

## The joint essential numerical range of operators: convexity and related results

by

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**Abstract.** Let  $W(\mathbf{A})$  and  $W_e(\mathbf{A})$  be the joint numerical range and the joint essential numerical range of an  $m$ -tuple of self-adjoint operators  $\mathbf{A} = (A_1, \dots, A_m)$  acting on an infinite-dimensional Hilbert space. It is shown that  $W_e(\mathbf{A})$  is always convex and admits many equivalent formulations. In particular, for any fixed  $i \in \{1, \dots, m\}$ ,  $W_e(\mathbf{A})$  can be obtained as the intersection of all sets of the form

$$\mathbf{cl}(W(A_1, \dots, A_{i+1}, A_i + F, A_{i+1}, \dots, A_m)),$$

where  $F = F^*$  has finite rank. Moreover, the closure  $\mathbf{cl}(W(\mathbf{A}))$  of  $W(\mathbf{A})$  is always star-shaped with the elements in  $W_e(\mathbf{A})$  as star centers. Although  $\mathbf{cl}(W(\mathbf{A}))$  is usually not convex, an analog of the separation theorem is obtained, namely, for any element  $\mathbf{d} \notin \mathbf{cl}(W(\mathbf{A}))$ , there is a linear functional  $f$  such that  $f(\mathbf{d}) > \sup\{f(\mathbf{a}) : \mathbf{a} \in \mathbf{cl}(W(\tilde{\mathbf{A}}))\}$ , where  $\tilde{\mathbf{A}}$  is obtained from  $\mathbf{A}$  by perturbing one of the components  $A_i$  by a finite rank self-adjoint operator. Other results on  $W(\mathbf{A})$  and  $W_e(\mathbf{A})$  extending those on a single operator are obtained.

**1. Introduction.** Let  $\mathcal{B}(\mathcal{H})$  denote the algebra of bounded linear operators acting on a complex Hilbert space  $\mathcal{H}$ . The *numerical range* of  $A \in \mathcal{B}(\mathcal{H})$  is defined as

$$W(A) = \{\langle A\mathbf{x}, \mathbf{x} \rangle : \mathbf{x} \in \mathcal{H}, \langle \mathbf{x}, \mathbf{x} \rangle = 1\},$$

which is useful in studying operators; see [10, 11, 22, 24] and [25, Chapter 1]. Let  $\mathcal{S}(\mathcal{H})$  denote the set of self-adjoint operators in  $\mathcal{B}(\mathcal{H})$ . Since every  $A \in \mathcal{B}(\mathcal{H})$  admits a decomposition  $A = A_1 + iA_2$  with  $A_1, A_2 \in \mathcal{S}(\mathcal{H})$ , we can identify  $W(A)$  with

$$\{(\langle A_1\mathbf{x}, \mathbf{x} \rangle, \langle A_2\mathbf{x}, \mathbf{x} \rangle) : \mathbf{x} \in \mathcal{H}, \langle \mathbf{x}, \mathbf{x} \rangle = 1\} \subseteq \mathbb{R}^2.$$

This leads to the *joint numerical range* of  $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$ ,

$$W(\mathbf{A}) = \{(\langle A_1\mathbf{x}, \mathbf{x} \rangle, \dots, \langle A_m\mathbf{x}, \mathbf{x} \rangle) : \mathbf{x} \in \mathcal{H}, \langle \mathbf{x}, \mathbf{x} \rangle = 1\} \subseteq \mathbb{R}^m,$$

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which has been studied by many researchers in order to understand the joint behavior of several operators  $A_1, \dots, A_m$ . One may see [1, 5, 12, 14, 15, 16, 19, 23, 28, 31, 33, 35] and their references for the background and many applications of the joint numerical range.

Let  $\mathcal{F}(\mathcal{H})$  and  $\mathcal{K}(\mathcal{H})$  be the sets of finite rank and compact operators in  $\mathcal{B}(\mathcal{H})$ . In the study of finite rank or compact perturbations of operators, researchers consider the *joint essential numerical range* of  $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$  defined by

$$W_e(\mathbf{A}) = \bigcap \{ \mathbf{cl}(W(\mathbf{A} + \mathbf{K})) : \mathbf{K} = (K_1, \dots, K_m) \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m \}.$$

Here  $\mathbf{cl}(S)$  denotes the closure of the set  $S$ . For  $m = 2$ ,  $W_e(\mathbf{A})$  can be identified with the *essential numerical range* of  $A = A_1 + iA_2 \in \mathcal{B}(\mathcal{H})$ , defined by

$$W_e(A) = \bigcap \{ \mathbf{cl}(W(A + K)) : K \in \mathcal{K}(\mathcal{H}) \}.$$

One may see [2, 3, 6, 7, 13, 18, 20, 21, 26, 27, 30, 32, 36, 37] for many interesting results on  $W_e(A)$  and  $W_e(\mathbf{A})$ .

In theoretical studies as well as applications, it is desirable to deal with  $\mathbf{A}$  such that  $W(\mathbf{A})$  or  $\mathbf{cl}(W(\mathbf{A}))$  is convex. For example, if  $\mathbf{cl}(W(\mathbf{A}))$  is convex, one can apply the separation theorem to show that  $\mathbf{0} \notin \mathbf{cl}(W(\mathbf{A}))$  if and only if there exist  $r > 0$  and  $\mathbf{c} = (c_1, \dots, c_m) \in \mathbb{R}^m$  such that  $(\sum_{i=1}^m c_i A_i) > rI_{\mathcal{H}}$ . Unfortunately,  $\mathbf{cl}(W(\mathbf{A}))$  is not always convex. Here are some results concerning the convexity of  $W(\mathbf{A})$  and  $\mathbf{cl}(W(\mathbf{A}))$ , and related to  $W_e(\mathbf{A})$  (for example, see [5, 10, 11, 36, 21, 29, 31] and their references).

- (P1) [31]  $W(A_1, \dots, A_m)$  is convex if
  - (a)  $\text{span}\{I, A_1, \dots, A_m\}$  has dimension at most 3, or
  - (b)  $\dim \mathcal{H} \geq 3$  and  $\text{span}\{I, A_1, \dots, A_m\}$  has dimension at most 4.
- (P2) [31] For any  $A_1, A_2, A_3 \in \mathcal{S}(\mathcal{H})$  such that  $\text{span}\{I, A_1, A_2, A_3\}$  has dimension 4, there is always an  $A_4 \in \mathcal{S}(\mathcal{H})$  for which  $W(A_1, \dots, A_4)$  is not convex.
- (P3) [31] If  $m \geq 4$  then there exists  $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$  such that  $W(\mathbf{A})$  is non-convex.
- (P4) For any positive integer  $m$  and any  $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ ,  $W_e(\mathbf{A})$  is a compact set contained in  $W(\mathbf{A})$ . If  $\text{span}\{I, A_1, \dots, A_m\}$  has dimension at most 4, then  $W_e(\mathbf{A})$  is convex.
- (P5) [36] For  $S \subseteq \mathbb{R}^m$ , let  $\text{Ext}(S)$  be the set of all points in  $S$  that do not lie in the open line segment joining two distinct points in  $S$ . Then  $\text{Ext}(\mathbf{cl}(W(\mathbf{A}))) \subseteq \text{Ext}(W(\mathbf{A})) \cup \text{Ext}(W_e(\mathbf{A}))$ .

We remark that (P1)–(P3) also hold if we replace  $W(\mathbf{A})$  by  $\mathbf{cl}(W(\mathbf{A}))$ . In view of (P2) and (P3), if  $m > 3$ , then for  $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$  and  $\mathbf{K} \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$  the set  $\mathbf{cl}(W(\mathbf{A} + \mathbf{K}))$  is usually non-convex. Since  $W_e(\mathbf{A})$

is the intersection of non-convex sets, one does not expect the set  $W_e(\mathbf{A})$  to be convex. This might be the reason why the convexity of  $W_e(\mathbf{A})$  is seldom discussed for  $m > 3$ . In fact, some researchers have studied different geometrical properties of  $W_e(\mathbf{A})$  under the assumption that  $W_e(\mathbf{A})$  is convex, and some have examined  $W_e(\mathbf{A})$  for different classes of operators without discussing their convexity; for example, see [6, 26, 27, 30, 32].

In this paper, we prove the rather unexpected result that  $W_e(\mathbf{A})$  is always convex. Moreover, it is shown that the closure  $\mathbf{cl}(W(\mathbf{A}))$  of  $W(\mathbf{A})$  is always star-shaped with the elements in  $W_e(\mathbf{A})$  as star centers. Many results relating  $W_e(\mathbf{A})$  and  $W(\mathbf{A})$  are also obtained. Our paper is organized as follows.

In Section 2, we extend the results of [21] by establishing several equivalent formulations of the essential joint numerical range for  $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ . One key obstacle for such an extension is the fact that  $W(\mathbf{A})$  may not be convex. To get around this problem, we show that  $\mathbf{cl}(W(\mathbf{A}))$  is star-shaped. The star-shapedness of  $\mathbf{cl}(W(\mathbf{A}))$  and the conditions equivalent to membership in  $W_e(\mathbf{A})$ , given in Section 2, lead to our main result that  $W_e(\mathbf{A})$  is convex and its elements are star centers of the set  $\mathbf{cl}(W(\mathbf{A}))$ , which is presented in Section 3. With the convexity theorem, we obtain additional descriptions of  $W_e(\mathbf{A})$  in Section 4 in terms of the perturbations of one of the components of  $\mathbf{A}$ , and also in terms of linear combinations of the components of  $\mathbf{A}$ . For example, we show that  $W_e(A_1, \dots, A_m)$  is equal to the sets

$$\bigcap \{ \mathbf{cl}(W(A_1, \dots, A_{i-1}, A_i + F, A_{i+1}, \dots, A_m)) : F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H}) \}$$

and

$$\left\{ (a_1, \dots, a_m) : \sum_{j=1}^m c_j a_j \in W_e \left( \sum_{j=1}^m c_j A_j \right) \text{ for all } (c_1, \dots, c_m) \in \Omega \right\},$$

where  $\Omega = \{(c_1, \dots, c_m) \in \mathbb{R}^m : \sum_{j=1}^m c_j^2 = 1\}$ . Also, we obtain an analog of the separation theorem for the not necessarily convex set  $\mathbf{cl}(W(\mathbf{A}))$ , namely, for any element  $\mathbf{d} \notin \mathbf{cl}(W(\mathbf{A}))$ , there is a linear functional  $f$  such that  $f(\mathbf{d}) > \sup\{f(\mathbf{a}) : \mathbf{a} \in \mathbf{cl}(W(\tilde{\mathbf{A}}))\}$ , where  $\tilde{\mathbf{A}}$  is obtained from  $\mathbf{A}$  by perturbing one of the components  $A_j$  by a finite rank self-adjoint operator. In Section 5, we present additional results on  $W(\mathbf{A})$  and  $W_e(\mathbf{A})$ . For instance,  $W_e(\mathbf{A}) = \mathbf{cl}(W(\mathbf{A}))$  if and only if the extreme points of  $W(\mathbf{A})$  are contained in  $W_e(\mathbf{A})$ ; the convex hull of  $\mathbf{cl}(W(\mathbf{A}))$  can always be realized as the joint essential numerical range of  $(\tilde{A}_1, \dots, \tilde{A}_m)$  for linear operators  $\tilde{A}_1, \dots, \tilde{A}_m$  acting on a separable Hilbert space.

In our discussion, we always assume that  $\mathcal{H}$  is infinite-dimensional. For any vector  $\mathbf{x} \in \mathcal{H}$  and  $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$ , we will use the notation

$$\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = (\langle A_1\mathbf{x}, \mathbf{x} \rangle, \dots, \langle A_m\mathbf{x}, \mathbf{x} \rangle).$$

Furthermore,  $\mathbb{R}^m$  will be used to denote the inner product space of  $1 \times m$  real vectors with the usual inner product  $\langle \mathbf{x}, \mathbf{y} \rangle$ .

**2. Equivalent conditions for  $W_e(\mathbf{A})$ .** Following [21, Theorem 5.1] and its corollary on a single operator  $A \in \mathcal{B}(\mathcal{H})$ , we obtain several conditions equivalent to membership in  $W_e(\mathbf{A})$ .

**THEOREM 2.1.** *Let  $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$ . The following conditions are equivalent for a real vector  $\mathbf{a} = (a_1, \dots, a_m)$ :*

- (1)  $\mathbf{a} \in W_e(\mathbf{A}) = \bigcap \{\text{cl}(W(\mathbf{A} + \mathbf{K})) : \mathbf{K} \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\}$ .
- (2)  $\mathbf{a} \in \bigcap \{\text{cl}(W(\mathbf{A} + \mathbf{F})) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\}$ .
- (3) *There is an orthonormal sequence  $\{\mathbf{x}_n\}_{n=1}^\infty \subset \mathcal{H}$  of vectors such that*

$$\lim_{n \rightarrow \infty} \langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle = \mathbf{a}.$$

- (4) *There is a sequence  $\{\mathbf{x}_n\}_{n=1}^\infty \subset \mathcal{H}$  of unit vectors converging weakly to  $\mathbf{0}$  in  $\mathcal{H}$  such that*

$$\lim_{n \rightarrow \infty} \langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle = \mathbf{a}.$$

- (5) *There is an infinite-dimensional projection  $P \in \mathcal{S}(\mathcal{H})$  such that*

$$P(A_j - a_j I)P \in \mathcal{K}(\mathcal{H}) \quad \text{for } j = 1, \dots, k.$$

Most of the argument in [21] can be applied here except for one crucial step, where the convexity of  $W(\mathbf{A})$  for  $m = 2$  is needed. Since  $W(\mathbf{A})$  may not be convex for  $m > 3$ , we need the following auxiliary result to overcome the obstacle. As a byproduct, it shows that  $\text{cl}(W(\mathbf{A}))$  is star-shaped.

**THEOREM 2.2.** *Let  $\mathbf{A}$  satisfy the hypothesis of Theorem 2.1, and let  $W_3(\mathbf{A})$  be the set of real vectors  $\mathbf{a}$  satisfying condition (3) of Theorem 2.1. Then  $W_3(\mathbf{A})$  is non-empty and closed. Moreover, each element  $\mathbf{a} \in W_3(\mathbf{A})$  is a star center of  $\text{cl}(W(\mathbf{A}))$ , i.e., for any  $\mathbf{b} \in \text{cl}(W(\mathbf{A}))$  we have  $(1-t)\mathbf{a} + t\mathbf{b} \in \text{cl}(W(\mathbf{A}))$  for all  $0 \leq t \leq 1$ .*

*Proof.* To prove that  $W_3(\mathbf{A})$  is non-empty, let  $\{\mathbf{x}_n\}_{n=1}^\infty$  be an orthonormal sequence of vectors in  $\mathcal{H}$ . Then the sequence  $\{\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle\}_{n=1}^\infty$  is bounded. By choosing a subsequence if necessary, we can assume that  $\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle$  converges. Hence,  $W_3(\mathbf{A})$  is non-empty.

Next, we show that  $W_3(\mathbf{A})$  is closed. Suppose  $\mathbf{a} \in \text{cl}(W_3(\mathbf{A}))$ . Then for each  $n \geq 1$ , there exists an orthonormal sequence  $\{\mathbf{x}_k^n\}_{k=1}^\infty$  such that

$$\lim_{k \rightarrow \infty} \langle \mathbf{A}\mathbf{x}_k^n, \mathbf{x}_k^n \rangle = \mathbf{a}^n \in \mathbb{R}^m \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbf{a}^n = \mathbf{a}.$$

Let  $\delta_n = 1/(4n^2)$ . By going to subsequences if necessary, we may assume that  $\|\langle \mathbf{A}\mathbf{x}_k^n, \mathbf{x}_k^n \rangle - \mathbf{a}\| < \delta_n$  for all  $n, k$ . We may also assume that  $\|A_1\|^2 + \dots + \|A_m\|^2 \leq 1$ . Then  $\|\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle\| \leq \|\mathbf{x}\| \|\mathbf{y}\|$  for all  $\mathbf{x}, \mathbf{y} \in \mathcal{H}$ .

Choose  $\mathbf{x}_1 = \mathbf{x}_1^1$ . Then  $\|\langle \mathbf{A}\mathbf{x}_1, \mathbf{x}_1 \rangle - \mathbf{a}\| < 1$ . Suppose we have chosen  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  orthonormal with  $\|\langle \mathbf{A}\mathbf{x}_k, \mathbf{x}_k \rangle - \mathbf{a}\| < 1/k$  for  $1 \leq k \leq n$ . Then choose  $N$  such that for all  $1 \leq k \leq n$ ,

$$|\langle \mathbf{x}_k, \mathbf{x}_N^{n+1} \rangle|, \|\langle \mathbf{A}\mathbf{x}_k, \mathbf{x}_N^{n+1} \rangle\| < \delta_{n+1}.$$

Let  $\mathbf{y} = \mathbf{x}_N^{n+1} - \sum_{k=1}^n \langle \mathbf{x}_N^{n+1}, \mathbf{x}_k \rangle \mathbf{x}_k$ . Then

$$\|\mathbf{y} - \mathbf{x}_N^{n+1}\| \leq n\delta_{n+1}, \quad \text{so} \quad 1 - n\delta_{n+1} \leq \|\mathbf{y}\| \leq 1 + n\delta_{n+1}.$$

Therefore,

$$\begin{aligned} \|\langle \mathbf{A}\mathbf{y}, \mathbf{y} \rangle - \mathbf{a}\| &\leq \|\langle \mathbf{A}(\mathbf{y} - \mathbf{x}_N^{n+1}), \mathbf{y} \rangle\| + \|\langle \mathbf{A}\mathbf{x}_N^{n+1}, \mathbf{y} - \mathbf{x}_N^{n+1} \rangle\| + \|\langle \mathbf{A}\mathbf{x}_N^{n+1}, \mathbf{x}_N^{n+1} \rangle - \mathbf{a}\| \\ &\leq \|\mathbf{y} - \mathbf{x}_N^{n+1}\|(\|\mathbf{y}\| + \|\mathbf{x}_N^{n+1}\|) + \delta_{n+1} \leq (2n+2)\delta_{n+1}. \end{aligned}$$

Let  $\mathbf{x}_{n+1} = \mathbf{y}/\|\mathbf{y}\|$ . Then

$$\|\mathbf{x}_{n+1} - \mathbf{y}\| = |1 - \|\mathbf{y}\|| \leq n\delta_{n+1}.$$

Hence,  $\{\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{x}_{n+1}\}$  is an orthonormal set and

$$\begin{aligned} \|\langle \mathbf{A}\mathbf{x}_{n+1}, \mathbf{x}_{n+1} \rangle - \mathbf{a}\| &\leq \|\mathbf{y} - \mathbf{x}_{n+1}\|(\|\mathbf{y}\| + \|\mathbf{x}_{n+1}\|) + (2n+2)\delta_{n+1} \\ &\leq (4n+3)\delta_{n+1} < 1/(n+1). \end{aligned}$$

To prove the last assertion, let  $\mathbf{a} \in W_3(\mathbf{A})$  and  $\mathbf{b} \in \mathbf{cl}(W(\mathbf{A}))$ . Suppose  $\{\mathbf{x}_n\}$  is an orthonormal sequence in  $\mathcal{H}$  such that  $\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle \rightarrow \mathbf{a}$ . For  $0 \leq t \leq 1$ , we are going to show that  $(1-t)\mathbf{a} + t\mathbf{b} \in \mathbf{cl}(W(\mathbf{A}))$ . Given  $\varepsilon > 0$ , let  $\mathbf{y}$  be a unit vector in  $\mathcal{H}$  such that  $\|\langle \mathbf{A}\mathbf{y}, \mathbf{y} \rangle - \mathbf{b}\| < \varepsilon$ . Choose  $n$  such that  $\|\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle - \mathbf{a}\| < \varepsilon$  and  $\|\langle \mathbf{A}\mathbf{y}, \mathbf{x}_n \rangle\| < \varepsilon$ . Choose  $\theta \in \mathbb{R}$  such that  $\langle e^{i\theta}\mathbf{y}, \mathbf{x}_n \rangle$  is imaginary. Let  $\mathbf{z} = \sqrt{t}e^{i\theta}\mathbf{y} + \sqrt{1-t}\mathbf{x}_n$ . Then

$$\langle \mathbf{z}, \mathbf{z} \rangle = t\langle \mathbf{y}, \mathbf{y} \rangle + (1-t)\langle \mathbf{x}_n, \mathbf{x}_n \rangle + 2\sqrt{t}\sqrt{1-t}(\langle e^{i\theta}\mathbf{y}, \mathbf{x}_n \rangle + \langle \mathbf{x}_n, e^{i\theta}\mathbf{y} \rangle) = 1$$

and

$$\begin{aligned} \|\langle \mathbf{A}\mathbf{z}, \mathbf{z} \rangle - ((1-t)\mathbf{a} + t\mathbf{b})\| &\leq (1-t)\|\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle - \mathbf{a}\| + t\|\langle \mathbf{A}\mathbf{y}, \mathbf{y} \rangle - \mathbf{b}\| \\ &\quad + \sqrt{t}\sqrt{1-t}\|\langle e^{i\theta}\mathbf{A}\mathbf{y}, \mathbf{x}_n \rangle + \langle \mathbf{A}\mathbf{x}_n, e^{i\theta}\mathbf{y} \rangle\| \leq 2\varepsilon. \end{aligned}$$

Therefore,  $(1-t)\mathbf{a} + t\mathbf{b} \in \mathbf{cl}(W(\mathbf{A}))$ . ■

The referee indicated that  $W_3(\mathbf{A})$  is clearly closed, and a short proof is possible. We include a detailed proof for the sake of completeness and easy reference.

*Proof of Theorem 2.1.* For  $j = 2, 3, 4, 5$ , let  $W_j(\mathbf{A})$  be the set of  $\mathbf{a}$  satisfying condition (j). Clearly, we have

$$W_5(\mathbf{A}) \subseteq W_3(\mathbf{A}) \subseteq W_4(\mathbf{A}) \subseteq W_6(\mathbf{A}) \subseteq W_2(\mathbf{A}).$$

Suppose  $\mathbf{a} \in W_2(\mathbf{A})$ . We are going to show that  $\mathbf{a} \in W_5(\mathbf{A})$ . Without loss of generality, we may assume  $\mathbf{a} = \mathbf{0}$ .

Since  $\mathbf{0} \in W_2(\mathbf{A}) \subseteq \text{cl}(W(\mathbf{A}))$ , there exists a unit vector  $\mathbf{x}_1 \in \mathcal{H}$  such that  $\|\langle \mathbf{A}\mathbf{x}_1, \mathbf{x}_1 \rangle\| < 1/2$ . Suppose we have an orthonormal set  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  such that  $\|\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle\| < 1/2^n$ . Let  $Q$  be the orthogonal projection of  $\mathcal{H}$  onto the subspace  $S$  spanned by  $\mathbf{x}_1, \dots, \mathbf{x}_n$  and let

$$\mathbf{B} = ((I - Q)A_1(I - Q)|_{S^\perp}, \dots, (I - Q)A_m(I - Q)|_{S^\perp}).$$

Let  $\mathbf{b} = (b_1, \dots, b_m) \in W_3(\mathbf{B})$  and  $\mathbf{b}I_S = (b_1I_S, \dots, b_mI_S)$ . Then for  $\bar{Q} = I - Q$ , we have

$$\mathbf{b}I_S \oplus \mathbf{B} = (b_1Q + \bar{Q}A_1\bar{Q}, \dots, b_mQ + \bar{Q}A_m\bar{Q}) = \mathbf{A} + \mathbf{F}$$

for some  $\mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$ . Therefore,  $\mathbf{0} \in \text{cl}(W(\mathbf{b}I_S \oplus \mathbf{B}))$ . Hence, there exists a unit vector  $\mathbf{x} \in \mathcal{H}$  such that  $\|\langle (\mathbf{A} + \mathbf{F})\mathbf{x}, \mathbf{x} \rangle\| < 1/2^{n+2}$ . Let  $\mathbf{x} = \mathbf{y} + \mathbf{z}$ , where  $\mathbf{y} \in S$  and  $\mathbf{z} \in S^\perp$ . Then  $\|\mathbf{y}\|^2 + \|\mathbf{z}\|^2 = \|\mathbf{x}\|^2 = 1$ . If  $\mathbf{z} = \mathbf{0}$ , then  $\langle (\mathbf{A} + \mathbf{F})\mathbf{x}, \mathbf{x} \rangle = \mathbf{b} \in W_3(\mathbf{B}) \subseteq \text{cl}(W(\mathbf{B}))$ . If  $\mathbf{z} \neq \mathbf{0}$ , then by Theorem 2.2, we have

$$\langle (\mathbf{A} + \mathbf{F})\mathbf{x}, \mathbf{x} \rangle = \|\mathbf{y}\|^2\mathbf{b} + \|\mathbf{z}\|^2\langle \mathbf{B}(\mathbf{z}/\|\mathbf{z}\|), \mathbf{z}/\|\mathbf{z}\| \rangle \in \text{cl}(W(\mathbf{B})).$$

So there exists a unit vector  $\mathbf{x}_{n+1} \in S^\perp$  such that

$$\|\langle (\mathbf{A} + \mathbf{F})\mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{B}\mathbf{x}_{n+1}, \mathbf{x}_{n+1} \rangle\| < \frac{1}{2^{n+2}},$$

and hence

$$\|\langle \mathbf{A}\mathbf{x}_{n+1}, \mathbf{x}_{n+1} \rangle\| = \|\langle \mathbf{B}\mathbf{x}_{n+1}, \mathbf{x}_{n+1} \rangle\| < \frac{1}{2^{n+1}},$$

because  $\langle \mathbf{F}\mathbf{x}_{n+1}, \mathbf{x}_{n+1} \rangle = \mathbf{0}$ . Inductively, we can choose an orthonormal sequence  $\{\mathbf{x}_n\}_{n=1}^\infty$  such that

$$(1) \quad \|\langle \mathbf{A}\mathbf{x}_n, \mathbf{x}_n \rangle\| < \frac{1}{2^n} \quad \text{for all } n \geq 1.$$

Let  $n_1 = 1$ . For every  $1 \leq i \leq m$ , we have

$$\sum_{n=1}^{\infty} |\langle A_i\mathbf{x}_{n_1}, \mathbf{x}_n \rangle|^2 \leq \|A_i\mathbf{x}_{n_1}\|^2 \quad \text{and} \quad \sum_{n=1}^{\infty} |\langle A_i\mathbf{x}_n, \mathbf{x}_{n_1} \rangle|^2 \leq \|A_i^*\mathbf{x}_{n_1}\|^2.$$

Hence, there exists  $n_2 > n_1$  such that

$$\sum_{n=n_2}^{\infty} |\langle A_i\mathbf{x}_{n_1}, \mathbf{x}_n \rangle|^2 < \frac{1}{2} \quad \text{and} \quad \sum_{n=n_2}^{\infty} |\langle A_i\mathbf{x}_n, \mathbf{x}_{n_1} \rangle|^2 < \frac{1}{2}$$

for all  $1 \leq i \leq m$ . Repeating this procedure, we get a strictly increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers such that for all  $1 \leq i \leq m$ , we have

$$(2) \quad \sum_{n=n_{k+1}}^{\infty} |\langle A_i\mathbf{x}_{n_k}, \mathbf{x}_n \rangle|^2 < \frac{1}{2^k} \quad \text{and} \quad \sum_{n=n_{k+1}}^{\infty} |\langle A_i\mathbf{x}_n, \mathbf{x}_{n_k} \rangle|^2 < \frac{1}{2^k}.$$

Formulas (1) and (2) imply that

$$(3) \quad \sum_{k,l=1}^{\infty} |\langle A_i \mathbf{x}_{n_k}, \mathbf{x}_{n_l} \rangle|^2 < \infty.$$

Let  $P$  be the orthogonal projection onto the subspace spanned by  $\{\mathbf{x}_{n_k}\}_{k=1}^{\infty}$ . Then it follows from (3) that  $PA_iP$  is compact for all  $1 \leq i \leq m$ . ■

### 3. Convexity and star-shapedness

**THEOREM 3.1.** *Let  $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ . Then  $W_e(\mathbf{A})$  is a compact convex subset of  $\mathbf{cl}(W(\mathbf{A}))$ . Moreover, each element in  $W_e(\mathbf{A})$  is a star center of the star-shaped set  $\mathbf{cl}(W(\mathbf{A}))$ .*

*Proof.* Because  $W_e(\mathbf{A})$  is the intersection of compact sets, it is compact. To prove the convexity, let  $\mathbf{a}, \mathbf{b} \in W_e(\mathbf{A})$  and  $0 \leq t \leq 1$ . Then for every  $\mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$ , we have  $\mathbf{a} \in W_e(\mathbf{A}) = W_e(\mathbf{A} + \mathbf{F})$  and  $\mathbf{b} \in W_e(\mathbf{A}) \subseteq \mathbf{cl}(W(\mathbf{A} + \mathbf{F}))$ . So, by Theorem 2.2, we have  $t\mathbf{a} + (1-t)\mathbf{b} \in \mathbf{cl}(W(\mathbf{A} + \mathbf{F}))$ . Hence,

$$t\mathbf{a} + (1-t)\mathbf{b} \in \bigcap \{ \mathbf{cl}(W(\mathbf{A} + \mathbf{F})) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m \} = W_e(\mathbf{A}).$$

By Theorems 2.1 and 2.2, we have the last assertion. ■

Note that  $W_e(\mathbf{A}) \cap W(\mathbf{A})$  may be empty. For example, if

$$A = \text{diag}(1, 1/2, 1/3, \dots)$$

acts on  $\ell^2$ , then  $W_e(A) = \{0\}$  and  $W(A) = (0, 1]$ . One may wonder whether a point  $\mathbf{a} \in W_e(\mathbf{A}) \cap W(\mathbf{A})$  is a star center of  $W(\mathbf{A})$ . This is not true, as shown by the example below. Moreover, the example shows that for  $m \geq 4$  there exists  $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$  such that  $\mathbf{cl}(W(\mathbf{A}))$  is convex whereas  $W(\mathbf{A})$  is not. Of course, this is impossible for  $m \leq 3$  as  $W(\mathbf{A})$  is always convex.

**EXAMPLE 3.2.** *Consider  $\mathcal{H} = \ell^2$  with canonical basis  $\{e_n : n \geq 1\}$ . Let  $\mathbf{A} = (A_1, \dots, A_4)$  with*

$$\begin{aligned} A_1 &= \text{diag}(1, 0, 1/3, 1/4, \dots), & A_2 &= \text{diag}(1, 0) \oplus \mathbf{0}, \\ A_3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbf{0}, & A_4 &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus \mathbf{0}. \end{aligned}$$

*Then  $(1, 1, 0, 0) \in W(\mathbf{A})$  and  $(0, 0, 0, 0) \in W(\mathbf{A}) \cap W_e(\mathbf{A})$ , but  $(1/2, 1/2, 0, 0) \notin W(\mathbf{A})$ . Hence,  $W(\mathbf{A})$  is not convex. However,  $\mathbf{cl}(W(\mathbf{A}))$  is convex.*

*Proof.* Note that  $(1, 1, 0, 0) = \langle \mathbf{A}e_1, e_1 \rangle \in W(\mathbf{A})$  and

$$(0, 0, 0, 0) = \langle \mathbf{A}e_2, e_2 \rangle = \lim_{n \rightarrow \infty} \langle \mathbf{A}e_n, e_n \rangle \in W(\mathbf{A}) \cap W_e(\mathbf{A}).$$

To show that  $(1/2, 1/2, 0, 0) \notin W(\mathbf{A})$ , consider a unit vector  $\mathbf{x} = \sum x_j e_j$  such that  $\sum_{n=1}^{\infty} |x_n|^2 = 1$ . If  $\langle A_1 \mathbf{x}, \mathbf{x} \rangle = \langle A_2 \mathbf{x}, \mathbf{x} \rangle = 1/2$ , then

$$|x_1|^2 + \sum_{n=3}^{\infty} |x_n|^2/n = |x_1|^2 = 1/2.$$

Thus,  $x_n = 0$  for all  $n \geq 3$  and  $|x_1|^2 = |x_2|^2 = 1/2$ . It then follows that  $(\langle A_3 \mathbf{x}, \mathbf{x} \rangle, \langle A_4 \mathbf{x}, \mathbf{x} \rangle) \neq (0, 0)$ . This proves that  $(1/2, 1/2, 0, 0) \notin W(\mathbf{A})$ . Hence,  $(0, 0, 0, 0) \in W_e(\mathbf{A}) \cap W(\mathbf{A})$  is not a star center of  $W(\mathbf{A})$ , and  $W(\mathbf{A})$  is not convex.

To see that  $\text{cl}(W(\mathbf{A}))$  is convex, note that  $\mathbf{0} \in W_e(\mathbf{A})$ . Thus, by Theorem 3.1, for every  $\mathbf{b} \in \text{cl}(W(\mathbf{A}))$  we have  $t\mathbf{0} + (1-t)\mathbf{b} \in \text{cl}(W(\mathbf{A}))$  for any  $t \in [0, 1]$ .

Let  $\mathbf{B} = (B_1, B_2, B_3, B_4)$ , where

$$\begin{aligned} B_1 &= \text{diag}(0, 1, 0), & B_2 &= \text{diag}(0, 1, 0), \\ B_3 &= [0] \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & B_4 &= [0] \oplus \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \end{aligned}$$

and  $\mathbf{C} = (C_1, C_2, C_3, C_4)$ , where  $C_1 = \text{diag}(1/3, 1/4, \dots) \oplus [0]$ ,  $C_2 = C_3 = C_4 = \text{diag}(0, 0, \dots) \oplus [0]$ . Then it is easy to verify that

$$W(\mathbf{B}) = \{(r, r, s, t) \in \mathbb{R}^4 : 4(r - 1/2)^2 + s^2 + t^2 \leq 1\}$$

and

$$W(\mathbf{C}) = \{(c, 0, 0, 0) : c \in [0, 1/3]\}$$

are both compact and convex. Hence,  $W(\mathbf{B} \oplus \mathbf{C}) = \text{conv}(W(\mathbf{B}) \cup W(\mathbf{C}))$  is compact and convex and

$$W(\mathbf{A}) \subseteq W(\mathbf{B} \oplus \mathbf{C}) \Rightarrow \text{cl}(W(\mathbf{A})) \subseteq W(\mathbf{B} \oplus \mathbf{C}).$$

On the other hand,  $\mathbf{B} \oplus \mathbf{C} = [0] \oplus \mathbf{A} \oplus [0]$ . Therefore,

$$W(\mathbf{B} \oplus \mathbf{C}) = \{t\mathbf{0} + (1-t)\mathbf{b} : \mathbf{b} \in W(\mathbf{A})\} \subseteq \text{cl}(W(\mathbf{A})).$$

So,  $\text{cl}(W(\mathbf{A})) = W(\mathbf{B} \oplus \mathbf{C})$  is convex. ■

**4. Other descriptions of  $W_e(\mathbf{A})$ .** For  $\mathbf{c} = (c_1, \dots, c_m) \in \mathbb{R}^m$  and  $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$ , let  $\mathbf{c} \cdot \mathbf{A} = \sum_{i=1}^m c_i A_i$ . Using the convexity of  $W_e(\mathbf{A})$ , we obtain additional conditions equivalent to membership in  $W_e(\mathbf{A})$  in terms of  $\mathbf{c} \cdot \mathbf{A} \in \mathcal{S}(\mathcal{H})$  so that the joint behavior of  $A_1, \dots, A_m$  can be understood from their linear combinations. For  $A \in \mathcal{S}(\mathcal{H})$  and a positive integer  $k$ , let

$$\lambda_k(A) = \inf\{\max \sigma(A + F) : F \in \mathcal{S}(\mathcal{H}) \text{ with } \text{rank}(F) < k\}.$$

**THEOREM 4.1.** *Let  $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$  and  $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^m$ . Then  $\mathbf{a} \in W_e(\mathbf{A})$  if and only if any one (and hence all) of the following conditions holds:*



- (1) For every  $\mathbf{c} \in \mathbb{R}^m$ ,  $\mathbf{c} \cdot \mathbf{a} \in W_e(\mathbf{c} \cdot \mathbf{A})$ .
- (2) For every  $\mathbf{c} \in \mathbb{R}^m$ ,  $\mathbf{c} \cdot \mathbf{a} \in \bigcap \{\text{cl}(W(\mathbf{c} \cdot \mathbf{A} + F)) : F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})\}$ .
- (3) For every  $\mathbf{c} \in \mathbb{R}^m$ , there is an orthonormal sequence  $\{\mathbf{x}_n\}_{n=1}^\infty \subset \mathcal{H}$  such that

$$\lim_{n \rightarrow \infty} \langle \mathbf{c} \cdot \mathbf{A} \mathbf{x}_n, \mathbf{x}_n \rangle = \mathbf{c} \cdot \mathbf{a}.$$

- (4) For every  $\mathbf{c} \in \mathbb{R}^m$ , there is a sequence  $\{\mathbf{x}_n\}_{n=1}^\infty \subset \mathcal{H}$  of unit vectors such that  $\{\mathbf{x}_n\}_{n=1}^\infty$  converges weakly to  $\mathbf{0}$  in  $\mathcal{H}$  and

$$\lim_{n \rightarrow \infty} \langle \mathbf{c} \cdot \mathbf{A} \mathbf{x}_n, \mathbf{x}_n \rangle = \mathbf{c} \cdot \mathbf{a}.$$

- (5) For every  $\mathbf{c} \in \mathbb{R}^m$ , there is an infinite-dimensional projection  $P \in \mathcal{S}(\mathcal{H})$  such that  $P(\mathbf{c} \cdot \mathbf{A} - \mathbf{c} \cdot \mathbf{a}I)P \in \mathcal{K}(\mathcal{H})$ .
- (6) For every  $\mathbf{c} \in \mathbb{R}^m$  and  $k \geq 1$ ,  $\lambda_k(\mathbf{c} \cdot \mathbf{A} - \mathbf{c} \cdot \mathbf{a}I) \geq 0$ .

*Proof.* By the convexity of  $W_e(\mathbf{A})$ , we can apply the separation theorem to Theorem 2.1 to show that  $\mathbf{a} \in W_e(\mathbf{A})$  if and only if any one of the conditions (1) to (5) holds.

To prove the equivalence of condition (6), suppose  $\mathbf{a} \in \mathbb{R}^m$ . Without loss of generality, we may assume that  $\mathbf{a} = \mathbf{0}$ . Suppose  $\mathbf{0}$  satisfies condition (6). Then for every  $\mathbf{c} \in \mathbb{R}^m$  and  $F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$  with  $\text{rank}(F) = k$ , we have

$$\lambda_1(\mathbf{c} \cdot \mathbf{A} + F) \geq \lambda_{k+1}(\mathbf{c} \cdot \mathbf{A}) \geq 0 \quad \text{and} \quad \lambda_1(-(\mathbf{c} \cdot \mathbf{A} + F)) \geq \lambda_{k+1}(-\mathbf{c} \cdot \mathbf{A}) \geq 0.$$

Hence,  $\mathbf{c} \cdot \mathbf{0} = 0 \in \text{cl}(W(\mathbf{c} \cdot \mathbf{A} + F))$ . Therefore, condition (2) is satisfied.

Conversely, if  $\mathbf{0}$  does not satisfy condition (6), then there exist  $\mathbf{c} \in \mathbb{R}^m$  and  $k \geq 1$  such that  $\lambda_k(\mathbf{c} \cdot \mathbf{A}) < 0$ . Thus there exists  $F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$  such that  $\mathbf{c} \cdot \mathbf{A} + F < 0$  and  $\mathbf{0}$  does not satisfy condition (2). ■

Let  $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ . Although the set  $\text{cl}(W(\mathbf{A}))$  may not be convex if  $m \geq 4$ , we have the following analog of the separation theorem for a convex set.

**THEOREM 4.2.** *Let  $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$  and  $\mathbf{d} = (d_1, \dots, d_m) \in \mathbb{R}^m$ . Then  $\mathbf{d} \notin W_e(\mathbf{A})$  if and only if any one (and hence all) of the following conditions holds:*

- (a) There exists  $\mathbf{K} \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$  such that  $\mathbf{d} \notin \text{cl}(W(\mathbf{A} + \mathbf{K}))$ .
- (b) There exists  $\mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$  with  $\mathbf{d} \notin \text{conv}(\text{cl}(W(\mathbf{A} + \mathbf{F})))$ .
- (c) There exist  $F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$ ,  $r > 0$  and  $\mathbf{c} = (c_1, \dots, c_m) \in \mathbb{R}^m$  such that

$$(4) \quad \left( \sum_{i=1}^m c_i (A_i - d_i I) \right) + F > r I_{\mathcal{H}}.$$

*Proof.* For simplicity, replace  $(A_1, \dots, A_m)$  by  $(A_1 - d_1 I, \dots, A_m - d_m I)$  and assume that  $\mathbf{d} = (0, \dots, 0)$ .

(c) $\Rightarrow$ (b). If (c) holds, we may perturb  $(c_1, \dots, c_m)$  so that  $c_j \neq 0$  for all  $j \in \{1, \dots, m\}$  and condition (4) still holds true. In particular,  $c_1 \neq 0$ . Then let  $\mathbf{F} = (F/c_1, 0, \dots, 0)$ . We have  $\mathbf{c} \cdot \mathbf{a} > r > 0$  for all  $\mathbf{a} \in W(\mathbf{A} + \mathbf{F})$ . Therefore,  $\mathbf{0} \notin \mathbf{conv}(\mathbf{cl}(W(\mathbf{A} + \mathbf{F})))$ .

Clearly, we have (b) $\Rightarrow$ (a), which implies that  $\mathbf{0} \notin W_e(\mathbf{A})$ .

Finally, suppose  $\mathbf{0} \notin W_e(\mathbf{A})$ . Then by Theorem 4.1(2), there exist a real vector  $\mathbf{c} = (c_1, \dots, c_m)$  and  $F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$  such that  $0 = \mathbf{c} \cdot \mathbf{0} \notin \mathbf{cl}(W(\mathbf{c} \cdot \mathbf{A} + F))$ . Since  $\mathbf{cl}(W(\mathbf{c} \cdot \mathbf{A} + F))$  is a closed subinterval  $[s, t]$  of  $\mathbb{R}$ , we may assume that  $0 < s \leq t$ . Let  $r = s/2$ . Then  $(\sum_{i=1}^m c_i A_i) + F > rI_{\mathcal{H}}$ . Hence, (c) holds. ■

Let  $\Omega = \{\mathbf{c} \in \mathbb{R}^m : \langle \mathbf{c}, \mathbf{c} \rangle = 1\}$ . By Theorem 4.2, we have the following result showing that  $W_e(\mathbf{A})$  can be expressed as the intersection of half-spaces.

**COROLLARY 4.3.** *Let  $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$ . Then*

$$\begin{aligned} W_e(\mathbf{A}) &= \bigcap_{\mathbf{c} \in \Omega} \{\mathbf{d} \in \mathbb{R}^m : \langle \mathbf{c}, \mathbf{d} \rangle \leq \max W_e(\mathbf{c} \cdot \mathbf{A})\} \\ &= \{\mathbf{d} \in \mathbb{R}^m : \langle \mathbf{c}, \mathbf{d} \rangle \in W_e(\mathbf{c} \cdot \mathbf{A}) \text{ for all } \mathbf{c} \in \Omega\}. \end{aligned}$$

For  $A \in \mathcal{B}(\mathcal{H})$ , let  $\sigma_e(A) = \bigcap \{\sigma(A + K) : K \in \mathcal{K}(\mathcal{H})\}$  be the essential spectrum of  $A$ . Then for  $A \in \mathcal{S}(\mathcal{H})$ , we have

$$W_e(A) = \mathbf{conv} \sigma_e(A).$$

Thus, one may replace  $\max W_e(\mathbf{c} \cdot \mathbf{A})$  by  $\max \sigma_e(\mathbf{c} \cdot \mathbf{A})$  in Corollary 4.3.

**COROLLARY 4.4.** *Let  $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$ . If  $\mathbf{d} \notin \mathbf{cl}(W(\mathbf{A}))$ , then for any  $i \in \{1, \dots, m\}$  there exists  $F \in \mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$  such that  $\mathbf{d} \notin \mathbf{conv}(\mathbf{cl}(W(\tilde{\mathbf{A}})))$ , where  $\tilde{\mathbf{A}} = (A_1, \dots, A_{i-1}, A_i + F, A_{i+1}, \dots, A_m)$ .*

*Proof.* If  $\mathbf{d} \notin \mathbf{cl}(W(\mathbf{A}))$ , then  $\mathbf{d} \notin W_e(\mathbf{A})$ . The result readily follows from the arguments in the last paragraph in the proof of Theorem 4.2. ■

It follows from Theorem 2.1 that the intersection of the non-convex sets  $\mathbf{cl}(W(\mathbf{A} + \mathbf{K}))$ , which equals  $W_e(\mathbf{A})$ , is a convex set. By Theorem 4.2 and Corollary 4.4, we see that one can replace  $\mathbf{cl}(W(\mathbf{A} + \mathbf{K}))$  by its convex hull in the intersection to obtain the same convex set  $W_e(\mathbf{A})$ . It is known that for any  $\mathbf{B} = (B_1, \dots, B_m) \in \mathcal{B}(\mathcal{H})^m$ ,

$$\mathbf{conv}(\mathbf{cl}(W(\mathbf{B}))) = \{(f(B_1), \dots, f(B_m)) : f \in \Xi\},$$

where  $\Xi$  is the set of linear functionals  $f$  on  $\mathcal{B}(\mathcal{H})$  satisfying  $1 = f(I) = \max\{f(X) : X \in \mathcal{B}(\mathcal{H}), \|X\| \leq 1\}$  (for example, see [10, 11]). So, it is easier to determine  $\mathbf{conv}(\mathbf{cl}(W(\mathbf{A} + \mathbf{K})))$  than  $\mathbf{cl}(W(\mathbf{A} + \mathbf{K}))$ . In fact, we have the following.

COROLLARY 4.5. *Let  $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$  and  $i \in \{1, \dots, m\}$ . Then*

$$\begin{aligned} W_e(\mathbf{A}) &= \bigcap \{ \mathbf{cl}(W(\mathbf{A} + \mathbf{F})) : \mathbf{F} \in \{0\}^{i-1} \times (\mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})) \times \{0\}^{m-i} \} \\ &= \bigcap \{ \mathbf{conv}(\mathbf{cl}(W(\mathbf{A} + \mathbf{F}))) : \mathbf{F} \in \{0\}^{i-1} \times (\mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})) \times \{0\}^{m-i} \}. \end{aligned}$$

*Proof.* Let  $\mathbf{F} \in \{0\}^{i-1} \times (\mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})) \times \{0\}^{m-i}$ . Clearly,

$$W_e(\mathbf{A}) \subseteq \mathbf{cl}(W(\mathbf{A} + \mathbf{F})) \subseteq \mathbf{conv}(\mathbf{cl}(W(\mathbf{A} + \mathbf{F}))).$$

So, we may take the intersection of the second and third sets over all  $\mathbf{F} \in \{0\}^{i-1} \times (\mathcal{F}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})) \times \{0\}^{m-i}$ , and get an inclusion involving the three sets in the corollary. Finally, if  $\mathbf{d} \notin W_e(\mathbf{A})$ , then  $\mathbf{d}$  will not belong to the third set by Corollary 4.4. So, the third set is a subset of  $W_e(\mathbf{A})$ . Hence, the three sets in the corollary are equal. ■

**5. Additional results.** The following result shows that  $W_e(\mathbf{A})$  is unchanged under certain operations on  $\mathbf{A}$ .

THEOREM 5.1. *Let  $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$ .*

(a) *Suppose  $\mathcal{H}_1$  is a closed subspace of  $\mathcal{H}$  such that  $\mathcal{H}_1^\perp$  is finite-dimensional. If  $X : \mathcal{H}_1 \rightarrow \mathcal{H}$  is such that  $X^*X = I_{\mathcal{H}_1}$ , then*

$$W_e(\mathbf{A}) = W_e(X^*A_1X, \dots, X^*A_mX).$$

(b) *For each  $j \in \{1, \dots, m\}$ , suppose  $P_j : \mathcal{H} \rightarrow \mathcal{H}$  is an orthogonal projection such that  $I - P_j$  has finite rank. Then*

$$W_e(\mathbf{A}) = W_e(P_1A_1P_1, \dots, P_mA_mP_m).$$

*Proof.* Use Theorem 2.1. ■

We will establish some additional relationships between the sets  $W_e(\mathbf{A})$  and  $W(\mathbf{A})$ . The next theorem generalizes the results of [29] and [14].

THEOREM 5.2. *Let  $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ . Then  $W_e(\mathbf{A}) = \mathbf{cl}(W(\mathbf{A}))$  if and only if  $\text{Ext}(W(\mathbf{A})) \subseteq W_e(\mathbf{A})$ .*

*Proof.* If  $W_e(\mathbf{A}) = \mathbf{cl}(W(\mathbf{A}))$ , then

$$\text{Ext}(W(\mathbf{A})) \subseteq W(\mathbf{A}) \subseteq W_e(\mathbf{A}).$$

Conversely, if  $\text{Ext}(W(\mathbf{A})) \subseteq W_e(\mathbf{A})$ , then by (P5),

$$\text{Ext}(\mathbf{cl}(W(\mathbf{A}))) \subseteq W_e(\mathbf{A}).$$

Hence,

$$\mathbf{cl}(W(\mathbf{A})) \subseteq \mathbf{conv}(\text{Ext}(\mathbf{cl}(W(\mathbf{A})))) \subseteq \mathbf{conv}(W_e(\mathbf{A})) = W_e(\mathbf{A}).$$

Since  $W_e(\mathbf{A}) \subseteq \mathbf{cl}(W(\mathbf{A}))$ , we have  $W_e(\mathbf{A}) = \mathbf{cl}(W(\mathbf{A}))$ . ■

For  $k \geq 1$ , let  $I_k$  denote the  $k \times k$  identity matrix. Then for  $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$ , we have

$$\mathbf{A} \otimes I_k = (A_1 \otimes I_k, \dots, A_m \otimes I_k) \in \mathcal{S}(\underbrace{\mathcal{H} \oplus \dots \oplus \mathcal{H}}_k)^m.$$

Similarly, let  $I_\infty$  denote the identity operator acting on  $\ell_2$ . Then for  $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$ , we have

$$\mathbf{A} \otimes I_\infty = (A_1 \otimes I_\infty, \dots, A_m \otimes I_\infty) \in \mathcal{S}(\underbrace{\mathcal{H} \oplus \mathcal{H} \oplus \dots}_{\text{infinitely many}})^m.$$

**THEOREM 5.3.** *Let  $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$ . Then for any positive integer  $k > \sqrt{m} - 1$ ,*

$$W(\mathbf{A} \otimes I_k) = \mathbf{conv}(W(\mathbf{A})).$$

Moreover,

$$W_e(\mathbf{A} \otimes I_\infty) = \mathbf{cl}(\mathbf{conv}(W(\mathbf{A}))).$$

*Proof.* Suppose  $k > \sqrt{m} - 1$ . By the result in [34], every  $\mathbf{a} \in \mathbf{conv}(W(\mathbf{A}))$  can be written as  $\mathbf{a} = \sum_{j=1}^k t_j \langle \mathbf{A} \mathbf{x}_j, \mathbf{x}_j \rangle$  for some unit vectors  $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathcal{H}$ . Thus, for  $\mathbf{x} = (\sqrt{t_1} \mathbf{x}_1, \dots, \sqrt{t_k} \mathbf{x}_k) \in \mathcal{H} \oplus \dots \oplus \mathcal{H}$ , we have  $\langle \mathbf{A} \otimes I_k \mathbf{x}, \mathbf{x} \rangle = \mathbf{a}$ . Conversely, if  $\mathbf{a} = \langle \mathbf{A} \otimes I_k \mathbf{x}, \mathbf{x} \rangle \in W(\mathbf{A} \otimes I_k)$ , one can decompose the unit vector  $\mathbf{x}$  into  $k$  parts  $\mathbf{y}_1, \dots, \mathbf{y}_k$  according to the structure of  $\mathcal{H} \otimes I_k$ . Then

$$\mathbf{a} = \sum_{j=1}^k \|\mathbf{y}_j\|^2 \langle A_j \mathbf{y}_j / \|\mathbf{y}_j\|, \mathbf{y}_j / \|\mathbf{y}_j\| \rangle \in \mathbf{conv}(W(\mathbf{A})).$$

If  $\mathbf{a} \in \mathbf{cl}(\mathbf{conv}(W(\mathbf{A})))$ , then there is a sequence  $\{\mathbf{x}_n\}$  of unit vectors in  $\mathcal{H}$  such that  $\langle \mathbf{A} \mathbf{x}_n, \mathbf{x}_n \rangle \rightarrow \mathbf{a}$ . Let

$$\tilde{\mathbf{x}}_n = \left( \underbrace{0, \dots, 0}_{n-1}, \mathbf{x}_n, 0, \dots \right) \in \mathcal{H} \oplus \mathcal{H} \oplus \dots.$$

Then  $\{\tilde{\mathbf{x}}_n\}$  is an orthonormal sequence in  $\mathcal{H} \oplus \mathcal{H} \oplus \dots$  and  $\langle \mathbf{A} \otimes I_\infty \tilde{\mathbf{x}}_n, \tilde{\mathbf{x}}_n \rangle \rightarrow \mathbf{a}$ . Therefore,  $\mathbf{a} \in W_e(\mathbf{A} \otimes I_\infty)$ . Since

$$W_e(\mathbf{A} \otimes I_\infty) \subseteq \mathbf{cl}(W(\mathbf{A} \otimes I_\infty)) = \mathbf{cl}\left(\bigcup_{k=1}^{\infty} W(\mathbf{A} \otimes I_k)\right) \subseteq \mathbf{cl}(\mathbf{conv}(W(\mathbf{A}))),$$

we get the reverse inclusion. ■

**COROLLARY 5.4.** *Let  $S$  be a compact convex subset of  $\mathbb{R}^m$ . Then there are  $\mathbf{A}, \tilde{\mathbf{A}} \in \mathcal{S}(\mathcal{H})^m$  with  $\mathcal{H} = \ell^2$  such that  $W(\mathbf{A})$  is convex and*

$$W(\mathbf{A}) \subseteq S = \mathbf{cl}(W(\mathbf{A})) = W_e(\tilde{\mathbf{A}}).$$

*Proof.* For  $j = 1, \dots, m$ , let  $A_j = \text{diag}(a_{1j}, a_{2j}, \dots)$  act on  $\ell^2$  with the standard canonical basis  $\{e_n : n \geq 1\}$  and be such that  $\{(a_{i1}, \dots, a_{im}) :$

$i \geq 1\}$  is a dense subset of  $S$ . Then for  $\mathbf{A} = (A_1, \dots, A_m)$  the set

$$W(\mathbf{A}) = \mathbf{conv}\{(a_{i1}, \dots, a_{im}) : i \geq 1\}$$

is convex, and  $\tilde{\mathbf{A}} = \mathbf{A} \otimes I_\infty$  satisfies the assertion by Theorem 5.3. ■

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