Moore–Penrose inverses of Gram operators on Hilbert C^*-modules

by

M. S. Moslehian (Mashhad), K. Sharifi (Shahrood and Tehran),
M. Forough (Mashhad) and M. Chakoshi (Mashhad)

Abstract. Let t be a regular operator between Hilbert C^*-modules and $t^\dagger$ be its Moore–Penrose inverse. We investigate the Moore–Penrose invertibility of the Gram operator t^*t. More precisely, we study some conditions ensuring that $t^\dagger = (t^*t)^\dagger t^* = t^*(tt^*)^\dagger$ and $(t^*t)^\dagger = t^\dagger t^{*\dagger}$. As an application, we get some results for densely defined closed operators on Hilbert C^*-modules over C^*-algebras of compact operators.

1. Introduction. Hilbert C^*-modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, takes its values in a C^*-algebra. Although Hilbert C^*-modules behave like Hilbert spaces in some ways, some fundamental Hilbert space properties like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not hold in general. A (right) *pre-Hilbert C^*-module* over a C^*-algebra $\mathscr{A}$ is a right $\mathscr{A}$-module $\mathscr{X}$ equipped with an $\mathscr{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$, $(x, y) \mapsto \langle x, y \rangle$, which is $\mathscr{A}$-linear in the second variable y as well as $\langle x, y \rangle = \langle y, x \rangle^*$ and $\langle x, x \rangle \geq 0$ with equality only when $x = 0$. A pre-Hilbert $\mathscr{A}$-module $\mathscr{X}$ is called a *Hilbert $\mathscr{A}$-module* if $\mathscr{X}$ is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$, where the latter norm is the norm of $\mathscr{A}$. Each C^*-algebra $\mathscr{A}$ can be regarded as a Hilbert $\mathscr{A}$-module via $\langle a, b \rangle = a^*b$ $(a, b \in \mathscr{A})$. A Hilbert $\mathscr{A}$-submodule W of a Hilbert $\mathscr{A}$-module $\mathscr{X}$ is an orthogonal summand if $W \oplus W^\perp = \mathscr{X}$, where $W^\perp$ denotes the orthogonal complement of W in $\mathscr{X}$.

Throughout this paper we assume that $\mathscr{A}$ is an arbitrary C^*-algebra (not necessarily unital) and $\mathscr{X}, \mathscr{Y}$ are Hilbert $\mathscr{A}$-modules. By an operator we mean a linear operator. We may deal with bounded and unbounded operators at the same time, so we will denote bounded operators by capital letters and unbounded operators by lower case letters. In addition, $\mathrm{Dom}(\cdot)$,

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Ker($\cdot$) and Ran($\cdot$) stand for the domain, kernel and range of operators, respectively. An *operator t between $\mathscr{X}$ and $\mathscr{Y}$* is a linear operator with Dom(t) $\subseteq$ $\mathscr{X}$ and Ran(t) $\subseteq$ $\mathscr{Y}$. It is $\mathscr{A}$-*linear* if $t(xa) = t(x)a$ for all $x \in$ Dom(t) and all $a \in \mathscr{A}$. The set of all $\mathscr{A}$-linear operators between $\mathscr{X}$ and $\mathscr{Y}$ is denoted by $\mathcal{L}(\mathscr{X}, \mathscr{Y})$. As usual, $\mathcal{L}(\mathscr{X})$ stands for $\mathcal{L}(\mathscr{X}, \mathscr{Y})$ if $\mathscr{X} = \mathscr{Y}$. An operator $t \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ for which Dom(t) is a dense submodule of $\mathscr{X}$ is called a *densely defined operator*. An operator $t \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ is called *closed* if its graph $G(t) = \{(x, t(x)) : x \in \text{Dom}(t)\}$ is a closed submodule of the Hilbert $\mathscr{A}$-module $\mathscr{X} \oplus \mathscr{Y}$ equipped with the C^*-inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle$. If $s \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ is an extension of $t \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$, we write $t \subseteq s$. As usual, this means that Dom(t) $\subseteq$ Dom(s) and $s(x) = t(x)$ for all $x \in$ Dom(t). If t has a closed extension, then it is called *closable* and the operator $\bar{t} \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ with $G(\bar{t}) = \overline{G(t)}$ is called the *closure* of t. A densely defined operator $t \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ is called *adjointable* if there exists a densely defined operator $t^* \in \mathcal{L}(\mathscr{Y}, \mathscr{X})$ with the domain Dom(t^*) $= \{y \in \mathscr{Y} :$ there exist $z \in \mathscr{X}$ such that $\langle t(x), y \rangle = \langle x, z \rangle$ for any $x \in$ Dom(t)$\}$ satisfying the property $\langle t(x), y \rangle = \langle x, t^*(y) \rangle$ for any $x \in$ Dom(t), $y \in$ Dom(t^*). This property ensures that Dom(t^*) is a submodule of $\mathscr{Y}$ and t^* is a closed $\mathscr{A}$-linear map. In the setting of Hilbert spaces any densely defined closed operator has a densely defined adjoint, but in the framework of Hilbert C^*-modules this does not occur in general. It is notable that any adjointable operator with domain $\mathscr{X}$ is automatically a bounded $\mathscr{A}$-linear map. We denote by $\mathbb{B}(\mathscr{X}, \mathscr{Y})$ the set of all adjointable operators from $\mathscr{X}$ into $\mathscr{Y}$. The set $\mathbb{B}(\mathscr{X}, \mathscr{X})$ is abbreviated by $\mathbb{B}(\mathscr{X})$.

If $s \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ and $t \in \mathcal{L}(\mathscr{Y}, \mathscr{Z})$ are densely defined operators between Hilbert C^*-modules, we define the composition operator ts by $(ts)(x) = t(s(x))$ for all $x \in$ Dom(ts), where Dom(ts) $= \{x \in$ Dom(s) $: s(x) \in$ Dom(t)$\}$. Then $ts \in \mathcal{L}(\mathscr{X}, \mathscr{Z})$, but ts is not necessarily densely defined. Suppose two densely defined operators t and s are adjointable; then it is easy to see that $s^* t^* \subseteq (ts)^*$. If T is a bounded adjointable operator, then $s^* T^* = (Ts)^*$. Damaville [3] proved that under certain conditions the product of two regular operators between Hilbert C^*-modules is regular. Regular operators on Hilbert C^*-modules were first introduced by Baaj and Julg [1] and extensively studied in [11].

DEFINITION 1.1. An operator $t \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ is said to be *regular* if t is densely defined, closed and adjointable and the range of $1 + t^*t$ is dense in $\mathscr{X}$. We denote the set of all regular operators in $\mathcal{L}(\mathscr{X}, \mathscr{Y})$ by $\mathcal{R}(\mathscr{X}, \mathscr{Y})$. We abbreviate $\mathcal{R}(\mathscr{X}, \mathscr{X})$ by $\mathcal{R}(\mathscr{X})$.

This definition is equivalent to the notion of regularity introduced by Woronowicz [18]. If t is regular, then t^* is regular, $t = t^{**}$ and also t^*t is regular and self-adjoint. It may occur that t^* is regular but t is not (see [12,

Propositions 2.2 and 2.3]). Also a densely defined operator t with a densely defined adjoint operator t^* is regular if and only if its graph is orthogonally complemented in $\mathscr{X} \oplus \mathscr{Y}$ (see e.g. [6, 11]). Suppose $t \in \mathcal{R}(\mathscr{X}, \mathscr{Y})$ and define $Q_t = (1 + t^*t)^{-1/2}$ and $F_t = tQ_t$. Then $\mathrm{Ran}(Q_t) = \mathrm{Dom}(t)$, $0 \leq Q_t = (1 - F_t^*F_t)^{1/2} \leq 1$ in $\mathbb{B}(\mathscr{X})$ and $F_t \in \mathbb{B}(\mathscr{X}, \mathscr{Y})$.

The following terminology is basic in our study (cf. [7]).

DEFINITION 1.2. *Let* $t \in \mathcal{R}(\mathscr{X}, \mathscr{Y})$. *An operator* $s \in \mathcal{R}(\mathscr{Y}, \mathscr{X})$ *is called a* Moore–Penrose inverse *of* t *if* $tst = t$, $sts = s$, $(ts)^* = \overline{ts}$ *and* $(st)^* = \overline{st}$.

If $t \in \mathcal{L}(\mathscr{H}, \mathscr{H}')$ is a densely defined closed operator between Hilbert spaces, then Pyt'ev [13] proved that there is a densely defined closed operator s satisfying the relations in Definition 1.2. Xu and Sheng [19] proved that an adjointable operator acting on the whole of a Hilbert C^*-module has a Moore–Penrose inverse if and only if it has closed range. In [7, Theorem 3.1], a very useful necessary and sufficient condition for a regular operator t to admit a unique Moore–Penrose inverse, denoted by $t^\dagger$, is given:

THEOREM 1.3. *If* $t \in \mathcal{R}(\mathscr{X}, \mathscr{Y})$, *then the following conditions are equivalent:*

   (i) *t and t^* have unique Moore–Penrose inverses which are adjoint to each other, $t^\dagger$ and $t^{\dagger*}$.*
   (ii) *$\mathscr{X} = \mathrm{Ker}(t) \oplus \overline{\mathrm{Ran}(t^*)}$ and $\mathscr{Y} = \mathrm{Ker}(t^*) \oplus \overline{\mathrm{Ran}(t)}$.*

In this situation, $\overline{t^*t^{\dagger*}}$ *and* $\overline{tt^\dagger}$ *are the projections onto* $\overline{\mathrm{Ran}(t^*)} = \overline{\mathrm{Ran}(t^*t)}$ *and* $\overline{\mathrm{Ran}(t)}$, *respectively.*

Recall that $\mathrm{Dom}(t^\dagger) := \mathrm{Ran}(t) \oplus \mathrm{Ker}(t^*)$ and $t^\dagger : \mathrm{Dom}(t^\dagger) \subseteq \mathscr{Y} \to \mathscr{X}$ is defined by $t^\dagger(t(x_1 + x_2) + x_3) = x_1$ for all $x_1 \in \mathrm{Dom}(t) \cap \overline{\mathrm{Ran}(t^*)}$, $x_2 \in \mathrm{Dom}(t) \cap \mathrm{Ker}(t)$ and $x_3 \in \mathrm{Ker}(t^*)$. The adjoint of $t^\dagger$ is defined similarly.

In view of [7, Corollary 3.4], every regular operator with closed range has a bounded adjointable Moore–Penrose inverse.

Let T be a bounded linear operator with closed range between Hilbert spaces. The *Gram operator* of T is defined to be T^*T. One interesting problem in matrix/operator theory is to investigate the Moore–Penrose inverse $(T^*T)^\dagger$. The equalities $(T^*T)^\dagger = T^\dagger T^{*\dagger}$ and $T^\dagger = T^*(TT^*)^\dagger = (T^*T)^\dagger T^*$ were proved in [4, 9] in the case when T is a bounded linear operator acting on a Hilbert space. In this paper, we generalize them to the case where t is a certain operator in the framework of Hilbert C^*-modules. We set conditions which ensure that $t^*(tt^*)^\dagger = t^\dagger$ and $t^\dagger = (t^*t)^\dagger t^*$ and $(t^*t)^\dagger = t^\dagger t^{*\dagger}$. We present an example showing that the equalities do not hold in general. Finally, we apply our results to densely defined closed operators on Hilbert C^*-modules over C^*-algebras of compact operators.

2. Moore–Penrose invertibility of the Gram operator. In this section we obtain unbounded versions of some results of [4] in the framework of Hilbert C^*-modules. Indeed, we study some conditions which ensure that $t^\dagger = (t^*t)^\dagger t^* = t^*(tt^*)^\dagger$ and $(t^*t)^\dagger = t^\dagger t^{*\dagger}$. Our results are also reformulated in terms of bounded adjointable operators.

LEMMA 2.1. *If $t \in \mathcal{R}(\mathscr{X}, \mathscr{Y})$ has closed range, then so does tt^*.*

Proof. If $\operatorname{Ran}(t)$ is closed, then $\operatorname{Ran}(t^*)$ is closed and $\mathscr{X} = \operatorname{Ker}(t) \oplus \operatorname{Ran}(t^*)$. Let $x \in \operatorname{Dom}(t)$. Then $x = z + t^*y$, for some $y \in \operatorname{Dom}(t^*)$ and $z \in \operatorname{Ker}(t) \subseteq \operatorname{Dom}(t)$. Therefore $t(x) = tt^*(y)$, that is, t and tt^* have the same range. ∎

THEOREM 2.2. *Suppose $t \in \mathcal{R}(\mathscr{X}, \mathscr{Y})$ and $\overline{\operatorname{Ran}(t^*)}$ and $\overline{\operatorname{Ran}(t)}$ are orthogonally complemented in $\mathscr{X}$ and $\mathscr{Y}$, respectively. Then*

(i) $t^*(tt^*)^\dagger \subseteq t^\dagger$,
(ii) $t^\dagger \subseteq (t^*t)^\dagger t^*$ *if and only if* $\operatorname{Ran}(t) \subseteq \operatorname{Dom}(t^*)$.

If in addition t has closed range, then

(iii) $t^*(tt^*)^\dagger = t^\dagger$,
(iv) $t^\dagger = (t^*t)^\dagger t^*$ *when* $\operatorname{Ran}(t) \subseteq \operatorname{Dom}(t^*)$.

Proof. The Moore–Penrose inverse of the regular operator tt^* exists by the orthogonal decompositions into direct summands and the fact that $\overline{\operatorname{Ran}(t)} = \overline{\operatorname{Ran}(tt^*)}$. To prove (i) we have $\operatorname{Dom}(t^*(tt^*)^\dagger) = \operatorname{Dom}((tt^*)^\dagger) = \operatorname{Ran}(tt^*) + \operatorname{Ker}(tt^*) \subseteq \operatorname{Ran}(t) + \operatorname{Ker}(t^*) = \operatorname{Dom}(t^\dagger)$. Let $x = tt^*(x_1 + x_2) + x_3 \in \operatorname{Dom}((tt^*)^\dagger)$ with $x_1 \in \operatorname{Dom}(tt^*) \cap \overline{\operatorname{Ran}(tt^*)}$, $x_2 \in \operatorname{Dom}(tt^*) \cap \operatorname{Ker}(tt^*)$ and $x_3 \in \operatorname{Ker}(tt^*) = \operatorname{Ker}(t^*) = \operatorname{Ker}(t^\dagger)$. Then $(tt^*)^\dagger(x) = x_1$. Therefore

$$t^\dagger(x) = t^\dagger(tt^*(x_1 + x_2) + x_3) = (t^\dagger t)t^*(x_1) + 0 + 0 = t^*(x_1) = t^*(tt^*)^\dagger(x),$$

that is, $t^*(tt^*)^\dagger = t^\dagger$ on $\operatorname{Dom}(t^*(tt^*)^\dagger)$.

Let the operator inclusion of (ii) hold. Then $\operatorname{Dom}(t^\dagger) \subseteq \operatorname{Dom}(t^*)$, which implies that $\operatorname{Ran}(t) \subseteq \operatorname{Dom}(t^*)$. Conversely, if $\operatorname{Ran}(t) \subseteq \operatorname{Dom}(t^*)$ and $x = t(x_1 + x_2) + x_3 \in \operatorname{Dom}(t^\dagger)$ with $x_1 \in \operatorname{Dom}(t) \cap \overline{\operatorname{Ran}(t^*)}$, $x_2 \in \operatorname{Dom}(t) \cap \operatorname{Ker}(t)$ and $x_3 \in \operatorname{Ker}(t^*)$, then $t^\dagger(x) = x_1$. Since $\overline{\operatorname{Ran}(t^*)} = \overline{\operatorname{Ran}(t^*t)}$, we get

$$((t^*t)^\dagger t^*)(t(x_1 + x_2) + x_3) = (t^*t)^\dagger (t^*t)x_1 + 0 + 0$$
$$= \overline{(t^*t)^\dagger(t^*t)}x_1 = x_1 = t^\dagger(x),$$

that is, $(t^*t)^\dagger t^* = t^\dagger$ on $\operatorname{Dom}(t^\dagger)$.

To demonstrate (iii) we suppose that t has closed range. Then tt^* has closed range and $\operatorname{Ran}(tt^*) = \operatorname{Ran}(t)$. In this case, $t^\dagger$ is everywhere defined. Hence,

$$\operatorname{Dom}(t^*(tt^*)^\dagger) = \operatorname{Ran}(tt^*) + \operatorname{Ker}(tt^*) = \operatorname{Ran}(t) + \operatorname{Ker}(t^*) = \operatorname{Dom}(t^\dagger) = \mathscr{Y}.$$

The result now follows from (i). Finally, if $t^\dagger$ is everywhere defined and $\mathrm{Ran}(t) \subseteq \mathrm{Dom}(t^*)$, then the inclusion of (ii) becomes an equality, which completes the proof. ∎

COROLLARY 2.3. *Suppose $t \in \mathcal{R}(\mathscr{X}, \mathscr{Y})$ has closed range and $\mathrm{Ran}(t) \subseteq$ $\mathrm{Dom}(t^*)$. Then*

(i) $t^\dagger = (t^*t)^\dagger t^* = t^*(tt^*)^\dagger$,
(ii) $(t^*t)^\dagger = t^\dagger t^{*\dagger}$.

Proof. According to [6, Proposition 1.2] and [11, Theorem 3.2], $\mathrm{Ran}(t)$ and $\mathrm{Ran}(t^*)$ are orthogonally complemented in $\mathscr{Y}$ and $\mathscr{X}$, respectively. These facts together with Theorem 2.2 imply the equalities of the first part.

The closedness of the range of t^* and part (iii) of Theorem 2.2 ensure that $t^{*\dagger} = t(t^*t)^\dagger$. Therefore

$$t^\dagger t^{*\dagger} = (t^*t)^\dagger t^* t(t^*t)^\dagger = (t^*t)^\dagger. \quad \blacksquare$$

COROLLARY 2.4. *Suppose $T \in \mathbb{B}(\mathscr{X}, \mathscr{Y})$ has closed range. Then*

(i) $T^\dagger = (T^*T)^\dagger T^* = T^*(TT^*)^\dagger$,
(ii) $(T^*T)^\dagger = T^\dagger T^{*\dagger}$.

This follows from the fact that $\mathbb{B}(\mathscr{X}, \mathscr{Y})$ is a subset of $\mathcal{R}(\mathscr{X}, \mathscr{Y})$. The set of all regular operators was studied from a topological point of view in [15–17].

The reader should be aware that the operator inclusions in Theorem 2.2 may be strict even for bounded operators.

EXAMPLE 2.5. Let V be the Volterra operator on $L^2[0,1]$, i.e., $(Vf)(x) = \int_0^x f(y)\,dy$. Then the adjoint of V is given by $(V^*f)(x) = \int_x^1 f(y)\,dy$. The operators V and V^* are bounded and injective, that is, $\mathrm{Ker}(V) = \mathrm{Ker}(V^*) = \{0\}$, and $\mathrm{Ran}(V)$ and $\mathrm{Ran}(V^*)$ are dense in $L^2[0,1]$. Indeed,

$$L^2[0,1] = \mathrm{Ker}(V^*) \oplus \overline{\mathrm{Ran}(V)} = \overline{\mathrm{Ran}(V)},$$
$$L^2[0,1] = \mathrm{Ker}(V) \oplus \overline{\mathrm{Ran}(V^*)} = \overline{\mathrm{Ran}(V^*)}.$$

Moreover, we have

$$(VV^*f)(x) = \int_0^x \left(\int_y^1 f(t)\,dt \right) dy = x \int_x^1 f(y)\,dy + \int_0^x y f(y)\,dy.$$

We claim that $\mathrm{Ran}(VV^*) \neq \mathrm{Ran}(V)$. To see this, we consider the identity function $f(x) = x$ in $\mathrm{Ran}(V)$. If $f = VV^*g$ for some $g \in L^2[0,1]$, then

$$f'(x) = \frac{d}{dx} \left(\int_0^x \left(\int_y^1 g(t)\,dt \right) dy \right) = \int_x^1 g(t)\,dt \quad \text{for all } x \in [0,1],$$

which implies that $f'(1) = 0$, a contradiction. This means that the Volterra integral equation $x = x \int_x^1 g(y)\,dy + \int_0^x y g(y)\,dy$ has no solution in $L^2[0,1]$.

Since $\mathrm{Dom}(V^*(VV^*)^\dagger) = \mathrm{Dom}((VV^*)^\dagger) = \mathrm{Ran}(VV^*) + \mathrm{Ker}(VV^*) = \mathrm{Ran}(VV^*) + \mathrm{Ker}(V^*) = \mathrm{Ran}(VV^*)$, we have $\mathrm{Dom}(V^*(VV^*)^\dagger) \subseteq \mathrm{Ran}(V) = \mathrm{Ran}(V) + \mathrm{Ker}(V^*) = \mathrm{Dom}(V^\dagger)$. The latter inclusion is strict since $\mathrm{Ran}(VV^*) \subset \mathrm{Ran}(V)$. Hence, $\mathrm{Dom}(V^*(VV^*)^\dagger) \subset \mathrm{Dom}(V^\dagger)$. This means that the operator inclusions in Theorem 2.2 may be strict even for bounded operators on Hilbert spaces. Another example can be found in the book of Ben-Israel and Greville [2, Chapter 9, Ex. 26].

The above example also shows that the assumption on closedness of the range of t in part (ii) of Corollary 2.3 cannot be removed.

THEOREM 2.6. *Suppose* $t \in \mathcal{R}(\mathscr{X})$, *and* $\overline{\mathrm{Ran}(t^*)}$ *and* $\overline{\mathrm{Ran}(t)}$ *are orthogonally complemented in* $\mathscr{X}$. *If* S *is a bounded adjointable operator which commutes with* t *and* t^*, *then* $St^\dagger \subseteq t^\dagger S$.

Proof. Suppose S commutes with t and t^* and $\omega > 0$. Then S commutes with $\omega 1 + tt^*$, hence also with the bounded operator $(\omega 1 + tt^*)^{-1}$. In view of commutativity of S with t^* and $(\omega 1 + tt^*)^{-1}$, boundedness of S and Theorem 2.8 of [14], we infer that

$$St^\dagger = S \lim_{\omega \to 0^+} t^*(\omega 1 + tt^*)^{-1} = \lim_{\omega \to 0^+} t^*(\omega 1 + tt^*)^{-1} S = t^\dagger S \quad \text{on } \mathrm{Dom}(t^\dagger). \blacksquare$$

PROPOSITION 2.7. *Suppose* $t \in \mathcal{R}(\mathscr{X})$, *and* $\overline{\mathrm{Ran}(t^*)}$ *and* $\overline{\mathrm{Ran}(t)}$ *are orthogonally complemented in* $\mathscr{X}$. *Then* t *is selfadjoint if and only if* $t = \overline{t^\dagger t}\, t^*$.

Proof. The assertion follows from the fact that $\overline{t^\dagger t}$ is the orthogonal projection onto $\overline{\mathrm{Ran}(t^\dagger)} = \mathrm{Ker}(t^{\dagger *})^\perp = \mathrm{Ker}(t)^\perp = \overline{\mathrm{Ran}(t^*)}$, which implies $t^* = \overline{t^\dagger t}\, t^*$. $\blacksquare$

We end our paper with the following useful observations. By an *arbitrary C^*-algebra of compact operators* we mean an algebra $\mathscr{A}$ of the form $c_0\text{-}\bigoplus_{i \in I} \mathbb{K}(\mathscr{H}_i)$, i.e., $\mathscr{A}$ is the c_0-direct sum of elementary C^*-algebras $\mathbb{K}(\mathscr{H}_i)$ of all compact operators acting on Hilbert spaces $\mathscr{H}_i$, $i \in I$. If $\mathscr{A}$ is an arbitrary C^*-algebra of compact operators, then for every pair of Hilbert $\mathscr{A}$-modules $\mathscr{X}, \mathscr{Y}$, every densely defined closed operator $t : \mathrm{Dom}(t) \subseteq \mathscr{X} \to \mathscr{Y}$ is automatically regular and has a Moore–Penrose inverse (cf. [6, 7, 5, 10]). The following results follow from [7, Theorem 3.8].

COROLLARY 2.8. *Suppose* $\mathscr{X}$ *and* $\mathscr{Y}$ *are Hilbert C^*-modules over an arbitrary C^*-algebra of compact operators and* $t \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ *is a densely defined closed operator. Then the conclusions of Theorems* 2.2, 2.6 *and Proposition* 2.7 *hold.*

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M. S. Moslehian
Department of Pure Mathematics
Center of Excellence in Analysis
on Algebraic Structures (CEAAS)
Ferdowsi University of Mashhad
P.O. Box 1159, Mashhad 91775, Iran
E-mail: moslehian@ferdowsi.um.ac.ir
moslehian@member.ams.org
http://profsite.um.ac.ir/~moslehian/

M. Forough
Department of Mathematics
Ferdowsi University of Mashhad
P.O. Box 1159, Mashhad, Iran
E-mail: mforough86@gmail.com

K. Sharifi
Department of Mathematics
Shahrood University of Technology
P.O. Box 3619995161-316, Shahrood, Iran
School of Mathematics
Institute for Research
in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran
E-mail: sharifi.kamran@gmail.com

M. Chakoshi
Tusi Mathematical Research Group
P.O. Box 1113, Mashhad, Iran
E-mail: machakoshi@yahoo.com