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Lineability of the set of holomorphic mappings with dense range

by

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Abstract. Let U be an open subset of a separable Banach space. Let $\mathcal{F}$ be the collection of all holomorphic mappings f from the open unit disc $\mathbb{D} \subset \mathbb{C}$ into U such that $f(\mathbb{D})$ is dense in U. We prove the lineability and density of $\mathcal{F}$ in appropriate spaces for different choices of U.

1. Introduction. Suppose that U is an open subset of a complex Banach space E and f is a holomorphic mapping from the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ into U. Then

$$\bigcup_{n=1}^{\infty} f\left(\frac{n}{n+1}\overline{\mathbb{D}}\right) = f(\mathbb{D}) \subset U$$

and each subset $f\big(\frac{n}{n+1}\overline{\mathbb{D}}\big)$ is compact in U. By Baire's theorem, if E is an infinite-dimensional Banach space, then $f(\mathbb{D})$ cannot be equal to U. Therefore, it seems natural to study "how big" the range of f can be. In this connection, in 1973, D. Patil posed the following question at the Conference on Infinite-Dimensional Holomorphy held at University of Kentucky: if E is separable, does there exist any holomorphic mapping f from $\mathbb{D}$ into the open unit ball B_E of E such that $f(\mathbb{D})$ is dense in B_E? In 1976, R. Aron obtained a positive answer to this problem for every separable Banach space E, both finite- and infinite-dimensional [1]. At the same time, J. Globevnik and W. Rudin independently obtained more general solutions:

THEOREM 1.1. *Let E be a separable complex Banach space.*

(a) (Globevnik [5]) *If U is a balanced open subset of E, then there is a holomorphic mapping $f : \mathbb{D} \to U$ such that $f(\mathbb{D})$ is dense in U.*
(b) (Rudin [9]) *If U is a connected open subset of E, then there is a holomorphic mapping $f : \mathbb{D} \to U$ such that $f(\mathbb{D})$ is dense in U.*

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Using some of the ideas from those papers, we will prove the lineability and density of the set of holomorphic mappings from $\mathbb{D}$ into a Banach space with dense range. Let us recall that a collection $\mathcal{F}$ of mappings is *lineable* if there exists an infinite-dimensional vector space contained in $\mathcal{F} \cup \{0\}$. As is known, the aim of results on lineability is to show that some collections of special or pathological mappings are not only non-empty, but even "very large", since they contain infinitely many linearly independent elements. The first theorem on lineability is probably due to Gurariy, who proved in 1966 that there is an infinite-dimensional vector space V of continuous functions on $[0,1]$ such that every $f \in V \setminus \{0\}$ is nowhere differentiable (see [6]). Let us see some other examples.

Example 1.2.

1. The set of differentiable nowhere monotone functions from $\mathbb{R}$ into $\mathbb{R}$ is lineable [2].
2. Let X be a subset of $[-\pi,\pi]$ of measure zero. The set of continuous functions on $[-\pi,\pi]$ whose Fourier series expansion is divergent at any point $t \in X$ is lineable [3].
3. The set of functions $f : \mathbb{C} \to \mathbb{C}$ such that $f(U) = \mathbb{C}$ for every open subset $U \subset \mathbb{C}$ is lineable [4].
4. If $1 \le p < q$, then $L^p[0,1] \setminus L^q[0,1]$ is lineable [8].
5. If $p > q \ge 1$, then the sets $L^p(\mathbb{R}) \setminus L^q(\mathbb{R})$ and $\ell_p \setminus \ell_q$ are lineable [8].

2. The results. Throughout this paper, the letter E will always denote a separable complex Banach space and B_E will be the open unit ball of E. If $R > 0$, let $D(0,R) = \{z \in \mathbb{C} : |z| < R\}$. If U is a subset of E, the symbol $\mathcal{H}(\mathbb{D}, U)$ will denote the set of all holomorphic mappings from $\mathbb{D}$ into E whose ranges are contained in U. The space $\mathcal{H}(\mathbb{D}, U)$ will always be endowed with the compact-open topology.

The main tool in the papers of Globevnik and Rudin was the interpolation of a dense sequence in E by a sequence of holomorphic mappings on $\mathbb{D}$. We recall this result as it appears in the paper of Rudin.

Proposition 2.1 (Rudin [9]). *Let U be a connected open subset of E and let $(x_n)_{n=1}^\infty$ be a dense sequence in U. Then there are three sequences, $(\delta_n)_{n=1}^\infty$, $(D_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$, with the following properties:*

(a) $0 < 2\delta_{n+1} < \delta_n$ *for all* n.
(b) D_n *is an open disc centered at* $e^{i/n} \in \partial\mathbb{D}$ *whose radius is less than* $1/n^2$.
(c) *Each* f_n *is a continuous mapping from* $\overline{\mathbb{D}}$ *into* U *and is holomorphic on* $\mathbb{D}$.
(d) $f_n(e^{i/n}) = x_n$ *and* $f_n(e^{i/k}) = 0$ *if* $n \neq k$.
(e) $\|f_n(z)\| < \delta_n$ *for every* $z \in \overline{\mathbb{D}} \setminus D_n$.

(f) $\sum_{n=1}^{\infty} f_n$ *is a continuous mapping from* $\overline{\mathbb{D}} \setminus \{1\}$ *into* U *and is holomorphic on* $\mathbb{D}$.

Using this proposition we can prove the main result of this paper.

THEOREM 2.2. *The set*

$$\{f \in \mathcal{H}(\mathbb{D}, E) : f(\mathbb{D}) \text{ is dense in } E\}$$

is lineable.

Proof. Let $(y_m)_{m=1}^{\infty}$ be a dense sequence in $E \setminus \{0\}$. Let $(x_n)_{n=1}^{\infty}$ be the following sequence obtained from $(y_m)_{m=1}^{\infty}$:

$$\underbrace{y_1,}\ \underbrace{y_2, y_1,}\ \underbrace{y_3, y_2, y_1,}\ \underbrace{y_4, y_3, y_2, y_1,}\ \underbrace{y_5, y_4, y_3, y_2, y_1}\ \cdots.$$

That is, $x_n = y_{j-k+1}$, where j and k are the unique natural numbers such that

$$1 \le k \le j \quad \text{and} \quad n = \frac{j(j-1)}{2} + k.$$

Let $(\delta_n)_{n=1}^{\infty}$, $(D_n)_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$ be sequences associated to $U = E$ and $(x_n)_{n=1}^{\infty}$ by Proposition 2.1. For every k, define $g_k : \overline{\mathbb{D}} \setminus \{1\} \to E$ by

$$g_k = \sum_{j=k}^{\infty} f_{j(j-1)/2+k}.$$

Note that if $k_1 \ne k_2$, $j_1 \ge k_1$ and $j_2 \ge k_2$, then

$$\frac{j_1(j_1-1)}{2} + k_1 \ne \frac{j_2(j_2-1)}{2} + k_2.$$

Let K be a compact subset of $\overline{\mathbb{D}} \setminus \{1\}$. As $e^{i/n} \to 1$ and the radius of D_n is less than $1/n^2$, there is $n_0 \in \mathbb{N}$ such that $K \cap D_n = \emptyset$ for all $n \ge n_0$. If $z \in K$, then

$$\sum_{n=n_0}^{\infty} \|f_n(z)\| \le \sum_{n=n_0}^{\infty} \delta_n \le 2\delta_{n_0} < \infty.$$

Therefore, the series which defines each function g_k converges uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{1\}$. Hence every g_k is continuous on $\overline{\mathbb{D}} \setminus \{1\}$ and holomorphic on $\mathbb{D}$.

Suppose that $k \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and $\lambda_1 g_1 + \cdots + \lambda_k g_k = 0$ on $\mathbb{D}$. The sum $\lambda_1 g_1 + \cdots + \lambda_k g_k$ is continuous on $\overline{\mathbb{D}} \setminus \{1\}$, so

$$\begin{aligned} 0 &= (\lambda_1 g_1 + \cdots + \lambda_k g_k)(e^{\frac{i}{k(k-1)/2+k}}) = \lambda_k f_{k(k-1)/2+k}(e^{\frac{i}{k(k-1)/2+k}}) \\ &= \lambda_k x_{k(k-1)/2+k}. \end{aligned}$$

Hence $\lambda_k = 0$ and $\lambda_1 g_1 + \cdots + \lambda_{k-1} g_{k-1} = 0$. If we repeat this argument, we get $\lambda_1 = \cdots = \lambda_k = 0$. Therefore, the sequence $(g_k)_{k=1}^{\infty}$ is linearly independent and $\operatorname{span}\{g_k : k \in \mathbb{N}\}$ is infinite-dimensional.

Finally, we have to prove that every non-zero element of $\operatorname{span}\{g_k : k \in \mathbb{N}\}$ has dense range. Pick $\alpha_1, \dots, \alpha_k \in \mathbb{C}$, where at least α_k is non-zero. Let $x \in E$ and $\varepsilon > 0$. Then there is m such that

$$\left\| y_m - \frac{1}{\alpha_k} x \right\| < \frac{\varepsilon}{2|\alpha_k|}.$$

Let

$$n = \frac{(k+m-1)(k+m-2)}{2} + k.$$

Then $x_n = y_m$ and f_n is a summand in g_k, so

$$g_k(e^{i/n}) = f_n(e^{i/n}) = x_n = y_m.$$

As f_n is not a summand in $g_1, \dots, g_{k-1}$, we have

$$g_1(e^{i/n}) = 0, \ \dots, \ g_{k-1}(e^{i/n}) = 0.$$

Since $g_1, \dots, g_k$ are continuous on $\overline{\mathbb{D}} \setminus \{1\}$, there is $w \in \mathbb{D}$ such that

$$\|(\alpha_1 g_1 + \dots + \alpha_k g_k)(w) - (\alpha_1 g_1 + \dots + \alpha_k g_k)(e^{i/n})\| < \varepsilon/2.$$

Hence

$$\begin{aligned}\|(\alpha_1 g_1 + \dots + \alpha_k g_k)(w) - x\| &< \varepsilon/2 + \|(\alpha_1 g_1 + \dots + \alpha_k g_k)(e^{i/n}) - x\| \\ &= \varepsilon/2 + \|\alpha_k x_n - x\| = \varepsilon/2 + \|\alpha_k y_m - x\| < \varepsilon.\end{aligned}$$

This shows that $(\alpha_1 g_1 + \dots + \alpha_k g_k)(\mathbb{D})$ is dense in E. ∎

It is clear that the set $\mathcal{H}(\mathbb{D}, B_E)$ does not contain any non-zero vector space. However, the next theorem shows that the subset of $\mathcal{H}(\mathbb{D}, B_E)$ of holomorphic mappings with dense range has a property similar to lineability. We use $\operatorname{co}(X)$ to denote the convex hull of a subset X.

THEOREM 2.3. *There is a linearly independent sequence $(h_k)_{k=1}^{\infty}$ of holomorphic mappings from $\mathbb{D}$ into B_E such that*

$$\operatorname{co}((h_k)_{k=1}^{\infty}) \subset \{f \in \mathcal{H}(\mathbb{D}, B_E) : f(\mathbb{D}) \textit{ is dense in } B_E\}.$$

Proof. By [1], there are holomorphic mappings $f_1 : \mathbb{D} \to B_{c_0}$ and $f_2 : B_{c_0} \to B_{\ell_2}$ such that $f_1(\mathbb{D})$ is dense in B_{c_0} and $f_2(B_{c_0})$ is dense in B_{ℓ_2}. We will define a sequence $(g_k)_{k=1}^{\infty}$ of holomorphic mappings from ℓ_2 into E and then h_k will be $g_k \circ f_2 \circ f_1$.

Denote by $(e_n)_{n=1}^{\infty}$ the canonical basis of ℓ_2. Let $(y_m)_{m=1}^{\infty}$ be a dense sequence in $B_E \setminus \{0\}$. Let $(x_n)_{n=1}^{\infty}$ be the sequence

$$\underbrace{y_1,}\ \underbrace{y_2, y_1,}\ \underbrace{y_3, y_2, y_1,}\ \underbrace{y_4, y_3, y_2, y_1,}\ \underbrace{y_5, y_4, y_3, y_2, y_1}\ \dots$$

For each $k \in \mathbb{N}$, define $g_k : \ell_2 \to E$ by

$$g_k((a_n)_{n=1}^{\infty}) = \sum_{n=k}^{\infty} a_n^2 x_n.$$

This series converges for every $(a_n)_{n=1}^\infty \in \ell_2$ because

$$\sum_{n=k}^\infty \|a_n^2 x_n\| \le \sum_{n=1}^\infty |a_n|^2 < \infty.$$

The mapping g_k is holomorphic on ℓ_2. Indeed, for $a = (a_n)_{n=1}^\infty$ and $b = (b_n)_{n=1}^\infty$ in ℓ_2, let

$$L_a(b) = \sum_{n=k}^\infty 2a_n b_n x_n.$$

Then $L_a : \ell_2 \to E$ is a continuous linear mapping and

$$\lim_{b\to 0} \frac{g_k(a+b) - g_k(a) - L_a(b)}{\|b\|} = 0.$$

If $k \in \mathbb{N}$ and $(a_n)_{n=1}^\infty \in B_{\ell_2}$, then

$$\|g_k((a_n)_{n=1}^\infty)\| \le \sum_{n=k}^\infty \|a_n^2 x_n\| \le \sum_{n=1}^\infty |a_n|^2 < 1;$$

that is, $g_k((a_n)_{n=1}^\infty) \in B_E$. The set B_E is convex, so if $t_1, \ldots, t_k \in [0,1]$ and $t_1 + \cdots + t_k = 1$, then

$$(t_1 g_1 + \cdots + t_k g_k)(B_{\ell_2}) \subset B_E.$$

Let $t_1, \ldots, t_k \in [0,1]$ be such that $t_1 + \cdots + t_k = 1$. In order to prove that $(t_1 g_1 + \cdots + t_k g_k)(B_{\ell_2})$ is dense in B_E, take $x \in B_E$ and $\varepsilon > 0$. Then there is $m \in \mathbb{N}$ such that

$$\|y_m - x\| < \varepsilon/2.$$

Let $n \in \mathbb{N}$ be such that $n \ge k$ and $x_n = y_m$. Then

$$g_1(e_n) = x_n = y_m, \quad \ldots, \quad g_k(e_n) = x_n = y_m.$$

As $g_1, \ldots, g_k$ are continuous on ℓ_2, there is $a \in B_{\ell_2}$ such that

$$\|(t_1 g_1 + \cdots + t_k g_k)(a) - (t_1 g_1 + \cdots + t_k g_k)(e_n)\| < \varepsilon/2.$$

Hence

$$\begin{aligned}\|(t_1 g_1 + \cdots + t_k g_k)(a) - x\| &< \varepsilon/2 + \|(t_1 g_1 + \cdots + t_k g_k)(e_n) - x\| \\ &= \varepsilon/2 + \|(t_1 y_m + \cdots + t_k y_m) - x\| \\ &= \varepsilon/2 + \|y_m - x\| < \varepsilon.\end{aligned}$$

This shows that $(t_1 g_1 + \cdots + t_k g_k)(B_{\ell_2})$ is dense in B_E.

For every k, let

$$h_k = g_k \circ f_2 \circ f_1 \in \mathcal{H}(\mathbb{D}, E).$$

If $k \in \mathbb{N}$, $t_1, \ldots, t_k \in [0,1]$ and $t_1 + \cdots + t_k = 1$, then $(t_1 h_1 + \cdots + t_k h_k)(\mathbb{D})$ is densely contained in B_E because $f_1(\mathbb{D})$ is dense in B_{c_0}, $f_2(B_{c_0})$ is dense in B_{ℓ_2} and $(t_1 g_1 + \cdots + t_k g_k)(B_{\ell_2})$ is dense in B_E.

To conclude, let us prove that $(h_k)_{k=1}^{\infty}$ is a linearly independent sequence. Suppose that $k \in \mathbb{N}$, $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ and $\lambda_1 h_1 + \dots + \lambda_k h_k = 0$ on $\mathbb{D}$; that is,

$$(\lambda_1 g_1 + \dots + \lambda_k g_k)(f_2 \circ f_1(z)) = 0$$

for all $z \in \mathbb{D}$. The set $f_2 \circ f_1(\mathbb{D})$ is dense in B_{ℓ_2}, so $(\lambda_1 g_1 + \dots + \lambda_k g_k)(a) = 0$ for all a in a dense subset of B_{ℓ_2}. Consequently,

$$\begin{aligned}
\|\lambda_1 x_1\| &= \|(\lambda_1 g_1 + \dots + \lambda_k g_k)(e_1)\| = 0, \\
\|\lambda_1 x_2 + \lambda_2 x_2\| &= \|(\lambda_1 g_1 + \dots + \lambda_k g_k)(e_2)\| = 0, \\
&\vdots \\
\|\lambda_1 x_k + \dots + \lambda_k x_k\| &= \|(\lambda_1 g_1 + \dots + \lambda_k g_k)(e_k)\| = 0.
\end{aligned}$$

As $x_1 \neq 0, \dots, x_k \neq 0$, we deduce that $\lambda_1 = 0, \dots, \lambda_k = 0$. ∎

PROPOSITION 2.4. *Let U be a subset of E. Then*

$$\{f \in \mathcal{H}(\mathbb{D}, U) : f(\mathbb{D}) \text{ is dense in } U\}$$

is a G_δ subset of $\mathcal{H}(\mathbb{D}, U)$.

Proof. Let $(y_m)_{m=1}^{\infty}$ be a dense sequence in U. Given $m, k \in \mathbb{N}$, let

$$F_{m,k} = \{f \in \mathcal{H}(\mathbb{D}, U) : \text{There is } z \in \mathbb{D} \text{ such that } \|f(z) - y_m\| < 1/k\}.$$

We will prove that each $F_{m,k}$ is an open subset of $\mathcal{H}(\mathbb{D}, U)$ and

$$\{f \in \mathcal{H}(\mathbb{D}, U) : f(\mathbb{D}) \text{ is dense in } U\} = \bigcap_{m,k \in \mathbb{N}} F_{m,k}. \tag{2.1}$$

Let $f \in F_{m,k}$. Then there is $z_0 \in \mathbb{D}$ such that $\|f(z_0) - y_m\| < 1/k$. The set

$$V_f = \{g \in \mathcal{H}(\mathbb{D}, U) : \|g(z_0) - f(z_0)\| < 1/k - \|f(z_0) - y_m\|\}$$

is a neighborhood of f in $\mathcal{H}(\mathbb{D}, U)$ in the compact-open topology. If $g \in V_f$, then

$$\|g(z_0) - y_m\| \leq \|g(z_0) - f(z_0)\| + \|f(z_0) - y_m\| < 1/k,$$

so $V_f \subset F_{m,k}$. This shows that $F_{m,k}$ is open in $\mathcal{H}(\mathbb{D}, U)$.

To prove (2.1), take $h \in \mathcal{H}(\mathbb{D}, U)$ such that $h(\mathbb{D})$ is dense in U. Then for every $m, k \in \mathbb{N}$ there is $z \in \mathbb{D}$ such that $\|h(z) - y_m\| < 1/k$; that is, $h \in \bigcap_{m,k \in \mathbb{N}} F_{m,k}$.

Conversely, suppose that $h \in \bigcap_{m,k \in \mathbb{N}} F_{m,k}$. Let $x \in U$ and $\varepsilon > 0$. As $(y_m)_{m=1}^{\infty}$ is dense in U, there are $k_0, m_0 \in \mathbb{N}$ such that

$$1/k_0 < \varepsilon/2 \quad \text{and} \quad \|y_{m_0} - x\| < 1/k_0.$$

As $h \in F_{m_0,k_0}$, there is $z \in \mathbb{D}$ such that $\|h(z) - y_{m_0}\| < 1/k_0$. Therefore,

$$\|h(z) - x\| \leq \|h(z) - y_{m_0}\| + \|y_{m_0} - x\| < 1/k_0 + 1/k_0 < \varepsilon,$$

which shows that $h(\mathbb{D})$ is dense in U. This completes the proof. ∎

THEOREM 2.5. *The set*
$$\{f \in \mathcal{H}(\mathbb{D}, E) : f(\mathbb{D}) \text{ is dense in } E\}$$
is a dense G_δ subset of $\mathcal{H}(\mathbb{D}, E)$.

Proof. Let K be a compact subset of $\mathbb{D}$, $g \in \mathcal{H}(\mathbb{D}, E)$ and $\varepsilon > 0$. Then there is a polynomial $P : \mathbb{C} \to E$ such that $\|P(z) - g(z)\| < \varepsilon/2$ for all $z \in K$ (see [7, Theorem 7.11]).

Let $(y_m)_{m=1}^\infty$, $(x_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ be as in the proof of Theorem 2.2. Then
$$f = \sum_{n=1}^\infty f_n$$
is a continuous mapping from $\overline{\mathbb{D}} \setminus \{1\}$ into E which is holomorphic on $\mathbb{D}$. If we define
$$t = \frac{\varepsilon}{2\sup_{z\in K} \|f(z)\| + 1},$$
then $\|(tf + P)(z) - g(z)\| < \varepsilon$ for all $z \in K$.

In order to prove that $(tf + P)(\mathbb{D})$ is dense in E, take $x \in E$ and $\delta > 0$. As $(y_m)_{m=1}^\infty$ is dense in E, there is $m \in \mathbb{N}$ such that
$$\left\| y_m - \left(\frac{1}{t}x - \frac{1}{t}P(1)\right)\right\| < \frac{\delta}{3t};$$
that is,
$$\|ty_m + P(1) - x\| < \delta/3.$$
Since P is continuous on $\mathbb{C}$ and $e^{i/n} \to 1$, there is $n_0 \in \mathbb{N}$ such that
$$\|P(e^{i/n}) - P(1)\| < \delta/3 \quad \text{for every } n \ge n_0.$$
Now, take $n \in \mathbb{N}$ such that $x_n = y_m$ and $n \ge n_0$. The mapping $tf + P$ is continuous on $\overline{\mathbb{D}} \setminus \{1\}$, so there is $w \in \mathbb{D}$ such that
$$\|(tf + P)(w) - (tf + P)(e^{i/n})\| < \delta/3.$$
Thus,
$$\begin{aligned}
\|(tf + P)(w) - x\| &< \delta/3 + \|tf(e^{i/n}) + P(e^{i/n}) - x\| \\
&= \delta/3 + \|tx_n + P(e^{i/n}) - x\| \\
&= \delta/3 + \|ty_m + P(e^{i/n}) - x\| \\
&\le \delta/3 + \|ty_m + P(1) - x\| + \|P(e^{i/n}) - P(1)\| \\
&< \delta.
\end{aligned}$$
This shows that $(tf + P)(\mathbb{D})$ is dense in E. ∎

The proof of the density of $\{f \in \mathcal{H}(\mathbb{D}, B_E) : f(\mathbb{D}) \text{ is dense in } B_E\}$ in $\mathcal{H}(\mathbb{D}, B_E)$ is much longer and we will need some preliminary results.

Proposition 2.6. *Let $0 < R < 1$ and $\varepsilon > 0$. There is a sequence $(a_n)_{n=1}^{\infty}$ of real numbers such that $0 < a_n < 1$ for all n, $\sum_{n=1}^{\infty}(1 - a_n) < \infty$ and $|b(z) - 1| < \varepsilon$ for every $z \in \overline{D}(0, R)$, where*

$$b(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - a_n z}$$

is the usual Blaschke product (*see* [10, p. 302]).

Proof. We can assume that $\varepsilon < 1$. Let $n \in \mathbb{N}$ and $\delta_n = \varepsilon/2^{n+1} \in (0, 1)$. As $R < 1$, we have $\delta_n R < \delta_n$, so

$$1 + R - \delta_n R > 1 + R - \delta_n > 0.$$

This implies that

$$\frac{1 + R - \delta_n}{1 + R - \delta_n R} < 1.$$

Therefore, there is $a_n \in \mathbb{R}$ such that

$$1 - \frac{1}{2^n} < a_n < 1 \quad \text{and} \quad \frac{1 + R - \delta_n}{1 + R - \delta_n R} < a_n < 1. \tag{2.2}$$

The first inequality in (2.2) implies that $\sum_{n=1}^{\infty}(1 - a_n) < \infty$, so the product

$$b(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - a_n z}$$

is well defined for every $z \in \mathbb{D}$. The second property in (2.2) implies that

$$1 + R - \delta_n < a_n + a_n R - \delta_n a_n R,$$

so

$$(1 - a_n)(1 + R) < \delta_n(1 - a_n R). \tag{2.3}$$

Let $z \in \mathbb{C}$ be such that $|z| \leq R$. Then

$$\begin{aligned} |(a_n - z) - (1 - a_n z)| &\leq (1 - a_n) + (1 - a_n)|z| \\ &\leq (1 - a_n) + (1 - a_n)R = (1 - a_n)(1 + R). \end{aligned} \tag{2.4}$$

We now use the inequalities (2.3) and (2.4):

$$|(a_n - z) - (1 - a_n z)| < \delta_n(1 - a_n R) \leq \delta_n(1 - a_n|z|) \leq \delta_n|1 - a_n z|.$$

Therefore,

$$\left| \frac{a_n - z}{1 - a_n z} - 1 \right| < \delta_n = \frac{\varepsilon}{2^{n+1}} \tag{2.5}$$

for every $n \in \mathbb{N}$ and every $z \in \mathbb{C}$ such that $|z| \leq R$.

Let $z \in \overline{D}(0, R)$. For each n, let

$$w_n = \frac{a_n - z}{1 - a_n z}.$$

Note that $|w_n| < 1$ because $|z| < 1$. In addition, there is $m \in \mathbb{N}$ such that

$$\Big|b(z) - \prod_{n=1}^{m} w_n\Big| < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}|b(z) - 1| &< \frac{\varepsilon}{2} + \Big|\prod_{n=1}^{m} w_n - 1\Big| \le \frac{\varepsilon}{2} + \Big|\prod_{n=2}^{m} w_n - 1\Big||w_1| + |w_1 - 1| \\ &\le \frac{\varepsilon}{2} + \Big|\prod_{n=2}^{m} w_n - 1\Big| + |w_1 - 1| \le \cdots \le \frac{\varepsilon}{2} + \sum_{n=1}^{m} |w_n - 1|.\end{aligned}$$

By (2.5), we obtain

$$|b(z) - 1| < \frac{\varepsilon}{2} + \sum_{n=1}^{m} \frac{\varepsilon}{2^{n+1}} < \varepsilon \quad \text{for all } z \in \overline{D}(0, R). \ \blacksquare$$

PROPOSITION 2.7. *Let $(y_m)_{m=1}^{\infty}$ be a dense sequence in B_E. Then for every $m, k \in \mathbb{N}$, the set*

$$F_{m,k} = \{f \in \mathcal{H}(\mathbb{D}, \overline{B}_E) : \text{There is } z \in \mathbb{D} \text{ such that } \|f(z) - y_m\| < 1/k\}$$

is dense in $\mathcal{H}(\mathbb{D}, \overline{B}_E)$.

Proof. Let $g \in \mathcal{H}(\mathbb{D}, \overline{B}_E)$, K a compact subset of $\mathbb{D}$ and $0 < \varepsilon < 1/(3k)$. We can assume that $K = \overline{D}(0, R)$ for some $0 < R < 1$. By Proposition 2.6, there is a sequence $(a_n)_{n=1}^{\infty}$ of real numbers such that $0 < a_n < 1$ for every n, the Blaschke product

$$b(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - a_n z}$$

defines a holomorphic function on $\mathbb{D}$, $|b(z)| < 1$ for all $z \in \mathbb{D}$ and $|b(z)-1| < \varepsilon$ if $|z| \le R$. As b is uniformly continuous on $\overline{D}(0, a_1)$, there is $\delta > 0$ such that if $z, w \in \overline{D}(0, a_1)$ and $|z - w| < \delta$, then

$$|b(z) - b(w)| < \varepsilon. \tag{2.6}$$

As before, $(x_n)_{n=1}^{\infty}$ will be the sequence

$$\underbrace{y_1,}\ \underbrace{y_2, y_1,}\ \underbrace{y_3, y_2, y_1,}\ \underbrace{y_4, y_3, y_2, y_1,}\ \underbrace{y_5, y_4, y_3, y_2, y_1}\ \cdots.$$

Let $(\delta_n)_{n=1}^{\infty}$, $(D_n)_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$ be the sequences associated to $U = B_E$ and $(x_n)_{n=1}^{\infty}$ by Proposition 2.1. Let $n \in \mathbb{N}$ be chosen so that

$$x_n = y_m, \quad R + 1/n^2 < 1, \quad \delta_n < \varepsilon, \quad 1/n^2 < \delta.$$

For each $z \in \mathbb{D}$, let

$$f(z) = b(a_1 e^{-i/n} z) g(z) + f_n(z).$$

The mapping f is holomorphic from $\mathbb{D}$ into E. If $z \in \mathbb{D} \setminus D_n$, then

$$\|f(z)\| \leq \|g(z)\| + \|f_n(z)\| \leq \|g(z)\| + \delta_n \leq 1 + \delta_n < 1 + \varepsilon.$$

If $z \in \mathbb{D} \cap D_n$, then $|z - e^{i/n}| < 1/n^2 < \delta$ by Proposition 2.1(b). Hence

$$|a_1 e^{-i/n} z - a_1| = a_1 |z - e^{i/n}| < \delta.$$

Moreover, $a_1 e^{-i/n} z \in \overline{D}(0, a_1)$. By (2.6),

$$|b(a_1 e^{-i/n} z)| = |b(a_1 e^{-i/n} z) - b(a_1)| < \varepsilon.$$

Consequently,

$$\|f(z)\| \leq |b(a_1 e^{-i/n} z)| + \|f_n(z)\| < \varepsilon + 1.$$

Therefore, $\|f(z)\| < 1 + \varepsilon$ for every $z \in \mathbb{D}$.

Let $z \in \overline{D}(0, R)$. Then $a_1 e^{-i/n} z$ also belongs to $\overline{D}(0, R)$, so

$$|b(a_1 e^{-i/n} z) - 1| < \varepsilon.$$

As $R + 1/n^2 < 1$, we have $\overline{D}(0, R) \cap D_n = \emptyset$, so $z \in \mathbb{D} \setminus D_n$ and $\|f_n(z)\| < \delta_n < \varepsilon$. Hence

$$\|f(z) - g(z)\| \leq |b(a_1 e^{-i/n} z) - 1| \cdot \|g(z)\| + \|f_n(z)\| < 2\varepsilon$$

for every $z \in \overline{D}(0, R)$.

Since f_n is continuous on $\overline{\mathbb{D}}$, there is $z_1 \in \mathbb{D} \cap D_n$ such that

$$\|f_n(z_1) - f_n(e^{i/n})\| < \varepsilon.$$

Again by (2.6),

$$|b(a_1 e^{-i/n} z_1)| = |b(a_1 e^{-i/n} z_1) - b(a_1)| < \varepsilon$$

and

$$\begin{aligned}\|f(z_1) - y_m\| &\leq \varepsilon + \|f_n(z_1) - y_m\| \leq 2\varepsilon + \|f_n(e^{i/n}) - y_m\| \\ &= 2\varepsilon + \|x_n - y_m\| = 2\varepsilon.\end{aligned}$$

Finally, we consider the function $\frac{1}{1+\varepsilon} f \in \mathcal{H}(\mathbb{D}, \overline{B}_E)$, which belongs to $F_{m,k}$ as

$$\begin{aligned}\left\| \frac{1}{1+\varepsilon} f(z_1) - y_m \right\| &= \left\| \frac{1}{1+\varepsilon} f(z_1) - f(z_1) \right\| + \|f(z_1) - y_m\| \\ &\leq \varepsilon + 2\varepsilon = 3\varepsilon < 1/k.\end{aligned}$$

If $z \in K = \overline{D}(0, R)$, then

$$\begin{aligned}\left\| g(z) - \frac{1}{1+\varepsilon} f(z) \right\| &\leq \|g(z) - f(z)\| + \left\| f(z) - \frac{1}{1+\varepsilon} f(z) \right\| \\ &< 2\varepsilon + \varepsilon = 3\varepsilon.\end{aligned}$$

This proves that $F_{m,k}$ is dense in $\mathcal{H}(\mathbb{D}, \overline{B}_E)$. ∎

THEOREM 2.8. *The set*

$$\{f \in \mathcal{H}(\mathbb{D}, B_E) : f(\mathbb{D}) \text{ is dense in } B_E\}$$

is a dense G_δ subset of $\mathcal{H}(\mathbb{D}, B_E)$.

Proof. As before, $(y_m)_{m=1}^{\infty}$ denotes a dense sequence in B_E and we set

$$F_{m,k} = \{f \in \mathcal{H}(\mathbb{D}, \overline{B}_E) : \text{There is } z \in \mathbb{D} \text{ such that } \|f(z) - y_m\| < 1/k\}.$$

In Propositions 2.4 and 2.7, we have seen that each $F_{m,k}$ is a dense open subset of $\mathcal{H}(\mathbb{D}, \overline{B}_E)$ and

$$\{f \in \mathcal{H}(\mathbb{D}, \overline{B}_E) : f(\mathbb{D}) \text{ is dense in } \overline{B}_E\} = \bigcap_{m,k\in\mathbb{N}} F_{m,k}. \tag{2.7}$$

The space $\mathcal{H}(\mathbb{D}, E)$, endowed with the compact-open topology, is a complete metric space. As $\mathcal{H}(\mathbb{D}, \overline{B}_E)$ is a closed subset of $\mathcal{H}(\mathbb{D}, E)$, it follows that $\mathcal{H}(\mathbb{D}, \overline{B}_E)$ is also a complete metric space. By Baire's theorem and (2.7), the set

$$\{f \in \mathcal{H}(\mathbb{D}, \overline{B}_E) : f(\mathbb{D}) \text{ is dense in } \overline{B}_E\}$$

is dense in $\mathcal{H}(\mathbb{D}, \overline{B}_E)$.

Finally, let $h \in \mathcal{H}(\mathbb{D}, B_E)$, K a compact subset of $\mathbb{D}$ and $\varepsilon > 0$. Then there is $f : \mathbb{D} \to \overline{B}_E$ such that $f(\mathbb{D})$ is dense in $\overline{B}_E$ and $\|f(z) - h(z)\| < \varepsilon$ for every $z \in K$. If there were $z_1 \in \mathbb{D}$ such that $\|f(z_1)\| = 1$, then $\|f(z)\| = 1$ for all $z \in \mathbb{D}$, so f would not have dense range. Therefore, we deduce that $f \in \mathcal{H}(\mathbb{D}, B_E)$. ∎

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