A subset $E$ of a discrete abelian group is a “Fatou-Zygmund interpolation set” ($FZI_0$ set) if every bounded Hermitian function on $E$ is the restriction of the Fourier-Stieltjes transform of a discrete, non-negative measure.

We show that every infinite subset of a discrete abelian group contains an $FZI_0$ set of the same cardinality (if the group is torsion free, a stronger interpolation property holds) and that $\varepsilon$-Kronecker sets are $FZI_0$ (with that stronger interpolation property).

1. Introduction and summary of results. In this paper we continue [6, 7, 10], where we studied $I_0$ and $\varepsilon$-Kronecker sets (definitions are below) in discrete abelian groups $\Gamma$, with compact dual groups $G$. In particular, in [7], we showed that many $I_0$ sets (including $\varepsilon$-Kronecker and Hadamard sets) had the property that every bounded function on them could be interpolated by the Fourier-Stieltjes transform of a discrete measure with arbitrarily small support.

In this paper we address the interpolation issue, but now ask that the interpolating measures be real or non-negative. Our main results show that such interpolation sets are plentiful, as is well known to be the case for other interpolation sets such as Sidon and $I_0$ sets.

1.1. Definitions and main results

Definition 1. A function $\varphi$ on a subset $E \subset \Gamma$ is Hermitian if $\varphi(\chi) = \overline{\varphi(\chi^{-1})}$ for all $\chi \in E$ with $\chi^{-1} \in E$.

Definition 2. Let $\varepsilon > 0$. A set $E \subset \Gamma$ is:

1. asymmetric if $\gamma \in E \cap E^{-1}$ implies $\gamma = \gamma^{-1}$, antisymmetric if $E \cap E^{-1} = \emptyset$, and symmetric if $E = E^{-1}$;
(2) Sidon (resp. $I_0$) if every bounded function $\varphi$ on $E$ is the restriction of a Fourier–Stieltjes transform of a measure (resp. of a discrete measure) [19, 14];

(3) $\varepsilon$-Kronecker if for every function $\varphi : E \rightarrow \mathbb{T}$, there exists $x \in G$ such that $|\langle \chi, x \rangle - \varphi(\chi)| < \varepsilon$ for all $\chi \in E$ \(^{(1)}\);

(4) $RI_0$ (resp. $FZI_0$) if every bounded Hermitian function $\varphi$ on $E$ is the restriction of a Fourier–Stieltjes transform of a real (resp. non-negative real), discrete measure.

In each case but the first, we append “($U$)” to the definition if the interpolating measures can be taken to be concentrated on the set $U \subset G$.

**Definition 3.** $E$ is $FZI_0(U)$ for all open $U$ with bounded constants \(^{(2)}\) if there is a constant $K$ such that for each open $U$ there is a finite set $\Delta$ such that for each bounded $\varphi : E \setminus \Delta \rightarrow \mathbb{C}$ there is a non-negative, discrete measure $\mu$ concentrated on $U$ such that $\hat{\mu} = \varphi$ on $E \setminus \Delta$ and $\|\mu\| \leq K$. We make analogous definitions for a set to be “$I_0(U)$ (or $RI_0(U)$) with bounded constants”.

Item (4) of Definition 2 is new (but not unanticipatable), and it is with these classes of sets and their relations to the other classes that this paper is concerned.

Clearly $I_0(U)$ sets are Sidon($U$) (the converse is shown to be not true in [17]), $FZI_0(U)$ sets are $RI_0(U)$, and any asymmetric $RI_0(U)$ set is $I_0(U)$. Less trivially, we show here that a set $E$ is $RI_0(U)$ if and only if $E \cup E^{-1}$ is $I_0(U)$, and hence there are $I_0$ sets that are not $RI_0$ (see Thm. 2.5 and Example 2.8). The class of $FZI_0$ sets is smaller again, as the singleton $\{0\} \subset \mathbb{Z}$ is $RI_0$ but not $FZI_0$. However, we do not know if $E$ being $RI_0$ and the identity not in $E$ implies $E$ is $FZI_0$. We also do not know if there are any non-trivial sets that are $I_0(U)$ with bounded constants that are not $FZI_0$. Sidon sets in the dual of a connected group are Sidon($U$) [3]; such interpolation can also be done with non-negative measures (see Florek [1] who improves upon previous results and gives a Sidon set version of our $FZI_0(U)$ results).

The main contributions of this paper improve upon the previous existence theorem for $I_0$ sets [13, 15], and are:

(1) Every infinite discrete abelian group $\Gamma$ contains an $FZI_0(U)$ set with bounded constants and of cardinality $\#\Gamma$ (Theorem 4.1).

\(^{(1)}\) See [5, 6, 10, 7, 21] for applications and properties of $\varepsilon$-Kronecker sets. Given and Kunen [5] use the term “$\varepsilon$-free”. For existence theorems for $\varepsilon$-Kronecker sets, see [4, Lem. 3.2], [5, Lem. 3.8], [11, Thm. 3.1], and [12, Thm. 4.1].

\(^{(2)}\) We generally omit the phrase “for all open $U$".
(2) Every infinite subset $E$ of the discrete abelian group $\Gamma$ contains an $FZI_0$ set of the same cardinality as $E$ (Theorem 4.4).

(3) If the dual $G$ is connected (i.e., $\Gamma$ is torsion free), then every infinite subset $E \subset \Gamma$ contains an $FZI_0(U)$ set with bounded constants of the same cardinality (Theorem 4.6). (Connectedness cannot be dispensed with here; see Remark 4.5.)

The first sets of these types were the Hadamard sets $E = \{n_j\} \subset \mathbb{N}$, where $\inf n_{j+1}/n_j \geq q > 1$. Hadamard sets with ratio $q$ are known to be $\varepsilon$-Kronecker for $\varepsilon > |1 - e^{i\pi/(q-1)}|$ ([15, Lem. 2.4(1)]) and $I_0(U)$ with bounded constants (3). Similar arguments can be used to show that they are $FZI_0(U)$ with bounded constants. More generally, in Theorem 3.1 we prove that $\varepsilon$-Kronecker sets are $FZI_0(U)$ with bounded constants if $\varepsilon < \sqrt{2}$ and this fact is used in obtaining our main results.

The terminology $FZ$ (for Fatou–Zygmund set) was adapted from [16] where it was used to denote sets with the property that every bounded function on the set can be interpolated by a non-negative (but not necessarily discrete) measure. Examples include asymmetric Sidon sets in duals of connected groups [3]. The notion “$I_0(U)$ with bounded constants” appears in [7], where it gives some insight into why some sets are $I_0(U)$.

For non-abelian versions of our results, see [8]. For further characterizations of and union results for $RI_0$ and $FZI_0$ sets, see [9].

2. Preliminaries

2.1. Notation. For a compact abelian group $G$, $G_d$ denotes the corresponding group with the discrete topology. The Bohr compactification of $\Gamma$ is denoted by $\overline{\Gamma}$. If $E \subset \Gamma$, $\overline{E}$ denotes the closure of $E$ in $\overline{\Gamma}$. Our groups are written multiplicatively, except for $\mathbb{Z}$ and $\mathbb{R}$, and $1 = 1_\Gamma$ denotes the identity of $\Gamma$ except for $\mathbb{Z}, \mathbb{R}$, where $0$ is used. We write $B(\ell^\infty(E))$ for the unit ball of $\ell^\infty(E)$. We write $M_d(U)$ for the discrete measures concentrated on $U \subseteq G$ and we let

$$D(N, U) = \left\{ \sum_{j=1}^N a_j \delta_{x_j} : |a_j| \leq 1, x_j \in U, 1 \leq j \leq N \right\}.$$ 

A superscript r or + on a space of measures refers to the real (respectively, positive) measures in that subset.

We have the following variation of a result in [7, Prop. 2.2].

**Proposition 2.1.** Let $G$ be a compact group, $U$ a $\sigma$-compact subset of $G$ and $E \subset \Gamma$. The following are equivalent:

(3) That Hadamard sets are $I_0$ was first proved in [20]; for other proofs see [14, 15]. The $I_0$ with bounded constants property was proved in [7].
(1) $E$ is $RI_0(U)$ (resp. $FZI_0(U)$).

(2) There is a constant $N$ such that for all Hermitian $\varphi \in B(\ell^{\infty}(E))$ there exists $\mu \in M^+_d(U)$ (resp. $M^+_d(U)$) with $\|\mu\| \leq N$ and $\bar{\mu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in E$.

(3) There exist $0 < \varepsilon < 1$ (equivalently, for every $0 < \varepsilon < 1$) and integer $N$ such that for all Hermitian $\varphi \in B(\ell^{\infty}(E))$ there exists $\mu \in M^+_d(U)$ (resp. $M^+_d(U)$) with $\|\mu\|_{M(G)} \leq N$ and $|\bar{\mu}(\gamma) - \varphi(\gamma)| < \varepsilon$ for all $\gamma \in E$.

(4) There exists $0 < \varepsilon < 1$ (equivalently, for all $0 < \varepsilon < 1$) and $N$ such that for all Hermitian $\phi \in B(l^{\infty}(E))$ there is some $\mu \in D^+(N, U)$ (resp. $D^+(N, U)$) with $|\bar{\mu}(\gamma) - \phi(\gamma)| < \varepsilon$ for all $\gamma \in E$.

(5) There exists $0 < \varepsilon < 1$ (equivalently, for all $0 < \varepsilon < 1$) such that for all Hermitian $\varphi \in B(\ell^{\infty}(E))$ there exists $\mu \in M^+_d(U)$ (resp. $M^+_d(U)$) with $|\bar{\mu}(\gamma) - \varphi(\gamma)| < \varepsilon$ for all $\gamma \in E$.

**Proof.** (2)$\Rightarrow$(1)$\Rightarrow$(5) and (4)$\Rightarrow$(3) are trivial. The implication (3)$\Rightarrow$(2) is an iteration argument which is essentially shown in [7]. A finite iteration argument shows the equivalences of “there exists” and “for all” in (3)–(5). We remark that for $RI_0(U)$ sets, the implication (1)$\Rightarrow$(2) is a standard application of the open mapping theorem. It remains only to show (5)$\Rightarrow$(4).

We give the proof for the $FZI_0(U)$ case.

There is no loss of generality in assuming $E$ is asymmetric since a Hermitian function has a unique Hermitian extension from $E$ to $E \cup E^{-1}$.

As $U$ is $\sigma$-compact we can write $U = \bigcup_{n=1}^{\infty} V_n$, where $V_n$ are compact, nested sets. Let $\mathbb{D}_\gamma = [-1, 1]$ if $\gamma = \gamma^{-1}$ and $\mathbb{D}_\gamma = \{t \in \mathbb{C} : |t| \leq 1\}$ if $\gamma \neq \gamma^{-1}$. Let $\mathbb{D}_E = \prod_{\gamma \in E} \mathbb{D}_\gamma$. Set

$$W_{n,k} = \{\phi \in \mathbb{D}_E : \exists \mu \in D^+(n, V_k) \text{ with } |\bar{\mu}(\gamma) - \phi(\gamma)| \leq \varepsilon/2 \forall \gamma \in E\}.$$

Given $\mu = \sum_{j=1}^{n} a_j \delta_{x_j}$ with $x_j \in U$, there exists $k$ such that $x_j \in V_k$ for all $j = 1, \ldots, n$. Hence

$$D^+(n, U) = \bigcup_{k=1}^{\infty} D^+(n, V_k).$$

Using (5) with suitably small $\varepsilon > 0$, we have $\bigcup_{n,k} W_{n,k} = \mathbb{D}_E$.

The closure of $V_k$ ensures that $W_{n,k}$ is closed and hence the Baire category theorem implies that some $W_{n,k}$ has non-empty interior. If we let $W_n = \bigcup_k W_{n,k}$, it follows that some $W_n$ has non-empty interior and therefore there is a finite set $F \subseteq E$ and a point $(z_1, \ldots, z_{|F|})$ such that $(z_1, \ldots, z_{|F|}) \times \mathbb{D}_{E \setminus F} \subseteq W_n$.

Consider the subset $S$ of $l^{\infty}(E)$ consisting of the Hermitian elements which vanish off $F$. As $F$ is finite, $S$ is a finite-dimensional, real subspace.

Take a basis of $S$, say $e_1, \ldots, e_l$, where $e_j \in B(l^{\infty}(E))$. Since all norms are
comparable on a finite-dimensional space, there is some $c > 0$ such that
\[
\left\| \sum b_j e_j \right\|_{\ell^\infty} \geq c \sum |b_j|.
\]
Each $\pm e_j$ is Hermitian, so again by (5) we can obtain $\mu_j, \nu_j \in M_0^+(U)$ such that
\[
|\hat{\mu}_j(\gamma) - e_j(\gamma)| < c\varepsilon/4n, \quad |\hat{\nu}_j(\gamma) - (-e_j)(\gamma)| < c\varepsilon/4n
\]
for all $\gamma \in E$. By taking suitably large partial sums we can assume there exists some $m$ such that $\mu_j, \nu_j \in D^+(m, U)$ for all $j$.

Let $\phi \in B(l^\infty(E))$ be Hermitian. Since $\phi$ coincides on $E \setminus F$ with an element of $W_n$, we can find $\mu \in D^+(n, U)$ such that $|\hat{\mu}(\gamma) - \phi(\gamma)| \leq \varepsilon/2$ for all $\gamma \in E \setminus F$. As $\mu$ is a positive measure and $E$ is asymmetric, $(\phi - \hat{\mu})|_F$ (extended by 0 off $F$) belongs to $S$ and therefore equals $\sum b_j e_j$ for some $b_j$ real. Write $b_j = b_j^+ - b_j^-$ where $b_j^+ \geq 0$. Notice
\[
c \sum |b_j| \leq \|(\phi - \hat{\mu})|_F\|_{\ell^\infty} \leq 1 + \|\mu\|_M(U) \leq 2n.
\]
For $\gamma \in E$,
\[
|\phi - \hat{\mu} - \left(\sum (b_j^+ \hat{\mu}_j + b_j^- \hat{\nu}_j)\right)(\gamma)|
\leq |(\phi - \hat{\mu})|_{E \setminus F} + \sum (b_j^+ (e_j - \hat{\mu}_j) + b_j^- (-e_j - \hat{\nu}_j))|
\leq \sup_{\gamma \in E \setminus F} |(\phi - \hat{\mu})(\gamma)| + \sup_{\gamma \in E} \left|\sum (b_j^+ (e_j - \hat{\mu}_j) + b_j^- (-e_j - \hat{\nu}_j))\right|
\leq \frac{\varepsilon}{2} + \frac{c\varepsilon}{4n} \sum |b_j| \leq \varepsilon.
\]
Finally, we note that
\[
\mu + \sum b_j^+ \mu_j + \sum b_j^- \nu_j \in D^+(N, U),
\]
with $N = n + m \dim F$, and as $N$ is independent of the choice of $\phi$ this completes the proof.

**Remark 2.2.** A similar Baire category theorem argument will show that $E \subseteq \Gamma$ is $I_0(U)$ if there exists some $\varepsilon > 0$ such that for every bounded $\phi$ on $E$ there is a discrete measure $\mu$, concentrated on $U$, such that $|\hat{\mu}(\gamma) - \varphi(\gamma)| < \varepsilon$ for all $\gamma \in E$. This improves [7] where it was shown that such a set $E$ is $I_0(U^2)$.

**Remark 2.3.** Proposition 2.1(3) implies that $\varepsilon$-Kronecker sets with $\varepsilon < 1$ are $FZI_0$. We will show in Theorem 3.1 that this is true whenever $\varepsilon < \sqrt{2}$ and that such sets are even $FZI_0(U)$ with bounded constants.

It is a classical result that $I_0$ sets are characterized by having the property that $\pm 1$-valued $E$-functions have continuous extensions to $\bar{\Gamma}$. A similar result holds for $RI_0(U)$ and $FZI_0(U)$ sets, as we show in Proposition 2.4 below.
Note that a set is $I_0(U)$, $RI_0(U)$ or $FZI_0(U)$, if and only if the same is true for any translate of $U$, so there is no loss of generality in assuming $U$ is a neighbourhood of the identity.

**Proposition 2.4.** Let $0 < \varepsilon < 1$, $E \subset \Gamma$ be asymmetric, and $U \subset G$ a symmetric, compact neighbourhood of the identity. Let $E_2 = \{\chi \in E : \chi^2 = 1\}$. Suppose that for each pair of Hermitian functions $r : E \rightarrow \{\pm 1\}$ and $s : E \setminus E_2 \rightarrow \{\pm i\}$ there are measures $\mu_1, \mu_2 \in M^+_d(U)$ (resp. $\in M^+_d(U)$) such that $|\widehat{\mu}_1(\chi) - r(\chi)| < \varepsilon$ for all $\chi \in E$ and $|\widehat{\mu}_2(\chi) - s(\chi)| < \varepsilon$ for all $\chi \in E \setminus E_2$. Then $E$ is $RI_0(U^2)$ (resp. $FZI_0(U^2)$).

**Proof.** We give the proof in the $FZI_0(U)$ case. Let $0 < \varepsilon_0 < 1/2$. First, suppose $\varphi \in B(\ell^\infty(E \setminus E_2))$ is imaginary and Hermitian. Put

$$s(\chi) = \begin{cases} i & \text{if } \Im \varphi(\chi) \geq 0, \\ -i & \text{if } \Im \varphi(\chi) < 0. \end{cases}$$

Since $E$ is asymmetric, $s$ is Hermitian. Thus, we can obtain $\mu_2$ as in the hypothesis. Then

$$\left|\frac{\widehat{\mu}_2(\chi)}{2} - \varphi(\chi)\right| < \frac{1 + \varepsilon}{2} \quad \text{for all } \chi \in E \setminus E_2.$$

Replacing $\varphi$ with $\varphi - i\Im \widehat{\mu}_2/2$, and iterating this process, we obtain $\nu_2 \in M^+_d(U)$ such that

$$|\widehat{\nu}_2 - \varphi| < \varepsilon_0 \quad \text{on } E \setminus E_2.$$

Now consider the general case, $\varphi(\chi) = a_\chi + b_\chi i$ on $E$. We find $\nu_2$ as above for $i\Im \varphi$ on $E \setminus E_2$. We note that $\varphi - \widehat{\nu}_2$ is Hermitian on $E$ since $\nu_2$ is a real measure. We set

$$r(\chi) = \begin{cases} 1 & \text{if } \Re \varphi(\chi) - \Re \widehat{\nu}_2(\chi) \geq 0, \\ -1 & \text{if } \Re \varphi(\chi) - \Re \widehat{\nu}_2(\chi) < 0, \end{cases}$$

for $\chi \in E$. Obtain $\mu_1$ as in the hypothesis. Then $\mu_1 + \widehat{\mu}_1 \in M^+_d(U)$ (4). Also $\widehat{\mu}_1(\chi) + (\widehat{\mu}_1)^\wedge(\chi) = 2\Re \widehat{\mu}_1(\chi)$. Hence

$$\left|\frac{\widehat{\mu}_1(\chi) + (\widehat{\mu}_1)^\wedge(\chi)}{4} - (\Re \varphi(\chi) - \Re \widehat{\nu}_2(\chi))\right| < \frac{1 + \varepsilon}{2} \quad \text{for all } \chi \in E.$$

As $\widehat{\mu}_1 + (\widehat{\mu}_1)^\wedge$ is real-valued, we iterate as before, and thus we find a $\nu_1 \in M^+_d(U)$ with a real-valued Fourier transform and satisfying

$$|\widehat{\nu}_1(\chi) - (\Re \varphi(\chi) - \Re \widehat{\nu}_2(\chi))| < \varepsilon_0.$$

Then $|\widehat{\nu}_1(\chi) + \widehat{\nu}_2(\chi) - \varphi(\chi)| < 2\varepsilon_0 < 1$.

We appeal to Proposition 2.1 to complete the argument. ■

\(^{(4)}\) $\widehat{\mu}(X) = \overline{\mu(X^{-1})}$ for a Borel set $X$.  

Of course, $E$ is $RI_0(U)$ (or $FZI_0(U)$) if and only if $E \cup E^{-1}$ is $RI_0(U)$ ($FZI_0(U)$), as the transform of a real measure is Hermitian. More is true, however.

**Theorem 2.5.** Let $E \subset \Gamma$ and $U \subset G$ be a symmetric neighbourhood of the identity. Then $E$ is $RI_0(U)$ if and only if $E \cup E^{-1}$ is $I_0(U)$.

**Proof.** Suppose that $E$ is $RI_0(U)$; we must show that $E \cup E^{-1}$ is $I_0(U)$. The argument is an adaptation of [7, proof of 2.11]. The novelty in the adaptation is to use “anti-Hermitian” functions: $\nu(\gamma) = -\nu(\gamma^{-1})$ for all relevant $\gamma$ and to decompose each bounded real-valued $\varphi : E \cup E^{-1} \rightarrow \mathbb{C}$ into the sum of Hermitian and anti-Hermitian functions, which will be the transform of measures of the form $\frac{1}{2}(\mu + \tilde{\mu})$ and $\frac{1}{2}i(\nu - \tilde{\nu})$, respectively, where $\mu, \nu \in M^*_d(U)$. We omit further details.

Now suppose that $E \cup E^{-1}$ is $I_0(U)$. Let $\varphi : E \rightarrow \mathbb{C}$ be the bounded Hermitian function to be interpolated on $E$ by an element of $M^*_d(U)$. Extend $\varphi$ to $E^{-1} \setminus E$ by

$$\varphi(\gamma^{-1}) = \overline{\varphi(\gamma)}, \quad \gamma \in E \setminus E^{-1}.$$  

Let $\mu \in M_d(U)$ be such that $\hat{\mu}(\gamma) = \varphi(\gamma)$ for $\gamma \in E \cup E^{-1}$. Such a $\mu$ exists because $E \cup E^{-1}$ is $I_0(U)$. Let $\nu = \frac{1}{2}(\mu + \bar{\mu})$. Then for $\gamma \in E$, $\hat{\nu}(\gamma) = \frac{1}{2}(\hat{\mu}(\gamma) + \bar{\mu}(\gamma^{-1})) = \varphi(\gamma)$, so $E$ is indeed $RI_0(U)$.  

**Corollary 2.6.** Suppose $E, F$ are $RI_0$ sets and that $E \cup E^{-1}$ and $F \cup F^{-1}$ have disjoint closures in $\Gamma$. Then $E \cup F$ is $RI_0$.

**Proof.** $E \cup E^{-1}$ and $F \cup F^{-1}$ are $I_0$ sets with disjoint closures. Hence their union is $I_0$ by [6, Lem. 2.1].

**Corollary 2.7.** Let $G$ be a connected group and suppose $E$ is $RI_0(U)$. If $\lambda \in \Gamma$, then for any neighbourhood $V$ of the identity of $G$, $E \cup \{\lambda\}$ is $RI_0(U \cdot V)$.

**Proof.** This follows since the union of an $I_0(U)$ set and a finite set is $I_0(U \cdot V)$ [7, 2.7].

The characterization of $RI_0$ given in the theorem can be used to prove the class of $RI_0$ sets is strictly smaller than the $I_0$ sets:

**Example 2.8** (A set in $\mathbb{Z}$ that is $I_0(U)$ for all open $U$, but not $RI_0$). Let $E_1 = \{10^j + 5j + 1 : j \geq 1\}$ and $E_2 = \{10^j + 1 : j \geq 1\}$. Then $E_1 \cup E_2$ is not $I_0$ [17, p. 178]. Let $E = E_1 \cup \overline{E_2}$. Then $E \cup \overline{E}$ is not $I_0$ and so $E$ is not $RI_0$ by the theorem.

Now suppose $U \supseteq (-4\pi/10^N, 4\pi/10^N)$ and put $b = \pi/10^N$. Except for a finite number of $n \in E$ (say, for all $n \in E \setminus \Delta$),

$$\hat{\delta}_b(n) = \begin{cases} e^{-i\pi/10^N} & \text{if } n = -10^j - 1, \\ e^{i\pi(5j+1)/10^N} & \text{if } n = 10^j + 5j + 1. \end{cases}$$
Put $\nu = e^{i\pi/10^N} \delta_b - \delta_0$. Then $\nu = 0$ on $E_2 \setminus \Delta$, $|\nu| \geq |1 - e^{i\pi/10^N}| > 0$ on $E_1 \setminus \Delta$ and $\nu \in M_d(-2\pi/10^N, 2\pi/10^N)$.

Let $\varphi \in \ell^\infty(E)$. As $E_1$ and $E_2$ are $\varepsilon$-Kronecker sets for some $\varepsilon < 1$, the tail of each is $I_0(-\pi/10^N, \pi/10^N)$ [7, Thm. 3.2]. Since $\nu$ is bounded away from 0 on $E_1 \setminus \Delta$ we can find $\omega_1, \omega_2 \in M_d(-\pi/10^N, \pi/10^N)$ such that $\omega_1 = \varphi$ on $E_2 \setminus \Delta$ ($\Delta$ renamed as necessary) and

$$\omega_2 = (\varphi - \omega_1)/\nu \quad \text{on } E_1 \setminus \Delta.$$ 

Take $\omega = \omega_1 + \omega_2 \ast \nu$. Then $\omega \in M_d(-3\pi/10^N, 3\pi/10^N)$ and interpolates $\varphi$ on $E \setminus \Delta$. Now apply [7, 2.8] to conclude that $E$ is $I_0(U)$.

Corollary 2.7 implies that a set is $RI_0(U)$ for all open sets $U$ if it is $RI_0(U)$ with bounded constants (in the connected case). In contrast, no non-empty set is $FZI_0(U)$ for all open sets $U$ since even singletons are not $FZI_0(U)$ for small neighbourhoods of the identity.

However, the class of $FZI_0$ sets is closed under the adjunction of finite sets.

**Proposition 2.9.** Suppose $E$ is $FZI_0$ and $F$ is a finite set not containing 1. Then $E \cup F$ is $FZI_0$.

**Proof.** There is no loss in assuming $F$ is a singleton $\{\gamma\}$ and $\gamma, \gamma^{-1} \notin E$. As $E \cap \Gamma = E$ (see [18]), there is a (Bohr) closed neighbourhood $V$ of 1 in $\Gamma$ such that $E \cap V \cdot V^{-1} = \emptyset$ and $E \cap \{\gamma, \gamma^{-1}\}V \cdot V^{-1} = \emptyset$. Put

$$f = \frac{1}{2m(V)} (1\gamma \cup V \ast 1\gamma^{-1} V^{-1} \cup V^{-1}).$$

Then $f(\gamma) \geq 1/2$ and $f = 0$ on $E$. As $f$ is positive definite, there exists $\mu \in M_d^+(G)$ such that $\hat{\mu} = f$ on $\Gamma$. It follows easily that $E \cup \{\gamma\}$ is $FZI_0$.

**3. $FZI_0(U)$ with bounded constants**

**3.1. $\varepsilon$-Kronecker sets are $FZI_0(U)$ with bounded constants**

**Theorem 3.1.** Let $0 < \varepsilon < \sqrt{2}$ and let $E$ be an $\varepsilon$-Kronecker subset of the discrete abelian group $\Gamma$. Let $U \subset G$ be open. Then $E$ is $FZI_0(U)$ with bounded constants.

**Proof.** Let $U$ be a neighbourhood of the identity. Let $W \subset U$ be a symmetric neighbourhood such that $W^2 \subset U$. By [7, Thm. 3.2], for any (fixed) $\varepsilon' > \varepsilon$ there is a finite subset $\Delta$ such that $E \setminus \Delta$ is $\varepsilon'$-Kronecker($W$). Choose $\delta = \delta(\varepsilon') \in (0, 1)$ such that $|\langle \gamma, x_0 \rangle - 1| < \varepsilon'$ implies $\Re\langle \gamma, x_0 \rangle \geq \delta$ and pick $b > 0$ such that $b + \sqrt{1 - \delta^2} < 1$. 

Let \( \varphi \in B(\ell^\infty(E \setminus \Delta)) \), say \( \varphi(\gamma) = a_\gamma + ib_\gamma \), for \( a_\gamma, b_\gamma \) real and \( \gamma \in E \setminus \Delta \). Let \( x_0 \in W \) satisfy

\[
|\langle \gamma, x_0 \rangle - 1| < \varepsilon' \quad \text{if} \quad a_\gamma \in [b, 1], \\
|\langle \gamma, x_0 \rangle + 1| < \varepsilon' \quad \text{if} \quad a_\gamma \in [-1, -b], \\
|\langle \gamma, x_0 \rangle - i| < \varepsilon' \quad \text{if} \quad a_\gamma \in (-b, b).
\]

Then

\[
|a_\gamma - \Re \langle \gamma, x_0 \rangle| < \max(1 - \delta, 1 - b, b + \sqrt{1 - \delta^2}) = \varepsilon_0 < 1.
\]

By iterating we can interpolate any real sequence on \( E \setminus \Delta \) with the Fourier–Stieltjes transform of some \( \mu \in M_d^+(W) \).

To interpolate \( \{ib_\gamma\}_{\gamma \in E \setminus \Delta} \) we argue as follows: Let \( x_1 \in W \) be such that

\[
|\langle \gamma, x_1 \rangle - r_\gamma i| < \varepsilon \quad \text{for all} \quad \gamma \in E \setminus \Delta,
\]

where \( r_\gamma = 1 \) if \( b_\gamma \geq 0 \) and \( r_\gamma = -1 \) if \( b_\gamma < 0 \). By the previous part of the proof there exists \( \mu \in M_d^+(W) \) such that \( \widehat{\mu}(\gamma) = -\Re \langle \gamma, x_1 \rangle \) for \( \gamma \in E \setminus \Delta \). For such \( \gamma \),

\[
\left|\frac{\mu + \delta x_1(\gamma)}{2} - ib_\gamma\right| = \left|\frac{3\langle \gamma, x_1 \rangle}{2} - b_\gamma\right| < 1 - \delta/2,
\]

where \( \delta \) is as above.

Consequently, if \( \sigma \in M_d^+(U) \) is a measure interpolating \( a_\gamma \), then

\[
\nu = \sigma + (\mu + \delta x_1)/2 \in M_d^+(U)
\]

satisfies \( |\widehat{\nu}(\gamma) - \varphi(\gamma)| < 1 - \delta/2 \). Now iterate. It is clear that the constant of interpolation depends only on \( \varepsilon \). ■

The following result should be contrasted with Example 5.1, which shows \( FZI_0 \) and \( RI_0 \) sets are not preserved under translation.

**Proposition 3.2.** Let \( E \) be an \( \varepsilon \)-Kronecker subset of the discrete abelian group \( \Gamma \). Then for all \( \gamma \) and \( \varepsilon' > \varepsilon \), there exists a finite set \( \Delta \) such that \( (E \setminus \Delta) \cdot \gamma \) is \( \varepsilon' \)-Kronecker and \( FZI_0(U) \) with bounded constants.

**Proof.** Choose a neighbourhood \( U \) of the identity such that \( \gamma \approx 1 \) on \( U \). There exists \( \Delta \) such that \( E \setminus \Delta \) is \( \varepsilon' \)-Kronecker(\( U \)); thus we can approximate any \( \varphi \in B(\ell^\infty((E \setminus \Delta) \cdot \gamma)) \) with \( \widehat{\delta_x} \), where \( x \in U \). ■

**3.2. Other \( FZI_0(U) \) sets with bounded constants.** In this subsection we collect examples of \( FZI_0(U) \) sets in non-classical abelian groups. These results will be used for the general existence theorems in Section 5.

**Proposition 3.3.** Let \( \varepsilon > 0 \) and let \( E = \{\chi_n\} \) be an infinite sequence in the (discrete group of) rationals, \( \mathbb{Q} \). Then \( E \) has an infinite \( \varepsilon \)-Kronecker subset and hence a subset that is \( FZI_0(U) \) with bounded constants.

**Proof.** Fix \( q \) such that \( 1 + \pi/\varepsilon < q \).
CASE 1: \( \{ \chi_n \} \) is unbounded in \( \mathbb{R} \). Then we can find a subsequence (not renamed) such that \( \chi_{n+1}/\chi_n > q \) for all \( n \). Such a set is \( \varepsilon \)-Kronecker by [15, 2.4(1)].

CASE 2: The \( \{ \chi_n \} \) accumulate at \( r \in \mathbb{R} \) in the usual topology of \( \mathbb{R} \). Then we can find a subsequence (not renamed) such that \( (\chi_{n+1} - r)/(\chi_n - r) < 1/3q \). Obviously all finite portions of such a set are \( \varepsilon/3 \)-Kronecker. By taking elements in the dual of \( \mathbb{R}_d \), the Bohr compactification of \( \mathbb{R} \), we see that \( \{ \chi_n - r : n \geq 1 \} \) is \( \varepsilon/2 \)-Kronecker. Given a \( 0 \)-neighbourhood \( U \) there is some integer \( N \) such that \( F = \{ \chi_n \}_{n=N}^{\infty} \) is \( \varepsilon/3 \)-Kronecker(\( U \)). But then \( F \) is \( \varepsilon \)-Kronecker in \( \mathbb{Q} \), and hence \( \text{FZI}_0(U) \) with bounded constants by Theorem 3.1.

For a prime \( p \), \( \mathcal{C}(p^{\infty}) \) denotes the discrete group generated by all elements of the form \( \chi = e^{2\pi ik/p^n} \in \mathbb{T} \), where \( 1 \leq k < p \) and \( n \geq 1 \), i.e., the infinite \( p \)-subgroup of \( \mathbb{T}_d \) consisting of all elements whose orders are a power of \( p \). Integers \( \ell \in \mathbb{Z} \) give rise to characters on \( \mathcal{C}(p^{\infty}) \): \( \langle \varrho, \ell \rangle = e^{2\pi ik\ell/p^n} \), where \( \varrho = e^{2\pi ik/p^n} \). Of course, any element of the coset \( \ell + p^n\mathbb{Z} \) gives rise to the same value.

**Proposition 3.4.** Let \( \varepsilon > 0 \) and let \( E = \{ \gamma_j \} \) be an infinite sequence in \( \mathcal{C}(p^{\infty}) \). Then \( E \) has an infinite \( \varepsilon \)-Kronecker subset and hence a subset that is \( \text{FZI}_0(U) \) with bounded constants.

**Proof.** Let \( 1 \leq C < \infty \) be such that \( |1 - e^{\pi i/p^C}| < \varepsilon/2 \). By passing to a subsequence \( \chi_j \) of the \( \gamma_j \), we may assume \( \chi_j = e^{2\pi ik_j/p^{n_j}} \) where \( n_1 \geq C \) and \( n_{j+1} - n_j \geq C \) for \( j \geq 1 \).

Let \( \varphi : \{ \chi_j \} \rightarrow \mathbb{T} \) be given. Then the possible values of a character at \( \chi_1 \) are \( e^{2\pi ik_1\ell/p^{n_1}} \). Since \( 1 \leq k_1 \leq p \), those values are spaced equidistantly on \( \mathbb{T} \) with distance between them equal to \( |1 - e^{2\pi ik_1\ell/p^{n_1}}| < \varepsilon/2 \), so one of them is at most \( |1 - e^{\pi i/p^{n_1}}| < \varepsilon/2 \) from \( \varphi(\chi_1) \). Choose \( \ell_1 \in \mathbb{Z} \) such that

\[
|e^{2\pi ik_1\ell_1/p^{n_1}} - \varphi(\chi_1)| < \varepsilon/2.
\]

The last inequality does not change if we replace \( \ell_1 \) by an element of the coset \( \ell_1 + p^{n_1}\mathbb{Z} \).

Now consider \( e^{2\pi ik_2(\ell_1 + \ell_2 p^{n_2})/p^{n_2}} \) for \( \ell_2 \in \mathbb{Z} \). Those quantities are equally spaced on the unit circle at spacing of \( |1 - e^{2\pi ik_2/p^{n_2-n_1}}| \). Since \( n_2 - n_1 \geq C \), the distance between those values is less than \( \varepsilon/2 \), so we can find \( \ell_2 \) such that

\[
|e^{2\pi ik_2(\ell_1 + \ell_2 p^{n_1})/p^{n_2}} - \varphi(\chi_2)| < \varepsilon/2.
\]

Those values do not change if we replace \( \ell_2 \) by an element of \( \ell_2 + p^{n_2}\mathbb{Z} \).

Continuing in this manner, we obtain a sequence in \( \mathbb{Z} \) which does the correct interpolation on each finite subset of \( E \). So any accumulation point
in the dual of \( \mathcal{C}(p^\infty) \) will interpolate \( \varphi \) with error less than or equal to \( \varepsilon/2 \) and from this we may conclude that \( E \) is \( \varepsilon \)-Kronecker. ■

We recall that a subset \( E \) of an abelian group \( L \) is called \( \textit{independent} \) if \( N \geq 1, x_1, \ldots, x_N \in E, n_1, \ldots, n_N \in \mathbb{Z} \) and \( \sum_{j=1}^{N} x_j^{n_j} = 1_L \) imply \( x_j^{n_j} = 1_L \) for \( 1 \leq j \leq N \).

**Proposition 3.5.** Let \( E \subset \Gamma \) be independent. Then \( E \) is \( \text{FZI}_0(U) \) for all open \( U \) with bounded constants.

**Proof.** Let \( \Lambda \) be the subgroup of \( \Gamma \) generated by \( E \). For each \( \chi \in E \), let \( \Lambda_\chi \) be the cyclic subgroup of \( \Gamma \) generated by \( \chi \), and \( H_\chi \) be the dual group of \( \Lambda_\chi \). The independence of \( E \) implies that \( \Lambda = \bigoplus_{\chi \in E} \Lambda_\chi \) and that the dual group, \( H \), of \( \Lambda \) is the direct product \( H = \prod_{\chi \in E} H_\chi \). We may assume that \( \Gamma = \Lambda \).

Let \( U \subset G \) be any neighbourhood of the identity. Then there exists a finite set \( \Delta \subset E \) such that \( \prod_{\chi \in E \setminus \Delta} H_\chi \subset U \). Let \( F \subset E \setminus \Delta \) be the elements of order 2 (if any). Without loss of generality, we may assume that \( U = \{1\} \times U_0 \times U_1 \), where \( U_0 = \prod_{\chi \in F} H_\chi \) and \( U_1 = \prod_{\chi \in E \setminus (\Delta \cup F)} H_\chi \).

Let \( \varphi : E \setminus \Delta \rightarrow \mathbb{T} \) be Hermitian. If \( F \) is non-empty, then the independence of \( F \) and the fact that \( \varphi \) takes only the values \( \pm 1 \) on \( F \) imply that for each \( \chi \in F \) there exists \( x_\chi \in H_\chi \) with \( \langle \chi, x_\chi \rangle = \varphi(\chi) \). Let \( x = \prod_{\chi \in F \setminus \Delta} x_\chi \) if \( F \) is non-empty, and let \( x \) be the identity of \( G \) otherwise.

For each \( \chi \in E \setminus F \), there exists \( x_\chi \) such that \( |\langle \chi, x_\chi \rangle - \varphi(\chi)| \leq 1 \), by the definition of \( H_\chi \). (The worst case occurs if \( \chi \) has order 3, and otherwise the above difference is at most \( \sqrt{2}/2 \).) This shows that an independent set containing only elements of order greater than 2 is \( \varepsilon \)-Kronecker for all \( \varepsilon > 1 \).

By Theorem 3.1 applied to \( E \setminus F \), there is a constant \( K \) (independent of \( U_1 \)), a finite set \( \Delta_1 \subset E \setminus F \) (depending only on \( U_1 \)), and a measure \( \nu_1 \in M_0^+(U_1) \) such that
\[
\hat{\nu}_1(\chi) = \varphi(\chi) \quad \text{on } E \setminus (F \cup \Delta_1), \quad \|\nu_1\| \leq K.
\]
Let \( \nu = \delta_x \times \nu_1 \). Then \( \hat{\nu}(\chi) = \varphi(\chi) \) for each \( \chi \in E \setminus (\Delta \cup \Delta_1) \). Hence \( E \setminus (\Delta \cup \Delta_1) \) is \( \text{FZI}_0(U) \) with constant \( K \). ■

4. Existence theorems for \( \text{FZI}_0 \) and \( \text{FZI}_0(U) \) sets

4.1. \( \Gamma \) has large \( \text{FZI}_0(U) \) sets with bounded constants

**Theorem 4.1.** Let \( \Gamma \) be an infinite discrete abelian group. Then \( \Gamma \) contains an \( \text{FZI}_0(U) \) set with bounded constants and having the same cardinality as \( \Gamma \).

The following corollary is due to [13]; another proof is in [15].

**Corollary 4.2.** Every discrete abelian group contains an \( I_0 \) set of the same cardinality.
Before proving Theorem 4.1, we need a lemma.

**Lemma 4.3.** Let $E \subset \Gamma$. Suppose that $E$ is uncountable. Then $E$ contains an independent set $A$ such that $\#A = \#E$.

**Proof of Lemma 4.3.** Let $\kappa = \#E$. Let $F$ be the torsion subgroup of the subgroup $H$ of $\Gamma$ that $E$ generates. Suppose $\kappa = \#(H/F)$. Let $A$ be a subset of $E$ such that the set of cosets $A \cdot F$ is maximal independent in the quotient $H/F$. As $E$ generates $H$, $E \cdot F$ generates $H/F$, and any maximal independent subset of $E \cdot F$ also generates $H/F$. Hence, the cardinalities of the four (types of) sets, $E \cdot F$, $H/F$, maximal independent subset of $H/F$, and maximal independent subset of $E \cdot F$, are all equal. Thus, $\#A = \kappa$ if $\kappa = \#(H/F)$. Of course, if $A \cdot F$ is independent in the torsion-free group $H/F$, then $A$ must be independent in $H$.

If $\kappa > \#(H/F)$ then $\kappa = \#F$. We apply verbatim all but the first two paragraphs of the proof of [15, Lem. 3.7], from which we conclude that $H$ contains an independent set $A'$ with $\#A' = \kappa$. Of course that cannot happen unless $E$ contains an independent set $A$ with $\#A = \kappa$.

**Proof of Theorem 4.1.** If $\Gamma$ is uncountable, then the theorem follows from Proposition 3.5 and Lemma 4.3 (with $E = \Gamma$).

If $\Gamma$ is countable and contains an infinite independent set, then the theorem follows from Proposition 3.5. We thus assume that $E$ is countable and does not contain an infinite independent set.

If $\Gamma$ has an element of infinite order, then $\Gamma$ contains a copy of $\mathbb{Z}$, which contains Hadamard sets of ratio greater than 3. These are $\varepsilon$-Kronecker sets with $\varepsilon < \sqrt{2}$ [6, Prop. 2-3] and consequently $FZI_0(U)$ with bounded constants, by Theorem 3.1.

We thus assume that $\Gamma$ is a countable torsion group with no infinite independent sets. By [2, 20.1], $\Gamma$ can be identified with a subgroup of the direct sum $\bigoplus_\beta \mathcal{C}(p_\beta^\infty)$. If that direct sum is minimal with respect to containing $\Gamma$, then the direct sum must be finite, since otherwise induction would show that $\Gamma$ contained an infinite independent set. Therefore the projection of $\Gamma$ onto one of the summands must be infinite. Applying Proposition 3.4 gives the desired set in $\mathcal{C}(p_\beta^\infty)$. Taking the corresponding characters in $\Gamma$ completes the proof.

**4.2. E has large FZI\_0 sets**

**Theorem 4.4.** Let $E$ be an infinite subset of the discrete abelian group $\Gamma$. Then $E$ has an infinite $FZI_0$ subset of the same cardinality as $E$.

**Proof.** If $E$ is uncountable, then $E$ contains a maximal independent subset $F$ of the same cardinality as $E$ by Lemma 4.3, and we may apply Proposition 3.5.
We thus assume that \( E = \{\chi_n\} \) is countable. By \([2, 20.1]\), \( \Gamma \) can be identified with a subgroup of a divisible group \( \Lambda \) which is a direct sum
\[
\bigoplus_\alpha \mathbb{Q}_\alpha \oplus \bigoplus_\beta \mathcal{C}(p_\beta^\infty),
\]
where the \( \mathbb{Q}_\alpha \) are copies of the rationals in \( \mathbb{R} \). We may assume \( \Gamma = \Lambda \). For convenience, let us write \( \Gamma = \bigoplus_\ell \Gamma_\ell \) where each \( \Gamma_\ell \) is one of \( \mathbb{Q} \) or \( \mathcal{C}(p_\ell^\infty) \), the direct sum is (at most) countable, and for each \( \ell \) there is some character \( \chi_n \in E \) whose projection onto \( \Gamma_\ell \) is not trivial.

Let \( I = \{\ell : \text{there is some } n \text{ such that the projection of } \chi_n \text{ onto } \Gamma_\ell \) is not trivial and not of order two\}.

**Case 1:** \( \# I \) is infinite. As each \( \chi_n \) has a non-trivial projection onto only finitely many factors \( \Gamma_\ell \), it follows that there must be an infinite subsequence \((\ell_n)\) such that the projection onto \( \Gamma_{\ell_n} \) is of order \( \geq 3 \), but the projections of \( \chi_m \) onto \( \Gamma_{\ell_n} \) are trivial for \( m < n \). As these \( \chi_n \) are of order at least three, they form an \( \varepsilon \)-Kronecker set for any \( \varepsilon > 1 \), and hence even an \( FZI_0(U) \) set with bounded constants.

**Case 2:** \( \# I \) is finite. We distinguish between two subcases.

(a) There is some index \( \ell \) such that the projection of \( \chi_n \) onto \( \Gamma_\ell \) is infinite. In this case apply Propositions 3.3 or 3.4 to find an infinite \( \varepsilon \)-Kronecker subset of these projections, and therefore of the original \( \chi_n \) (for any \( \varepsilon > 0 \)).

(b) Otherwise it follows that the projections of \( \chi_n \) onto \( \bigoplus_{\ell \in I} \Gamma_\ell \) form a finite set and therefore infinitely many of them have the same projection, say \( \gamma \). We restrict ourselves to this set of projections (again, not renamed). If \( \ell \not\in I \), then \( \Gamma_\ell = \mathcal{C}(2^\infty) \) and the projection of \( \chi_n \) onto \( \Gamma_\ell \) is either trivial or is (the unique element) of order two. As the projections of these \( \chi_n \) onto \( \bigoplus_{\ell \in I} \Gamma_\ell \) coincide, their projections onto \( \bigoplus_{\ell \in I'} \Gamma_\ell \) must be distinct.

Hence the complement \( I^c \) must be infinite and therefore we can obtain an infinite independent set as in Case 1, but consisting of elements of order 2, say \( \{\gamma_j\} \). Such a set is \( FZI_0(U) \) with bounded constants: we need only to interpolate real functions \( \varphi \), since the characters are of order 2. The corresponding characters are of the form \( \gamma \oplus \gamma_j \). If \( \gamma \) is of order one or two it is obvious that \( \{\gamma \oplus \gamma_j\} \) is \( FZI_0 \). To see that \( \{\gamma \oplus \gamma_j\} \) is \( FZI_0 \) otherwise, we apply Proposition 2.4: Given any choice of signs \( \{r_j\} \) we can find \( \mu \in M^+_d(\bigoplus_{\ell \in I^c} \Gamma_\ell) \) such that \( \hat{\mu}(\gamma_j) = r_j \). As \( \gamma^2 \neq 1 \) we can choose \( \nu \in M^+_d(\bigoplus_{\ell \in I^c} \Gamma_\ell) \) such that \( \hat{\nu}(\gamma) = i \). Then \( \delta_1 \times \mu(\gamma \oplus \gamma_j) = r_j \) and \( \nu \times \mu(\gamma \oplus \gamma_j) = ir_j \).

**Remark 4.5.** \( E \) need not contain an \( FZI_0(U) \) set for small \( U \). Take, for instance, \( G = \mathbb{Z}_3 \times \mathbb{D}_2 \) and let \( g \in \hat{\mathbb{Z}}_3, g \neq 1 \). Let \( F \subset \hat{\mathbb{D}}_2 \) be an independent set. The set \( E = \{g\} \times F \) is \( FZI_0 \) (it is just a special case of Case 2(b) above) but no infinite subset is \( FZI_0(\{0\} \times \mathbb{D}_2) \) as it is clearly only...
possible to interpolate real-valued sequences by positive measures supported on \( \{0\} \times \mathbb{D}_2 \).

### 4.3. Existence of \( FZI_0(U) \) subsets with bounded constants

**Theorem 4.6.** Let \( G \) be a compact connected abelian group. Then every infinite \( E \subset \Gamma \) contains an \( FZI_0(U) \) set with bounded constants of the same cardinality.

**Corollary 4.7.** \( \Gamma \) contains a subset of cardinality \( \# \Gamma \) that is \( I_0(U) \) for all open \( U \).

**Proof.** If \( E \) is uncountable, then \( E \) has an independent set of the same cardinality and Proposition 3.5 completes the proof.

We may thus assume that \( E \), and therefore \( \Gamma \), is countable. Because \( \Gamma \) has no elements of finite order, \( \Gamma \) is contained in the countable direct sum of copies of \( \mathbb{Q} \), and we may assume that \( \Gamma \) is such a sum.

Suppose that the projection of \( E \) on any one of the factors \( \mathbb{Q} \) is infinite. Then by Proposition 3.3, we have an infinite \( FZI_0(U) \) set with bounded constants.

Otherwise, \( E \) has non-zero projections on an infinite number of the factors. We again have an infinite independent set of elements whose order is infinite, so we have an \( FZI_0(U) \) set with bounded constants by Proposition 3.5.

### 5. Translation of \( FZI_0 \) sets

In contrast to the situation for \( I_0 \) sets, translation does not in general preserve \( RI_0 \) or \( FZI_0 \) sets.

**Example 5.1 (A translate of an \( FZI_0 \) set which is not \( RI_0 \)).** Let \( E_1 = \{16^j + 4j : j \geq 1\} \), \( E_2 = \{-16^j - 2 : j \geq 1\} \), and \( E = E_1 \cup E_2 \). The two sets \( E_1, E_2 \) are \( FZI_0(U) \) with bounded constants, being \( \varepsilon \)-Kronecker for \( \varepsilon < 1 \). If we evaluate \( \hat{\delta}_0 + \hat{\delta}_{\pi/2} \) on \( E \), we get 2 on \( E_1 \) and 0 on \( E_2 \). Standard arguments show that \( E \) is \( FZI_0(U) \). But \( E+1 \) is not even \( RI_0 \) because \( (E+1) \cup (-E-1) \) is not \( I_0 \).

One reason for the interest in sets that are \( FZI_0(U) \) with bounded constants is that under this (additional) assumption \( FZI_0 \) is preserved under translation.

**Proposition 5.2.** Suppose \( E \) is an antisymmetric \( FZI_0(U) \) set with bounded constants and suppose \( F \) is a finite, asymmetric set.

1. Suppose there exists a neighbourhood \( V \subset G \) such that \( F \) is \( FZI_0(V) \). Then there is a finite set \( \Delta \) such that \( (E \setminus \Delta) \cdot F \) is \( FZI_0(V) \).
2. If \( F^{-1} \cap E = \emptyset \) then \( E \cdot F \) is \( FZI_0 \).

**Proof.** (2) follows from (1), since a finite set is \( FZI_0(G) \) and the union of a finite set with an \( FZI_0 \) set is \( FZI_0 \) (Prop. 2.9).
(1) Assume $F = \{\lambda_1, \ldots, \lambda_M\}$. For each $k = 1, \ldots, M$ let $\mu_k = \sum_{j=1}^{\infty} a_{jk} \delta_{x_{jk}} \in M_d^+(V)$ be such that
\[
\hat{\mu}_k(\lambda_i) = \begin{cases} 
1 & \text{if } i = k, \\
\hat{\mu}_k(\lambda_i) = 0 & \text{otherwise.}
\end{cases}
\]
(Here we use the fact that $F$ is asymmetric, as well as $FZI_0(V)$.) Choose $N_k$ such that $\|\mu_k - \sum_{j=1}^{N_k} a_{jk} \delta_{x_{jk}}\| < \varepsilon/M$.

Let $K$ be as in the definition of $FZI_0(U)$ with bounded constants (Definition 3) and let $\varepsilon_{jk} = 2^{-j}/(KMa_{jk})$. Choose neighbourhoods $U_{jk} \subset V$ of $x_{jk}$ such that $|\lambda_i(x_{jk}) - \lambda_i(y)| < \varepsilon_{jk}$ for $y \in U_{jk}$, $i = 1, \ldots, M$.

Let $\varphi \in \ell^\infty(E \cdot F)$ be a given Hermitian function of norm one. Select finite sets $\Delta_{jk} \subset \Gamma$ and measures $\nu_{jk} \in M_d^+(U_{jk})$, of norm at most $K$, such that $\hat{\nu}_{jk}(\chi) = \varphi(\chi \lambda_k)$ for $\chi \in E \setminus \Delta_{jk}$. Put $\Delta = \bigcup_{k=1}^{M} \bigcup_{j=1}^{N_k} \Delta_{jk}$ and let $\mu = \sum_{j,k} a_{jk} \nu_{jk} \in M_d^+(V)$. From (5) we know that for $\chi \in E \setminus \Delta$ and $\lambda_i \in F$, $\hat{\nu}_{jk}(\chi \lambda_i) = \lambda_i(x_{jk}) \hat{\nu}_{jk}(\chi) + E_{ijk}$ with error term $E_{ijk}$ satisfying $|E_{ijk}| \leq K\varepsilon_{jk}$.

Thus
\[
|\hat{\mu}(\chi \lambda_i) - \varphi(\chi \lambda_i)| \leq \left| \sum_{k=1}^{M} \sum_{j=1}^{N_k} a_{jk} \lambda_i(x_{jk}) \varphi(\chi \lambda_k) - \varphi(\chi \lambda_i) \right| + \sum_{j,k} \frac{\varepsilon 2^{-j} a_{jk}}{M}.
\]
As $|\sum_{j=1}^{N_k} a_{jk} \lambda_i(x_{jk}) - \hat{\mu}_k(\lambda_i)| \leq \varepsilon/M$ and $\hat{\mu}_k(\lambda_i) = 1$ if $i = k$ and 0 else, it follows that $|\hat{\mu}(\chi \lambda_i) - \varphi(\chi \lambda_i)| \leq 2\varepsilon$. Thus $(E \setminus \Delta) \cdot F$ is $FZI_0(V)$.\]

A set can be $FZI_0(U)$ for all open $U$ without bounded constants:

**Example 5.3.** The set $\{9^j\} \cup \{9^j + 3j + 1\}$ is $FZI_0(U)$ for all open $U$, but not with bounded constants.

**Proof.** To see this assume $U = [-\pi/9^N, \pi/9^N]$. With $a = \pi/9^N$ we get $\hat{\delta}_a(9^j) = -1$ for $j > N$ and $\hat{\delta}_a(9^j + 3j + 1) \geq \varepsilon > 0$. Since the transform of $\delta_0 + \delta_a$ is 0 on $\{9^j\}$ and bounded away from zero on $\{9^j + 3j + 1\}$, it follows that $\{9^j\}_{j>N} \cup \{9^j + 3j + 1\}_{j>N}$ is $FZI_0(U)$. This set is not even $I_0(U)$ with bounded constants as $E \cup (E + 1)$ is not $I_0$ [7, 3.1].\]

**References**


[8] —, —, $I_0$ sets for compact, connected groups: interpolation with measures that are non-negative or of small support, preprint, 2006.


