-T^*)^m A)\\ &= \mathcal{R}((I-T^*)^q A^{1/2}(I-T^*)^{m-q} A^{1/2}) \subset \mathcal{R}((I-T^*)^q A^{1/2}),\end{aligned}$$

whence

$$\mathcal{R}((I-T^*)^q A^{1/2}) = \mathcal{R}((I-T^*)^m A) =: \mathcal{R}_*, \qquad 1 \leq q \leq m.$$

It follows that $\mathcal{R}_* = \mathcal{R}((I-T^*)^{m-1}A^{1/2})$ is a closed invariant subspace for $I-T^*$ and $(I-T^*)\mathcal{R}_* = \mathcal{R}_*$, that is, $I-T^*$ is surjective on $\mathcal{R}_*$. Hence, we also have

$$\mathcal{R}((I-T^*)^{m+1}A) = (I-T^*)\mathcal{R}_* = \mathcal{R}_*,$$

and in particular $\mathcal{R}((I-T^*)^{m+1}A)$ is a closed subspace. So, one finds by recurrence that all subspaces $\mathcal{R}((I-T^*)^q A^{1/2})$ and $\mathcal{R}((I-T^*)^p A)$ coincide, for any $q \geq 1$ and $p \geq m$.

Now, since $\mathcal{R}((I-T^*)^j A^{1/2})$ is closed for $1 \leq j \leq m-1$, Proposition 2.1 shows that

$$\mathcal{H} = \mathcal{R}((I-T^*)^j A^{1/2}) \oplus \mathcal{N}((I-T^*)A^{1/2}), \tag{2.5}$$

and using this decomposition and the fact that T is a regular A-contraction we obtain

$$\mathcal{R}((I-T^*)^{j+1}A) = (I-T^*)A^{1/2}\mathcal{R}((I-T^*)^j A^{1/2}) = \mathcal{R}((I-T^*)A^{1/2}).$$

Hence all subspaces $\mathcal{R}((I-T^*)^q A^{1/2})$ and $\mathcal{R}((I-T^*)^p A)$ coincide for any $q \geq 1$ and $p \geq 2$, that is, we have the relations (2.4), if the range $\mathcal{R}((I-T^*)A)$ is closed for some $m \geq 2$.

Conversely, suppose that $\mathcal{R}((I-T^*)^m A^{1/2})$ is closed for some integer $m \geq 1$. Then by Proposition 2.1 we have

$$\mathcal{H} = \mathcal{R}((I-T^*)^m A^{1/2}) \oplus \mathcal{N}((I-T^*)^m A^{1/2}),$$

and from this we get $\mathcal{R}((I-T^*)^{2m}A) = \mathcal{R}((I-T^*)^m A^{1/2})$. In particular, $\mathcal{R}((I-T^*)^{2m}A)$ is closed, where $2m \geq 2$.

Next, we assume that $\mathcal{R}((I-T^*)A)$ is closed. Then, as above,

$$\overline{\mathcal{R}((I-T^*)A^{1/2})} = \mathcal{R}((I-T^*)A) \subset \mathcal{R}((I-T^*)A^{1/2}),$$

hence $\mathcal{R}((I-T^*)A^{1/2}) = \mathcal{R}((I-T^*)A)$, and this range is closed. So, from the decomposition (2.5) for $j=1$ we obtain

$$\mathcal{R}((I-T^*)^2 A) = \mathcal{R}((I-T^*)A^{1/2}) = \mathcal{R}((I-T^*)A),$$

this range being closed. Finally, using the above remark, we conclude that all ranges from (2.4) coincide. ∎

In the ergodic case we have the following theorem, partially suggested by Proposition 4.5 from [6].

Theorem 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be A-ergodic.*

(i) *$\mathcal{R}(A^{1/2}-A^{1/2}T)$ is closed if and only if $\mathcal{R}((A^{1/2}-A^{1/2}T)^2)$ is closed. In this case, the ranges coincide.*
(ii) *$\mathcal{R}(A^{1/2}-T^*A^{1/2})$ is closed if and only if $\mathcal{R}((A^{1/2}-T^*A^{1/2})^2)$ is closed. In this case, the ranges coincide.*

(iii) *If* $\mathcal{R}(A^{1/2}-A^{1/2}T)=\mathcal{R}(A)$ *and this range is closed, then the operators* $(A^{1/2}-A^{1/2}T)|_{\mathcal{R}(A)}$ *and* $I-T^*|_{\mathcal{R}(A)}$ *are invertible in* $\mathcal{B}(\mathcal{R}(A))$ *and*

$$[(A^{1/2}-A^{1/2}T)|_{\mathcal{R}(A)}]^{-1}A^{1/2}h = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1} j(A^{1/2}|_{\mathcal{R}(A)})^{-1}A^{1/2}T^{n-j-1}h \tag{2.6}$$

and

$$((I-T^*)|_{\mathcal{R}(A)})^{-1}A^{1/2}h=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1} jT^{*(n-j-1)}A^{1/2}h, \tag{2.7}$$

for any $h\in\mathcal{H}$.

Proof. (i) Let $B=A^{1/2}-A^{1/2}T$ and $\mathcal{R}=\mathcal{R}(B)$. First suppose that the subspace $\mathcal{R}$ is closed. Then

$$\mathcal{R}=B\mathcal{H}=B(\mathcal{R}\oplus\mathcal{N})=B\mathcal{R}=\mathcal{R}(B^2),$$

hence $\mathcal{R}(B^2)$ is closed. Conversely, if $\mathcal{R}(B^2)$ is closed then

$$\mathcal{R}=B(\overline{\mathcal{R}}\oplus\mathcal{N})=B\overline{\mathcal{R}}\subset\overline{B^2\mathcal{H}}=\mathcal{R}(B^2)\subset\mathcal{R},$$

therefore $\mathcal{R}=\mathcal{R}(B^2)$, and consequently $\mathcal{R}$ is closed.

(ii) Let $\mathcal{R}_*=\mathcal{R}(B^*)$. If $\mathcal{R}_*$ is closed then, as $\mathcal{N}=\mathcal{N}_*$, we have

$$\mathcal{R}_*=B^*(\mathcal{R}_*\oplus\mathcal{N}_*)=B^*\mathcal{R}_*=\mathcal{R}(B^{*2}),$$

hence $\mathcal{R}_*=\mathcal{R}(B^{*2})$, and in particular $\mathcal{R}(B^{*2})$ is closed. Conversely, if $\mathcal{R}(B^{*2})$ is closed then as T is A-ergodic one has $\mathcal{H}=\overline{\mathcal{R}(B^*)}\oplus\mathcal{N}_*$, so

$$\mathcal{R}_*=B^*\mathcal{H}=B^*\overline{\mathcal{R}(B^*)}\subset\overline{\mathcal{R}(B^{*2})}=\mathcal{R}(B^{*2})\subset\mathcal{R}_*,$$

that is, $\mathcal{R}_*=\mathcal{R}(B^{*2})$ and $\mathcal{R}_*$ is closed.

(iii) Now suppose that $\mathcal{R}=\mathcal{R}(A)$ and that this range is closed. Then $\mathcal{R}(A)=\mathcal{R}(A^{1/2})$. Indeed, as $\mathcal{N}(A)=\mathcal{N}(A^{1/2})$ we have $\overline{\mathcal{R}(A^{1/2})}=\overline{\mathcal{R}(A)}=\mathcal{R}(A)\subset\mathcal{R}(A^{1/2})$, hence $\mathcal{R}(A^{1/2})$ is closed and so $\mathcal{R}(A)=\mathcal{R}(A^{1/2})$. Hence $A^{1/2}$ is invertible in $\mathcal{B}(\mathcal{R}(A))$. Also, if $\widehat{T}$ is the contraction on $\mathcal{R}(A)$ given in (1.6), from the assumption $\mathcal{R}=\mathcal{R}(A)$ we infer that $\mathcal{R}(I-\widehat{T})=\mathcal{R}(A)$, or equivalently $\mathcal{N}(I-\widehat{T})=\{0\}$. So, $I-\widehat{T}$ is invertible in $\mathcal{B}(\mathcal{R}(A))$ and by Proposition 4.5 of [6] we have, for $h'\in\mathcal{H}$,

$$(I-\widehat{T})^{-1}A^{1/2}h'=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1} j\widehat{T}^{n-j-1}A^{1/2}h'=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1} jA^{1/2}T^{n-j-1}h'.$$

Since $\mathcal{R}(A)=\mathcal{R}(A^{1/2})$, for any $h\in\mathcal{H}$ there exists $k\in\mathcal{H}$ such that $A^{1/2}h=(I-\widehat{T})Ak=(A^{1/2}-A^{1/2}T)A^{1/2}k$. Hence $\mathcal{R}(A)$ is an invariant (in fact, reducing) subspace for $A^{1/2}-A^{1/2}T$ and the operator $V:=(A^{1/2}-A^{1/2}T)|_{\mathcal{R}(A)}$

is surjective on $\mathcal{R}(A)$. It is also injective because if $VA^{1/2}h = 0$ then $A^{1/2}h \in \mathcal{R}(A) \cap \mathcal{N} = \{0\}$ by our assumption that $\mathcal{R}(A) = \mathcal{N}^{\perp}$, therefore $A^{1/2}h = 0$. Thus, V is invertible in $\mathcal{B}(\mathcal{R}(A))$, and for h and k as above we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} jA^{1/2}T^{n-j-1}h = (I - \widehat{T})^{-1}A^{1/2}h = Ak = A^{1/2}V^{-1}A^{1/2}h,$$

whence the formula (2.6) immediately follows.

Next, we remark that the subspace $\mathcal{R}_* = \mathcal{R}(A^{1/2} - T^*A^{1/2})$ is closed, in fact $\mathcal{R}_* = \mathcal{R} = \mathcal{R}(A)$. Indeed, as $\mathcal{R}(I - \widehat{T}) = \mathcal{R}$ is closed, Theorem 1 of [8] shows $\mathcal{R}(I - \widehat{T}^*)$ is also closed, where $\widehat{T}^* = (\widehat{T})^*$, hence

$$\begin{aligned}\mathcal{R}_* &= (I - T^*)A\mathcal{H} = A^{1/2}(I - \widehat{T}^*)A^{1/2}\mathcal{H} = A^{1/2}\mathcal{R}(I - \widehat{T}^*)\\ &= A^{1/2}\mathcal{R}(I - \widehat{T}) = A^{1/2}\mathcal{R} = A^{1/2}\mathcal{R}(A),\end{aligned}$$

$A^{1/2}$ being invertible on $\mathcal{R}(A)$. Since $\overline{\mathcal{R}_*} = \overline{\mathcal{R}}$, it follows that $\mathcal{R}_* = \mathcal{R} = \mathcal{R}(A)$. This implies that the operator $W := I - T^*|_{\mathcal{R}(A)}$ is surjective on $\mathcal{R}(A)$. It is also injective because

$$\mathcal{R}(A) \cap \mathcal{N}(I - T^*) = \mathcal{R}(W) \cap \mathcal{N}(W) = \{0\}$$

by Theorem 3.2 of [14]. Thus, W is invertible in $\mathcal{B}(\mathcal{R}(A))$ and Proposition 4.5 of [6] yields

$$W^{-1}A^{1/2}h = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} jT^{*(n-j-1)}A^{1/2}h \quad (h \in \mathcal{H}),$$

which is the formula (2.7). ■

Remark 2.5. Let T be an A-contraction on $\mathcal{H}$ and $\mathcal{R}, \mathcal{R}_*$ be as in the previous proof. Then the condition $\overline{\mathcal{R}} = \overline{\mathcal{R}(A)}$ is equivalent to $\mathcal{N}_* = \mathcal{N}(A)$. In this case it follows that T is A-ergodic, because $\mathcal{N}(A)$ reduces A, hence we also have $\overline{\mathcal{R}_*} = \overline{\mathcal{R}(A)}$. Moreover, by the same argument as in the previous proof, $\mathcal{R} = \mathcal{R}(A)$ implies $\mathcal{R}_* = \mathcal{R}(A)$, and therefore $\mathcal{R} = \mathcal{R}(A) = \mathcal{R}_*$.

If $\mathcal{R} = \mathcal{R}(A)$, then by Theorem 2.4 of [14] we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} A^{1/2}T^jh = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^*A^{1/2}h = 0 \qquad (h \in \mathcal{H}).$$

3. Uniform quasi-A-ergodicity. In this section we investigate certain conditions under which for a given A-contraction T on $\mathcal{H}$ there exists an operator $Q \in \mathcal{B}(\mathcal{H})$ such that $n^{-1}\sum_{j=0}^{n-1} A^{1/2}T^j$ converges uniformly to Q in $\mathcal{B}(\mathcal{H})$. We then say that T is *uniformly quasi-A-ergodic*. In this case, we infer from (1.7) that $Q = P_*A^{1/2}$, where P_* is the orthogonal projection onto the null-space $\mathcal{N}_*$. Obviously, the limit $Q = P_*A^{1/2}$ of such an A-contraction T

is a self-adjoint operator if and only if T is A-ergodic. An A-contraction can be uniformly quasi-A-ergodic and not A-ergodic, as in the following

EXAMPLE 3.1. Let $A, T \in \mathcal{B}(\mathbb{C}^2)$ be the operators given by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $AT = T^*AT = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, therefore T is an A-contraction on $\mathbb{C}^2$. Since A is invertible and

$$A^{1/2} = \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}, \quad A^{1/2}T = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix},$$

the operator $\widehat{T} = A^{1/2}TA^{-1/2}$ on $\mathbb{C}^2$ has the form

$$\widehat{T} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Furthermore, since $\mathcal{N}(A) = \{0\}$ we have

$$\mathcal{N}_* = \mathcal{N}(I - \widehat{T}) = \{(\lambda, 2\lambda) : \lambda \in \mathbb{C}\},$$

and $\widehat{T}$ is just the orthogonal projection corresponding to this null-space. Then, as $T^2 = T$, we obtain

$$\frac{1}{n}\sum_{j=0}^{n-1} A^{1/2}T^j - \widehat{T}A^{1/2} = \frac{1}{n}A^{1/2}[I + (n-1)T] - A^{1/2}T = \frac{1}{n}A^{1/2}(I - T) \to 0$$

as $n \to \infty$. Hence T is uniformly quasi-A-ergodic. On the other hand, we have $\mathcal{N} = \{0\} \oplus \mathbb{C}$, so $\mathcal{N} \neq \mathcal{N}_*$, and consequently T is not A-ergodic.

PROPOSITION 3.2. *Let T be an A-contraction on $\mathcal{H}$.*

(i) *If $\widehat{T}$ is orthogonally uniformly ergodic on $\overline{\mathcal{R}(A)}$, then T is uniformly quasi-A-ergodic on $\mathcal{H}$.*

(ii) *If A is an orthogonal projection in $\mathcal{B}(\mathcal{H})$, then the converse of* (i) *holds true.*

Proof. (i) If $P_0 \in \mathcal{B}(\overline{A\mathcal{H}})$ is the orthogonal projection onto $\mathcal{N}(I - \widehat{T})$, then for $n \geq 1$ we have (since $\|A\| = 1$)

$$\left\|\frac{1}{n}\sum_{j=0}^{n-1} A^{1/2}T^j - P_*A^{1/2}\right\| \leq \sup_{\|h\|\leq 1} \left\|\left(\frac{1}{n}\sum_{j=0}^{n-1} \widehat{T}^j - P_0\right)A^{1/2}h\right\|$$

$$\leq \left\|\frac{1}{n}\sum_{j=0}^{n-1} \widehat{T}^j - P_0\right\|.$$

Hence, if the contraction $\widehat{T}$ is orthogonally uniformly ergodic on $\overline{\mathcal{R}(A)}$, then T is uniformly quasi-A-ergodic on $\mathcal{H}$.

(ii) Suppose that A is an orthogonal projection in $\mathcal{B}(\mathcal{H})$. As $A = A^2$, for $n \geq 1$ we obtain

$$\begin{aligned}\left\|\frac{1}{n}\sum_{j=0}^{n-1}\widehat{T}^j - P_0\right\| &= \sup_{h\in A\mathcal{H},\|h\|\leq 1}\left\|\left(\frac{1}{n}\sum_{j=0}^{n-1}\widehat{T}^j - P_0\right)Ah\right\| \\ &= \sup_{h\in A\mathcal{H},\|h\|\leq 1}\left\|\left(\frac{1}{n}\sum_{j=0}^{n-1}AT^j - P_*A\right)h\right\| \leq \left\|\frac{1}{n}\sum_{j=0}^{n-1}AT^j - P_*A\right\|.\end{aligned}$$

Consequently, if T is uniformly quasi-A-ergodic on $\mathcal{H}$, then $\widehat{T}$ is orthogonally uniformly ergodic on $\mathcal{R}(A)$. ■

Remark 3.3. It is easy to see that, if A is an orthogonal projection in $\mathcal{B}(\mathcal{H})$, then the matrix form of the A-contractions T on $\mathcal{H}$ relative to the decomposition $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{R}(A)$ is

$$T = \begin{pmatrix} S & R \\ 0 & C \end{pmatrix},$$

where $S \in \mathcal{B}(\mathcal{N}(A))$ and $R \in \mathcal{B}(\mathcal{R}(A), \mathcal{N}(A))$ are arbitrary operators, while C is a contraction on $\mathcal{R}(A)$. Furthermore, we have $CAh = ATh = \widehat{T}Ah$ for $h \in \mathcal{H}$, therefore $C = \widehat{T}$. We also observe that $ATh = \widehat{T}Ah = ATAh$ for $h \in \mathcal{H}$, which means $AT = ATA$. Hence T is a regular A-contraction if A is an orthogonal projection.

Conversely, we can show that a regular A-contraction T on $\mathcal{H}$ is also a Q-contraction, Q being the orthogonal projection onto $\overline{\mathcal{R}(A)}$. In addition, in this case the two contractions $\widehat{T}$ on $\overline{\mathcal{R}(A)} = \mathcal{R}(Q)$ associated to T relative to A and Q (as in (1.6)) coincide. Thus, from Proposition 3.2 it follows that if T is uniformly quasi-Q-ergodic then T is also uniformly quasi-A-ergodic. The converse is also true if $\mathcal{R}(A)$ is closed, that is, T uniformly quasi-A-ergodic implies T uniformly quasi-Q-ergodic; this is a consequence of Proposition 3.2 and Corollary 4.2 below.

Now, from the previous proposition and Theorem 1 of [9] we infer the following

Corollary 3.4. *If T is an A-contraction on $\mathcal{H}$ such that the range $\mathcal{R}(A^{1/2}(I-T)^m)$ is closed for some integer $m \geq 1$, then T is uniformly quasi-A-ergodic.*

Proof. Firstly, we have

$$\begin{aligned}\mathcal{R}(A^{1/2}(I-T)^m) &= (I-\widehat{T})^m A^{1/2}\mathcal{H} \subset \mathcal{R}((I-\widehat{T})^m) \\ &\subset \overline{\mathcal{R}((I-\widehat{T})^m)} = \overline{\mathcal{R}(A^{1/2}(I-T)^m)}.\end{aligned}$$

Thus, if $\mathcal{R}(A^{1/2}(I-T)^m)$ is closed, then so is $\mathcal{R}((I-\widehat{T})^m)$, and Theorem 1 of [9] implies that the contraction $\widehat{T}$ is orthogonally uniformly ergodic on $\overline{\mathcal{R}(A)}$. Then Proposition 3.2 shows that T is uniformly quasi-A-ergodic on $\mathcal{H}$. ∎

REMARK 3.5. If T is a regular A-contraction, we can prove the assertion of the above corollary directly, as follows. First notice that in view of Theorem 2.3 we can suppose that $\mathcal{R}=\mathcal{R}(B)$ is closed, where $B=A^{1/2}-A^{1/2}T$. Since $\mathcal{H}=\mathcal{R}\oplus\mathcal{N}$, we have $\mathcal{R}=B\mathcal{H}=B\mathcal{R}$ so the operator B is surjective on $\mathcal{R}$. But it is also injective on $\mathcal{R}$ because $\mathcal{N}(B|_{\mathcal{R}})\subset\mathcal{R}\cap\mathcal{N}=\{0\}$. Hence $C=B|_{\mathcal{R}}$ is invertible in $\mathcal{B}(\mathcal{R})$.

Now we prove that $B_n:=n^{-1}\sum_{j=0}^{n-1}A^{1/2}T^j$ $(n\geq 1)$ converges uniformly to $A^{1/2}P$ as $n\to\infty$, P being the orthogonal projection onto the null-space $\mathcal{N}$ (that is, $P=P_*$ in this case). Let $h\in\mathcal{H}$ with $\|h\|=1$. We write $h=h_0+h_1$ where $h_0\in\mathcal{R}$ and $h_1\in\mathcal{N}$. Then $B_nh=B_nh_0+A^{1/2}h_1=B_nh_0+A^{1/2}Ph$, and since C is invertible, one has $h_0=CC^{-1}h_0=Ch_2$. Thus, we deduce that

$$
\begin{aligned}
\|B_nh-A^{1/2}Ph\|=\|B_nh_0\|&=\left\|\frac{1}{n}\sum_{j=0}^{n-1}A^{1/2}T^j(A^{1/2}-A^{1/2}T)h_2\right\|\\
&=\frac{1}{n}\left\|\sum_{j=0}^{n-1}(AT^j-AT^{j+1})h_2\right\|=\frac{1}{n}\,\|A(I-T^n)C^{-1}h_0\|\\
&\leq\frac{2}{n}\,\|C^{-1}\|,
\end{aligned}
$$

because $\|h_0\|\leq\|h\|\leq 1$. Consequently, $\|B-A^{1/2}P\|\to 0$ as $n\to\infty$, i.e. T is uniformly quasi-A-ergodic on $\mathcal{H}$. In fact, T is A-ergodic, being a regular A-contraction on $\mathcal{H}$.

We also notice that, concerning the converse to the assertion of Corollary 3.4, we will show in the following section a partial result, using a stronger concept than uniform quasi-A-ergodicity for A-contractions.

4. Uniform A-ergodicity. We say that an A-contraction T on $\mathcal{H}$ is *uniformly A-ergodic* (briefly, T is *uniformly A-ergodic*) if it is A-ergodic and uniformly quasi-A-ergodic.

Concerning the converse to the assertion of Corollary 3.4 one has

THEOREM 4.1. *If $T\in\mathcal{B}(\mathcal{H})$ is uniformly A-ergodic and $\mathcal{R}(A)$ is closed, then $\mathcal{R}(A^{1/2}-A^{1/2}T)=\mathcal{R}(A^{1/2}-T^*A^{1/2})$ and this range is closed.*

Proof. Let $\mathcal{R}, B_n$ and P be as in Remark 3.5. Suppose that $\mathcal{R}(A)$ is closed and $\|B_n-PA^{1/2}\|\to 0$ as $n\to\infty$.

First we show that $\mathcal{R}$ is closed. Since $\mathcal{R}(A) = \mathcal{R}(A^{1/2})$, it follows that $\mathcal{R} = \mathcal{R}(I - \widehat{T})$, hence the subspace $\overline{\mathcal{R}}$ reduces $\widehat{T}$. We define $S := \widehat{T}|_{\overline{\mathcal{R}}}$, which is a contraction on $\overline{\mathcal{R}}$. As $A^{1/2}$ is invertible in $\mathcal{B}(A^{1/2}\mathcal{H})$, for $r \in \overline{\mathcal{R}}$ and $n \geq 1$ we obtain

$$\frac{1}{n}\sum_{j=0}^{n-1} S^j r = \frac{1}{n}\sum_{j=0}^{n-1} A^{1/2}T^j A^{-1/2} r = B_n A^{-1/2} r.$$

But $A^{-1/2}r \in \overline{\mathcal{R}}$, because T being A-ergodic one has

$$\overline{\mathcal{R}} = \overline{(A - AT)\mathcal{H}} = A^{1/2}\overline{\mathcal{R}},$$

hence $A^{1/2}$ is invertible on $\overline{\mathcal{R}}$. Also, one obtains

$$B_n\overline{\mathcal{R}} \subset \bigvee_{j=0}^{n-1} S^j A^{1/2}\overline{\mathcal{R}} = \bigvee_{j=0}^{n-1} S^j\overline{\mathcal{R}} \subset \overline{\mathcal{R}},$$

therefore $\overline{\mathcal{R}}$ is invariant for B_n, $n \geq 1$. As $P\overline{\mathcal{R}} = \{0\}$, we obtain for $r \in \overline{\mathcal{R}}$ and $n \geq 1$,

$$\left\| \frac{1}{n}\sum_{j=0}^{n-1} S^j r \right\| = \|(B_n - PA^{1/2})A^{-1/2}r\| \leq \|A^{-1/2}|_{\overline{\mathcal{R}}}\| \, \|B_n - PA^{1/2}\| \, \|r\|,$$

whence it follows that $n^{-1}\sum_{j=0}^{n-1} S^j \to 0$ $(n \to \infty)$ uniformly in $\mathcal{B}(\overline{\mathcal{R}})$. So, there exists an integer $n_0 \geq 1$ such that

$$\left\| \frac{1}{n_0}\sum_{j=0}^{n_0-1} S^j \right\| < 1,$$

hence $I - n_0^{-1}\sum_{j=0}^{n_0-1} S^j$ is an invertible operator in $\mathcal{B}(\overline{\mathcal{R}})$. Since

$$I - \frac{1}{n_0}\sum_{j=0}^{n_0-1} S^j = \frac{1}{n_0}(I - S)\sum_{j=1}^{n_0-1}\sum_{i=0}^{j-1} S^i,$$

the invertibility of the product and the fact that the factors on the right side commute imply that the operator $I - S$ is invertible in $\mathcal{B}(\overline{\mathcal{R}})$. Therefore $(I - S)\overline{\mathcal{R}} = \overline{\mathcal{R}}$, which leads to

$$(A^{1/2} - A^{1/2}T)\overline{\mathcal{R}} = (I - \widehat{T})A^{1/2}\overline{\mathcal{R}} = (I - \widehat{T})\overline{\mathcal{R}} = (I - S)\overline{\mathcal{R}} = \overline{\mathcal{R}},$$

and next

$$\overline{\mathcal{R}} = (A^{1/2} - A^{1/2}T)\overline{\mathcal{R}} \subset (A^{1/2} - A^{1/2}T)\mathcal{H} = \mathcal{R}.$$

Consequently, the subspace $\mathcal{R}$ is closed.

Now, we show that the ranges $\mathcal{R}_* = \mathcal{R}(A^{1/2} - T^*A^{1/2})$ and $\mathcal{R}$ coincide. Being in the A-ergodic case, we know that $\overline{\mathcal{R}_*} = \overline{\mathcal{R}}$, hence $\mathcal{R}_* \subset \mathcal{R}$. Since

$T^*A^{1/2}h = A^{1/2}\widehat{T}^*h$ for any $h \in \mathcal{R}(A)$, we obtain

$$(A^{1/2} - T^*A^{1/2})\overline{\mathcal{R}}_* = A^{1/2}(I - \widehat{T}^*)\overline{\mathcal{R}}_* = A^{1/2}(I - S^*)\overline{\mathcal{R}} = A^{1/2}\overline{\mathcal{R}} = \mathcal{R},$$

whence

$$\mathcal{R} = (A^{1/2} - T^*A^{1/2})\overline{\mathcal{R}}_* \subset (A^{1/2} - T^*A^{1/2})\mathcal{H} = \mathcal{R}_*.$$

Finally, we get $\mathcal{R} = \mathcal{R}_*$, and the proof is finished. ∎

Corollary 4.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be A-ergodic with $\mathcal{R}(A)$ closed. The following assertions are equivalent*:

(i) *T is uniformly A-ergodic*;
(ii) *$\widehat{T}$ is orthogonally uniformly ergodic on $\mathcal{R}(A)$*;
(iii) *$\mathcal{R}(A^{1/2}(I - T)^m)$ is closed for some integer $m \geq 1$*;
(iv) *$\mathcal{R}(A - AT)$ is closed*;
(v) *$\mathcal{R}(A - T^*A)$ is closed*;
(vi) *$\mathcal{R}(A^{1/2} - T^*A^{1/2})$ is closed*;
(vii) *For every $h \in \overline{\mathcal{R}(A - AT)}$ one has*

$$\sup_{n\geq 1}\left\|\sum_{j=0}^{n} A^{1/2}T^j h\right\| < \infty. \tag{4.1}$$

Furthermore, if these conditions are satisfied we have

$$\begin{aligned}\mathcal{R}(A - AT) &= \mathcal{R}(A^{1/2} - A^{1/2}T) = \mathcal{R}(A^{1/2} - T^*A^{1/2})\\ &= \mathcal{R}(A - T^*A).\end{aligned}\tag{4.2}$$

Proof. Clearly, (i) implies (iii) (with $m = 1$) by Theorem 4.1, and (iii) is equivalent to (ii) by Theorem 1 of [9].

On the other hand, $\mathcal{R}(A)$ is a closed reducing subspace for $A^{1/2}$, therefore $A_0 := A^{1/2}|_{\mathcal{R}(A)}$ is an invertible operator on $\mathcal{R}(A)$. Since

$$\mathcal{R}(A - AT) = A_0\mathcal{R}(I - \widehat{T}) = A^{1/2}\mathcal{R}(I - \widehat{T}^*) = \mathcal{R}(A - T^*A),$$

because $A^{1/2}\widehat{T}^*h = T^*A^{1/2}h$ for $h \in \mathcal{R}(A)$, it follows from the above remark that (iii), (iv) and (v) are equivalent. Also, (v) implies

$$\mathcal{R}_* := \mathcal{R}(A^{1/2} - T^*A^{1/2}) \subset \mathcal{R}(A - T^*A) \subset \mathcal{R}_*,$$

hence $\mathcal{R}_* = \mathcal{R}(A - T^*A)$, and in particular $\mathcal{R}_*$ is a closed subspace. So, (v) implies (vi) and

$$\mathcal{R} := \mathcal{R}(A^{1/2} - A^{1/2}T) = \mathcal{R}(A - AT) = \mathcal{R}(A - T^*A).$$

Since we also have

$$\mathcal{R} = \overline{\mathcal{R}} = \overline{\mathcal{R}}_* \supset \mathcal{R}_* \supset (A^{1/2} - T^*A^{1/2})A^{1/2}\mathcal{H} = \mathcal{R},$$

it follows that $\mathcal{R} = \mathcal{R}_*$. Conversely, if the range $\mathcal{R}_*$ is closed, then

$$\mathcal{R}(A-T^*A)=A^{1/2}\mathcal{R}_*=\overline{A_0\mathcal{R}_*}=\overline{\mathcal{R}(A-T^*A)},$$

that is, $\mathcal{R}(A-T^*A)$ is closed, and consequently (vi) implies (v).

Now, (ii) implies by Corollary 1 of [8] that

$$\sup_{n\geq 1}\Big\|\sum_{j=0}^{n}\widehat{T}^j k\Big\|<\infty$$

for every $k\in\overline{\mathcal{R}(I-\widehat{T})}=\overline{\mathcal{R}}=\overline{\mathcal{R}(A-AT)}$, the last equality holding because T is A-ergodic. As $A^{1/2}k\in\overline{\mathcal{R}(I-\widehat{T})}$ for k as above, we can put $A^{1/2}k$ instead of k in the previous condition, to obtain (4.1). Therefore, (ii) implies (vii). Finally, the implication (vii)$\Rightarrow$(i) follows from the following proposition (where the closedness of $\mathcal{R}(A)$ is not necessary). ∎

Proposition 4.3. *If the A-contraction T on $\mathcal{H}$ is A-ergodic and satisfies the condition* (4.1), *then T is uniformly A-ergodic.*

Proof. Let T be as in the hypothesis and $\mathcal{R}$, B_n and P be as in Remark 3.5. Since the subspace $\overline{\mathcal{R}}$ is reducing for the operators $A^{1/2}T^j$, $j\geq 0$ (see [14]), it is also reducing for B_n, $n\geq 1$. Then from the condition (4.1) we infer that the sequence $\{nB_n|_{\overline{\mathcal{R}}}\}_{n\geq 1}$ is bounded in $\mathcal{B}(\overline{\mathcal{R}})$, therefore $\{B_n\}$ uniformly converges to zero in $\mathcal{B}(\overline{\mathcal{R}})$. Let $h\in\mathcal{H}$ with $\|h\|=1$. We put $h=h_0+h_1$ where $h_0\in\overline{\mathcal{R}}$ and $h_1\in\mathcal{N}=\overline{\mathcal{R}}^{\perp}$. Then $B_nh=B_nh_0+A^{1/2}h_1=B_nh_0+A^{1/2}Ph$. Therefore

$$\|(B_n-A^{1/2}P)h\|=\|B_nh_0\|\leq\|B_n|_{\overline{\mathcal{R}}}\|,$$

whence $\|B_n-A^{1/2}P\|\leq\|B_n|_{\overline{\mathcal{R}}}\|$, and hence $\|B_n-A^{1/2}P\|\to 0$ as $n\to\infty$. In conclusion, T is uniformly A-ergodic. ∎

Since in the A-ergodic case the above subspace $\overline{\mathcal{R}}$ reduces the operators $A^{1/2}T^j$ $(j\geq 0)$, the condition (4.1) is equivalent to

$$\sup_{n\geq 1}\Big\|\Big(\sum_{j=0}^{n}A^{1/2}T^j\Big)\Big|_{\overline{\mathcal{R}}}\Big\|<\infty. \tag{4.3}$$

Proposition 4.4. *Let T be a regular A-contraction on $\mathcal{H}$. If the range $\mathcal{R}(A^{1/2}-A^{1/2}T)$ is closed then the condition* (4.3) *is satisfied.*

Proof. Suppose that $\mathcal{R}=\mathcal{R}(A^{1/2}-A^{1/2}T)$ is a closed subspace. Then the operator $A^{1/2}-A^{1/2}T$ is invertible in $\mathcal{B}(\mathcal{R})$, being surjective by Theorem 2.4(i), and injective by (1.5), on $\mathcal{R}$. Therefore, there exists a constant $c>0$ such that $\|k\|\leq c\|h\|$ for every $h=(A^{1/2}-A^{1/2}T)k\in\mathcal{R}$ with $k\in\mathcal{H}$. Thus, using the fact that T is a regular A-contraction, we obtain for h as above

$$\Big\|\sum_{j=0}^{n} A^{1/2}T^j h\Big\| = \Big\|\sum_{j=0}^{n} A(T^j - T^{j+1})k\Big\| \le \|A(I-T^{n+1})k\|$$
$$\le \|A^{1/2}k\| + \|A^{1/2}T^{n+1}k\| \le 2c\|h\|.$$

Hence the condition (4.3) holds true. ∎

REMARK 4.5. For an A-contraction T on $\mathcal{H}$ one has in general $R_0 := \mathcal{R}(A-AT) \neq \mathcal{R}(A^{1/2}-A^{1/2}T) = \mathcal{R}$. But R_0 and $\mathcal{R}$ are contained in $\overline{\mathcal{R}(A)}$ and $\mathcal{R}_0 = \tilde{A}_0\mathcal{R}$, where $\tilde{A}_0 = A^{1/2}|_{\overline{\mathcal{R}(A)}}$ is an injective operator on $\overline{\mathcal{R}(A)}$. Hence $\mathcal{R} = (\tilde{A}_0)^{-1}\mathcal{R}_0$ is a closed subspace if $\mathcal{R}_0$ is. But if $\mathcal{R}(A)$ is closed then $\tilde{A}_0 = A_0$ is an invertible operator on $\mathcal{R}(A)$, and in this case $\mathcal{R}_0$ is closed if $\mathcal{R}$ is.

Next, we investigate the case when the range $\mathcal{R}_0$ is closed in the following theorem which completes Theorem 2.4 and Corollary 4.2.

THEOREM 4.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be A-ergodic with $\mathcal{R} = \mathcal{R}(A-AT)$ closed. Then the relations* (4.2) *hold true and*:

(i) *The subspace $\mathcal{R}$ is reducing for T, the operators A and $I-T$ are invertible on $\mathcal{R}$ and*

$$(I-T|_{\mathcal{R}})^{-1}r = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n-1} jT^{n-j-1}r \quad (r \in \mathcal{R}). \tag{4.4}$$

(ii) *The operators $T_0 := T|_{\mathcal{H}_0}$ where $\mathcal{H}_0 = \mathcal{R} \oplus \mathcal{N}(I-T)$, and $T_* := T^*|_{\overline{\mathcal{R}(A)}}$, are orthogonally uniformly ergodic on $\mathcal{H}_0$ and $\overline{\mathcal{R}(A)}$, respectively. Moreover, we have the relations*

$$\mathcal{R} = \mathcal{R}(I-T_0) = \mathcal{R}(I-T_*), \tag{4.5}$$

$$\overline{\mathcal{R}(A)} = \mathcal{R} \oplus \mathcal{N}(I-T^*), \quad \mathcal{N}(I-T^*) = \overline{A^{1/2}\mathcal{N}} = \mathcal{N}(I-T_*), \tag{4.6}$$

$$\overline{\mathcal{R}(I-T)} = \mathcal{R} \oplus \mathcal{N}(A), \quad \mathcal{N}(A) = \overline{(I-T)\mathcal{N}}, \tag{4.7}$$

and

$$\mathcal{N}(I-T) = \mathcal{N} \cap [(I-T^*)\mathcal{N}(A)]^{\perp} = \mathcal{N}(I-T_0). \tag{4.8}$$

Proof. (i). Since $\mathcal{R}(A-AT)$ is closed, $\mathcal{R} = \mathcal{R}(A^{1/2}-A^{1/2}T)$ is also closed by Remark 4.5, so $\mathcal{R} = \mathcal{R}(A-AT)$. Then $A^{1/2}\mathcal{R} = \mathcal{R}$, hence $A\mathcal{R} = \mathcal{R}$ and A is an invertible operator on $\mathcal{R}$, because $\mathcal{R} \subset \mathcal{R}(A^{1/2})$ and $A^{1/2}$ is injective on $\mathcal{R}(A^{1/2})$. On the other hand, since $\mathcal{R}$ is closed, Corollary 3.4 implies that T is uniformly A-ergodic.

Let $B_n := n^{-1}\sum_{j=0}^{n-1} A^{1/2}T^j$, $n \ge 1$. The subspace $\mathcal{R}$ is invariant for T^*, because $T^*\mathcal{R} = T^*A^{1/2}\mathcal{R} \subset \mathcal{R}$ and $\mathcal{R} = \overline{\mathcal{R}}$ is reducing for $A^{1/2}T$, T

being A-ergodic. Hence $\mathcal{R}$ is invariant for B_n^*, and putting $A_1 = A^{1/2}|_{\mathcal{R}}$ and $\tilde{T} := T^*|_{\mathcal{R}}$, for $r \in \mathcal{R}$ with $\|r\| = 1$ and $n \geq 1$ we obtain

$$\left\| \frac{1}{n} \sum_{j=0}^{n-1} \tilde{T}^j r \right\| = \|B_n^* A_1^{-1} r\| \leq \|A_1^{-1}\| \, \frac{\|B_n^* A_1^{-1} r\|}{\|A_1^{-1} r\|}$$
$$\leq \|A_1^{-1}\| \, \|B_n^*|_{\mathcal{R}}\| \leq \|A_1^{-1}\| \, \|B_n^* - A^{1/2} P\|.$$

Since $\|B_n^* - A^{1/2}P\| \to 0$ as $n \to \infty$, we have $n^{-1} \sum_{j=0}^{n-1} \tilde{T}^j \to 0$ as $n \to \infty$, uniformly in $\mathcal{B}(\mathcal{R})$. Thus the operator $I - \tilde{T}$ is invertible in $\mathcal{B}(\mathcal{R})$, therefore $(I - \tilde{T})\mathcal{R} = \mathcal{R}$. This implies

$$\mathcal{R} = (I - T^*)A^{1/2}\mathcal{R} = (A^{1/2} - T^*A^{1/2})\mathcal{R} \subset \mathcal{R}(A^{1/2} - T^*A^{1/2}) \subset \overline{\mathcal{R}} = \mathcal{R},$$

whence $\mathcal{R} = \mathcal{R}(A^{1/2} - T^*A^{1/2}) = (A^{1/2} - T^*A^{1/2})\mathcal{R}$. Next we get

$$\overline{\mathcal{R}(A - T^*A)} = \mathcal{R} = (A^{1/2} - T^*A^{1/2})\mathcal{R} = (A - T^*A)A_1^{-1}\mathcal{R} \subset \mathcal{R}(A - T^*A),$$

which leads to $\mathcal{R} = \mathcal{R}(A - T^*A)$. Hence the relations (4.2) are proved.

Now we remark that

$$\mathcal{R} = (A^{1/2} - A^{1/2}T)\mathcal{H} = (A^{1/2} - A^{1/2}T)(\mathcal{R} \oplus \mathcal{N}) = (A^{1/2} - A^{1/2}T)\mathcal{R}.$$

Since $A^{1/2}$ is invertible on $\mathcal{R}$, we get $(I - T)\mathcal{R} = \mathcal{R}$, therefore the subspace $\mathcal{R}$ is reducing for T and the operator $I - T$ is surjective on $\mathcal{R}$. Clearly, it is also injective on $\mathcal{R}$, because $\mathcal{N}(I - T|_{\mathcal{R}}) \subset \mathcal{R} \cap \mathcal{N} = \{0\}$. We infer that it is invertible on $\mathcal{R}$, and Proposition 4.5 of [6] implies that $(I - T|_{\mathcal{R}})^{-1}$ is given by the limit from (4.4). Thus we have proved assertion (i).

(ii) First we show that $\mathcal{N}(A) = \overline{(I - T)\mathcal{N}}$. Clearly, $(I - T)\mathcal{N} \subset \mathcal{N}(A)$. Next, if $h_0 \in \mathcal{N}(A)$ and h is orthogonal to $(I - T)\mathcal{N}$, then $(I - T^*)h_0$ is orthogonal to $\mathcal{N}$, therefore $(I - T^*)h_0 \in \mathcal{R}$, and also $h_0 \in \mathcal{R}$ because $I - T^*$ is invertible on $\mathcal{R}$. Hence $h_0 = 0$, and we have $\mathcal{N}(A) = \overline{(I - T)\mathcal{N}}$.

Now, using the decomposition $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$, we obtain

$$\overline{\mathcal{R}(I - T)} = (I - T)\mathcal{R} \oplus \overline{(I - T)\mathcal{N}} = \mathcal{R} \oplus \mathcal{N}(A),$$

whence $\mathcal{N}(I - T^*) = \mathcal{N} \ominus \mathcal{N}(A) = \overline{A^{1/2}\mathcal{N}}$ (see [12, 14]), and consequently

$$\overline{\mathcal{R}(A)} = \mathcal{R} \oplus \mathcal{N}(I - T^*).$$

Let us prove that the operator $T_* := T^*|_{\overline{\mathcal{R}(A)}}$ is orthogonally uniformly ergodic. Let $h \in \overline{\mathcal{R}(A)}$ with $\|h\| = 1$. Putting $h = h_0 + h_1$ with $h_0 \in \mathcal{R}$ and $h_1 \in \mathcal{N}(I - T^*) = \mathcal{N}(I - T_*)$, for $n \geq 1$ we have

$$n^{-1} \sum_{j=0}^{n-1} T_*^j h = n^{-1} \sum_{j=0}^{n-1} T_*^j h_0 + h_1.$$

So, if $P_1 \in \mathcal{B}(\overline{A\mathcal{H}})$ is the orthogonal projection onto $\mathcal{N}(I - T_*)$ and $A_1 :=$

$A^{1/2}|_{\mathcal{R}}$, we infer that

$$\left\| \frac{1}{n} \sum_{j=0}^{n-1} T_*^j h - P_1 h \right\| = \left\| \frac{1}{n} \sum_{j=0}^{n-1} T_*^j h_0 \right\| = \|(B_n^* - PA^{1/2})A_1^{-1} h_0\|$$
$$\leq \|A_1^{-1}\| \, \|B_n - A^{1/2}P\|,$$

P being as usual the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}$. Since T is uniformly A-ergodic, we conclude that $\|n^{-1} \sum_{j=0}^{n-1} T_*^j - P_1\| \to 0$ $(n \to \infty)$, hence T_* is orthogonally uniformly ergodic on $\overline{\mathcal{R}(A)}$. This implies by Theorem 1 of [8] that $\mathcal{R}(I - T_*)$ is closed, in fact $\mathcal{R}(I - T_*) = \mathcal{R}$ because $\mathcal{N}(I - T_*) = \mathcal{N}(I - T^*)$. Also, we infer that

$$\overline{\mathcal{R}(I - T^*)} = \mathcal{R} \vee \overline{(I - T^*)\mathcal{N}(A)},$$

whence

$$\mathcal{N}(I - T) = \mathcal{N} \cap [(I - T^*)\mathcal{N}(A)]^{\perp}.$$

Now, the subspace $\mathcal{H}_0 = \mathcal{R} \oplus \mathcal{N}(I - T)$ is clearly invariant for T, and the operator $T_0 := T|_{\mathcal{H}_0}$ is orthogonally uniformly ergodic on $\mathcal{H}_0$. Indeed, if $h = r + k \in \mathcal{H}_0$ with $r \in \mathcal{R}$ and $k \in \mathcal{N}(I - T)$, and if $P_0 \in \mathcal{B}(\mathcal{H}_0)$ is the orthogonal projection onto $\mathcal{N}(I - T)$, then for $n \geq 1$ we have as above

$$\left\| \frac{1}{n} \sum_{j=0}^{n-1} T_0^j h - P_0 h \right\| = \left\| n^{-1} \sum_{j=0}^{n-1} T_0^j r \right\| = \left\| A_1^{-1} \left(\frac{1}{n} \sum_{j=0}^{n-1} A_1 T_0^j r - A_1 P r \right) \right\|$$
$$\leq \|A_1^{-1}\| \, \|(B_n - A^{1/2}T)r\|$$
$$\leq \|A_1^{-1}\| \, \|B_n - A^{1/2}P\| \, \|r\|.$$

Since T is uniformly A-ergodic on $\mathcal{H}$ we obtain

$$\left\| \frac{1}{n} \sum_{j=0}^{n-1} T_0^j - P_0 \right\| \to 0 \quad (n \to \infty),$$

and consequently T_0 is orthogonally uniformly ergodic on $\mathcal{H}_0$. Finally, we deduce from Theorem 1 of [8] that $\mathcal{R}(I - T_0)$ is closed, in fact $\mathcal{R}(I - T_0) = \mathcal{R}$ because $\mathcal{N}(I - T_0) = \mathcal{N}(I - T)$. Thus, assertion (ii) and all relations (4.5)–(4.8) are proved, and the proof is finished. ∎

Corollary 4.7. *If $T \in \mathcal{B}(\mathcal{H})$ is A-ergodic with A injective and the range $\mathcal{R}(A - AT)$ is closed, then T is orthogonally uniformly ergodic on $\mathcal{H}$.*

Proof. If $\mathcal{N}(A) = \{0\}$ then $\mathcal{N}(I - T) = \mathcal{N}$, hence we have $\mathcal{H}_0 = \mathcal{H}$ in the above theorem. ∎

Also, we infer immediately from Theorem 4.6, Corollary 4.2 and Proposition 4.5 of [6] the following

Corollary 4.8. *Let T be an A-contraction on $\mathcal{H}$ such that $\overline{\mathcal{R}} = \mathcal{R}(A)$. The following assertions are equivalent*:

(i) $\mathcal{R} = \mathcal{R}(A)$;
(ii) T *is uniformly A-ergodic on* $\mathcal{H}$;
(iii) $I - T|_{\mathcal{R}}$ *is invertible in* $\mathcal{B}(\mathcal{R})$;
(iv) $(A^{1/2} - A^{1/2}T)|_{\mathcal{R}}$ *is invertible in* $\mathcal{B}(\mathcal{R})$;
(v) $\sup_{n\geq 1} \| \sum_{j=0}^{n} A^{1/2}T^j \| < \infty$;
(vi) $\sup_{n\geq 1} \sup_{Ah\neq 0} \|B_n(T)h\|/\|A^{1/2}h\| < \infty$, *where*

$$B_n(T) = n^{-1} \sum_{j=1}^{n-1} jA^{1/2}T^{n-j-1}, \quad n \geq 1.$$

Let us remark that if $(I-T^*)\mathcal{N}(A) = \mathcal{N}(A)$ in Theorem 4.6 then $\mathcal{N}(I-T) = \mathcal{N}(I-T^*)$, and in this case T is orthogonally mean ergodic on $\mathcal{H}$, T being also orthogonally uniformly ergodic on its reducing subspace $\overline{\mathcal{R}(A)} = \mathcal{H}_0$. In general, T is not orthogonally mean ergodic even if T is uniformly A-ergodic, as we can see in the following example. But if T is orthogonally uniformly ergodic and is an A-contraction then T is uniformly A-ergodic.

Example 4.9. Let $A, T \in \mathcal{B}(\mathbb{C}^3)$ be the operators given by

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then T is a regular A-contraction on $\mathbb{C}^3$ and we have

$$\mathcal{R} = \mathcal{R}(A - AT) = \{0\} \oplus \mathbb{C} \oplus \{0\}, \quad \mathcal{N}(A) = \{(\lambda, 0, -\lambda) : \lambda \in \mathbb{C}\}$$

and

$$\mathcal{N}(I - T) = \mathbb{C} \oplus \{0\} \oplus \{0\} \neq \{(\lambda, 0, \lambda) : \lambda \in \mathbb{C}\} = \mathcal{N}(I - T^*).$$

Thus, T is not orthogonally uniformly ergodic. But T is uniformly A-ergodic. Indeed, we first observe that $T = TP$, P being the orthogonal projection onto the subspace $\mathcal{N} = \mathbb{C} \oplus \{0\} \oplus \mathbb{C} = \mathcal{N}_*$. Also, since $T^2 = T$, it follows that $A^{1/2}T^j = A^{1/2}T = A^{1/2}TP = A^{1/2}P$ $(j \geq 1)$, because $A^{1/2}T = A^{1/2}$ on $\mathcal{N}$. Then we obtain

$$\frac{1}{n} \sum_{j=0}^{n-1} A^{1/2}T^j - A^{1/2}P = \frac{1}{n} A^{1/2}(I - T) \to 0 \quad (n \to \infty).$$

Finally, we also remark that $[(I - T^*)\mathcal{N}(A)] \cap \mathcal{N}(A) = \{0\}$, but the two subspaces are not orthogonal.

This example gives a regular A-contraction T which is uniformly A-ergodic with $\mathcal{R} \neq \mathcal{R}(A)$ and $\mathcal{R}$ closed, such that T is not orthogonally mean ergodic.

Now, we can get a non-regular A-contraction T on $\mathbb{C}^3$ with A non-injective and $\mathcal{R} = \mathcal{R}(A)$, such that T is orthogonally uniformly ergodic, hence also uniformly A-ergodic (see Corollaries 4.7 and 4.8).

EXAMPLE 4.10. Let $A, T \in \mathcal{B}(\mathbb{C}^3)$ be the operators given by

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that T is non-regular A-ergodic on $\mathbb{C}^3$ with $\mathcal{N} = \mathcal{N}(A) = \{(\lambda, 0, -\lambda) : \lambda \in \mathbb{C}\}$. Hence $\mathcal{R} = \mathcal{R}(A)$ and by Corollary 4.8, T is uniformly A-ergodic. Since $T^2 = 0$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1} T^j = \frac{1}{n}(I+T) \to 0 \quad (n \to \infty),$$

and the orthogonal projection onto $\mathcal{N}(I-T)$ is the null operator, because $(I-T^*)\mathcal{N}(A) = \mathcal{N}(A)$ which implies $\mathcal{N}(I-T) = \mathcal{N} \ominus \mathcal{N}(A) = \{0\}$. So, T is orthogonally uniformly ergodic on $\mathbb{C}^3$ and in this case $\mathcal{H}_0 \subsetneq \mathbb{C}^3$, $\mathcal{H}_0$ being the corresponding subspace from Theorem 4.6.

This example also shows that, in general, $\mathcal{H}_0$ is not the largest invariant subspace for T with the property that $T|_{\mathcal{H}_0}$ is orthogonally uniformly ergodic.

The finite-dimensional examples presented above show some new phenomena concerning the concept of uniform (quasi-) A-ergodicity. Of course, such examples can also be given in the infinite-dimensional case, where we can get classes of operators which are uniformly A-ergodic without being orthogonally uniformly ergodic, even operators which are not power bounded. We will exhibit such classes of operators below.

Firstly, let $T \in \mathcal{B}(\mathcal{H})$ be hyponormal, that is, satisfying $T^*T \le TT^*$, such that $\mathcal{N}(I-T) \neq 0$. Then $\mathcal{N}(I-T)$ reduces T ($I-T$ is also hyponormal), and if $A_0 \in \mathcal{B}(\mathcal{N}(I-T))$ is an arbitrary positive operator and $A = A_0 \oplus 0 \in \mathcal{B}(\mathcal{H})$, we have $T^*AT = A$, that is, T is an A-isometry. Since $AT = TA$, it follows that T is just a regular A-isometry, hence T is A-ergodic. Moreover, T is even uniformly A-ergodic because $n^{-1}\sum_{j=0}^{n-1} A^{1/2}T^j = A^{1/2}$ for any $n \ge 1$. Here $\mathcal{R}(A - AT) = \{0\}$ so it is a closed subspace, but neither $\mathcal{R}(A)$ nor $\mathcal{R}(I-T)$ is closed in general. Also, $\{n^{-1}T^n\}$ may not converge to 0 in the strong operator topology (even for some unilateral weighted shifts; see [7]). Thus, by the well known results on uniform ergodicity ([6], [8], [9], [15], [16]) it follows that such operators T are not orthogonally uniformly ergodic, and clearly, they are not power bounded in general.

Now let T be a regular A-contraction for a positive operator A on $\mathcal{H}$. Then from Corollary 3.4 and Remark 4.5 we see that the closedness of $\mathcal{R}(A - AT)$

ensures that T is uniformly A-ergodic. But the first relation in (4.7) shows that $\mathcal{R}(I-T)$ is not closed, hence T is not orthogonally uniformly ergodic on $\mathcal{H}$, in general. For T as above it is clear that $\mathcal{R}_0 = \mathcal{R}(A - AT) \subset \mathcal{R}(A^{1/2} - A^{1/2}T) = \mathcal{R}$, and as we have seen, $\mathcal{R}$ is closed if $\mathcal{R}_0$ is. Conversely, $\mathcal{R}_0$ is closed if $\mathcal{R}$ is, and in fact $\mathcal{R}_0 = \mathcal{R}$ in this case. Indeed, when $\mathcal{R}$ is closed, by Theorem 2.4(i) we have (using the condition $AT = A^{1/2}TA^{1/2}$)

$$\mathcal{R} = \mathcal{R}((A^{1/2} - A^{1/2}T)^2) = \mathcal{R}[(A - AT)(I - T)] \subset \mathcal{R}(A - AT) = \mathcal{R}_0,$$

hence $\mathcal{R} = \mathcal{R}_0$.

By Remark 3.3 the class of all regular A-contractions on $\mathcal{H}$ for any $0 \leq A \in \mathcal{B}(\mathcal{H})$ can be identified with the class of P-contractions on $\mathcal{H}$, P being any orthogonal projection in $\mathcal{B}(\mathcal{H})$. On the other hand, if T is a P-contraction then $\mathcal{R}(P)$ is an invariant subspace for T^* and if we put $T_1 = T^*|_{\mathcal{R}(P)}$ then T is a *lifting* of $T_0 = T_1^*$, which means (see [5]) that $PT = T_0P$. Conversely, if $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M} \subset \mathcal{H}$ is an invariant subspace for T^* such that $T^*|_{\mathcal{M}}$ is a contraction on $\mathcal{M}$, then T is a lifting for $(T^*|_{\mathcal{M}})^*$ and one has ($P_{\mathcal{M}} \in \mathcal{B}(\mathcal{H})$ being the orthogonal projection onto $\mathcal{M}$)

$$T^* P_{\mathcal{M}} T = P_{\mathcal{M}}(T^*|_{\mathcal{M}})(T^*|_{\mathcal{M}})^* P_{\mathcal{M}} \leq P_{\mathcal{M}},$$

hence T is a $P_{\mathcal{M}}$-contraction on $\mathcal{H}$. Thus, by the above remark, the class of all regular A-contractions on $\mathcal{H}$ can be identified with the class of all liftings T (in $\mathcal{B}(\mathcal{H})$) of contractions on invariant subspaces of T^*.

Liftings of contractions have been studied and applied in operator theory (see [5]), and we now see that such operators are ergodic in the sense of A-contractions. In view of the matrix representation of T in Remark 3.3 (where S and R are arbitrary operators), it is clear that a lifting of a contraction is not power bounded in general.

Finally, we mention a class of A-isometries having bounded powers.

According to [10] we say that $0 \neq T \in \mathcal{B}(\mathcal{H})$ is a *quasi-isometry* if it is a T^*T-isometry. Clearly, such a T has bounded powers, and $\|T\| = 1$ if and only if T is hyponormal ([10]). Also, a contractive quasi-isometry T is regular if and only if T is quasinormal (that is, $T^*T^2 = TT^*T$). A quasi-isometry T has a matrix representation on $\mathcal{H} = \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T^*)$ of the form

$$T = \begin{pmatrix} V & S \\ 0 & 0 \end{pmatrix} \tag{4.9}$$

where $V = T|_{\overline{\mathcal{R}(T)}}$ is an isometry and S is a bounded linear operator from $\mathcal{N}(T^*)$ into $\overline{\mathcal{R}(T)}$. Using this representation we find that $\|T\| = 1$ if and only if $V^*S = 0$ and $\|S\| \leq 1$. But if we suppose only $V^*S = 0$, then $T^*T^2 = T$ and it is easy to see that

$$\mathcal{N}(T^*T - T) = \mathcal{N}(I - V) \oplus \mathcal{N}(S).$$

Since $T^*T = I_{\overline{\mathcal{R}(T)}} \oplus S^*S$ in this case, we infer that

$$T^*T\mathcal{N}(T^*T - T) = \mathcal{N}(I - V) \subset \mathcal{N}(T^*T - T),$$

which (since T is power bounded) implies that T is T^*T-ergodic. On the other hand, since

$$I - T = \begin{pmatrix} I - V & -S \\ 0 & I \end{pmatrix}, \quad T^*T - T = \begin{pmatrix} I - V & -S \\ 0 & S^*S \end{pmatrix}$$

it follows (by Lemma 3.14 of [10]) that $\mathcal{R}(T^*T - T)$ is closed if and only if $\mathcal{R}(I-T)$ and $\mathcal{R}(S)$ are closed. Now in view of the condition $V^*S = 0$ we deduce (by Theorem 3.11 [10]) that $\mathcal{R}(S)$ is closed, if and only if $\mathcal{R}(T)$ is closed, and so we conclude that $\mathcal{R}(T^*T-T)$ is closed if and only if $\mathcal{R}(T)$ and $\mathcal{R}(I-T)$ are closed. Since T is power bounded one has $n^{-1}\|T^n\| \to 0$ $(n \to \infty)$, and so the fact that $\mathcal{R}(I - T)$ is closed means that T is uniformly ergodic (as in [9], [6], [8]). But it is easy to see that $\mathcal{N}(I - T) = \mathcal{N}(I - V) = \mathcal{N}(I - T^*)$ when $V^*S = 0$, hence one has (1.2) and T is even orthogonally uniformly ergodic if $\mathcal{R}(I-T)$ is closed, and in this case T is also uniformly T^*T-ergodic. This shows that T can be uniformly T^*T-ergodic without $\mathcal{R}(T^*T-T)$ being closed, as it happens if $\mathcal{R}(I - T)$ is closed but $\mathcal{R}(T)$ is not.

Acknowledgments. The author would like thank Professors Mostafa Mbekhta and Jaroslav Zemánek for stimulating discussions on this subject during the Conference on Operator Theory in Timişoara, July 2004 and 2006. We are also grateful to the referee for a careful reading of the manuscript, interesting suggestions and useful comments.

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Received November 15, 2005
Revised version March 31, 2009 (5799)