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ε-Kronecker and I_0 sets in abelian groups, IV: interpolation by non-negative measures

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Abstract. A subset E of a discrete abelian group is a "Fatou–Zygmund interpolation set" (FZI_0 set) if every bounded Hermitian function on E is the restriction of the Fourier–Stieltjes transform of a discrete, non-negative measure.

We show that every infinite subset of a discrete abelian group contains an FZI_0 set of the same cardinality (if the group is torsion free, a stronger interpolation property holds) and that ε-Kronecker sets are FZI_0 (with that stronger interpolation property).

1. Introduction and summary of results. In this paper we continue [6, 7, 10], where we studied I_0 and ε-Kronecker sets (definitions are below) in discrete abelian groups Γ, with compact dual groups G. In particular, in [7], we showed that many I_0 sets (including ε-Kronecker and Hadamard sets) had the property that every bounded function on them could be interpolated by the Fourier–Stieltjes transform of a discrete measure with arbitrarily small support.

In this paper we address the interpolation issue, but now ask that the interpolating measures be real or non-negative. Our main results show that such interpolation sets are plentiful, as is well known to be the case for other interpolation sets such as Sidon and I_0 sets.

1.1. *Definitions and main results*

DEFINITION 1. A function φ on a subset $E \subset \Gamma$ is *Hermitian* if $\varphi(\chi) = \overline{\varphi(\chi^{-1})}$ for all $\chi \in E$ with $\chi^{-1} \in E$.

DEFINITION 2. Let $\varepsilon > 0$. A set $E \subset \Gamma$ is:

(1) *asymmetric* if $\gamma \in E \cap E^{-1}$ implies $\gamma = \gamma^{-1}$, *antisymmetric* if $E \cap E^{-1} = \emptyset$, and *symmetric* if $E = E^{-1}$;

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(2) *Sidon* (resp. I_0) if every bounded function φ on E is the restriction of a Fourier–Stieltjes transform of a measure (resp. of a discrete measure) [19, 14];
(3) ε-*Kronecker* if for every function $\varphi : E \to \mathbb{T}$, there exists $x \in G$ such that $|\langle \chi, x\rangle - \varphi(\chi)| < \varepsilon$ for all $\chi \in E$ (1);
(4) RI_0 (resp. FZI_0) if every bounded Hermitian function φ on E is the restriction of a Fourier–Stieltjes transform of a real (resp. non-negative real), discrete measure.

In each case but the first, we append "(U)" to the definition if the interpolating measures can be taken to be concentrated on the set $U \subset G$.

DEFINITION 3. E is *$FZI_0(U)$ for all open U with bounded constants* (2) if there is a constant K such that for each open U there is a finite set Δ such that for each bounded $\varphi : E \setminus \Delta \to \mathbb{C}$ there is a non-negative, discrete measure μ concentrated on U such that $\widehat{\mu} = \varphi$ on $E \setminus \Delta$ and $\|\mu\| \le K$. We make analogous definitions for a set to be "$I_0(U)$ (or $RI_0(U)$) with bounded constants".

Item (4) of Definition 2 is new (but not unanticipatable), and it is with these classes of sets and their relations to the other classes that this paper is concerned.

Clearly $I_0(U)$ sets are Sidon(U) (the converse is shown to be not true in [17]), $FZI_0(U)$ sets are $RI_0(U)$, and any asymmetric $RI_0(U)$ set is $I_0(U)$. Less trivially, we show here that a set E is $RI_0(U)$ if and only if $E \cup E^{-1}$ is $I_0(U)$, and hence there are I_0 sets that are not RI_0 (see Thm. 2.5 and Example 2.8). The class of FZI_0 sets is smaller again, as the singleton $\{\mathbf{0}\} \subset \mathbb{Z}$ is RI_0 but not FZI_0. However, we do not know if E being RI_0 and the identity not in E implies E is FZI_0. We also do not know if there are any non-trivial sets that are $I_0(U)$ with bounded constants that are not FZI_0. Sidon sets in the dual of a connected group are Sidon(U) [3]; such interpolation can also be done with non-negative measures (see Florek [1] who improves upon previous results and gives a Sidon set version of our $FZI_0(U)$ results).

The main contributions of this paper improve upon the previous existence theorem for I_0 sets [13, 15], and are:

(1) Every infinite discrete abelian group Γ contains an $FZI_0(U)$ set with bounded constants and of cardinality $\#\Gamma$ (Theorem 4.1).

(1) See [5, 6, 10, 7, 21] for applications and properties of ε-Kronecker sets. Given and Kunen [5] use the term "ε-free". For existence theorems for ε-Kronecker sets, see [4, Lem. 3.2], [5, Lem. 3.8], [11, Thm. 3.1], and [12, Thm. 4.1].

(2) We generally omit the phrase "for all open U".

(2) Every infinite subset E of the discrete abelian group Γ contains an FZI_0 set of the same cardinality as E (Theorem 4.4).
(3) If the dual G is connected (i.e., Γ is torsion free), then every infinite subset $E \subset \Gamma$ contains an $FZI_0(U)$ set with bounded constants of the same cardinality (Theorem 4.6). (Connectedness cannot be dispensed with here; see Remark 4.5.)

The first sets of these types were the *Hadamard sets* $E = \{n_j\} \subset \mathbb{N}$, where $\inf n_{j+1}/n_j \geq q > 1$. Hadamard sets with ratio q are known to be ε-Kronecker for $\varepsilon > |1-e^{i\pi/(q-1)}|$ ([15, Lem. 2.4(1)]) and $I_0(U)$ with bounded constants ([3]). Similar arguments can be used to show that they are $FZI_0(U)$ with bounded constants. More generally, in Theorem 3.1 we prove that ε-Kronecker sets are $FZI_0(U)$ with bounded constants if $\varepsilon < \sqrt{2}$ and this fact is used in obtaining our main results.

The terminology FZ (for Fatou–Zygmund set) was adapted from [16] where it was used to denote sets with the property that every bounded function on the set can be interpolated by a non-negative (but not necessarily discrete) measure. Examples include asymmetric Sidon sets in duals of connected groups [3]. The notion "$I_0(U)$ with bounded constants" appears in [7], where it gives some insight into why some sets are $I_0(U)$.

For non-abelian versions of our results, see [8]. For further characterizations of and union results for RI_0 and FZI_0 sets, see [9].

2. Preliminaries

2.1. *Notation.* For a compact abelian group G, G_{d} denotes the corresponding group with the discrete topology. The Bohr compactification of Γ is denoted by $\overline{\Gamma}$. If $E \subset \Gamma$, $\overline{E}$ denotes the closure of E in $\overline{\Gamma}$. Our groups are written multiplicatively, except for $\mathbb{Z}$ and $\mathbb{R}$, and $\mathbf{1} = \mathbf{1}_\Gamma$ denotes the identity of Γ except for $\mathbb{Z}, \mathbb{R}$, where $\mathbf{0}$ is used. We write $B(\ell^\infty(E))$ for the unit ball of $\ell^\infty(E)$. We write $M_{\mathrm{d}}(U)$ for the discrete measures concentrated on $U \subseteq G$ and we let

$$D(N,U) = \Big\{ \sum_{j=1}^{N} a_j \delta_{x_j} : |a_j| \leq 1,\, x_j \in U,\, 1 \leq j \leq N \Big\}.$$

A superscript r or + on a space of measures refers to the real (respectively, positive) measures in that subset.

We have the following variation of a result in [7, Prop. 2.2].

PROPOSITION 2.1. *Let G be a compact group, U a σ-compact subset of G and $E \subset \Gamma$. The following are equivalent*:

([3]) That Hadamard sets are I_0 was first proved in [20]; for other proofs see [14, 15]. The I_0 with bounded constants property was proved in [7].

(1) *E is $RI_0(U)$ (resp. $FZI_0(U)$).*
(2) *There is a constant N such that for all Hermitian $\varphi \in B(\ell^\infty(E))$ there exists $\mu \in M_{\mathrm{d}}^{\mathrm{r}}(U)$ (resp. $M_{\mathrm{d}}^{+}(U)$) with $\|\mu\| \le N$ and $\widehat{\mu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in E$.*
(3) *There exist $0 < \varepsilon < 1$ (equivalently, for every $0 < \varepsilon < 1$) and integer N such that for all Hermitian $\varphi \in B(\ell^\infty(E))$ there exists $\mu \in M_{\mathrm{d}}^{\mathrm{r}}(U)$ (resp. $M_{\mathrm{d}}^{+}(U)$) with $\|\mu\|_{M(G)} \le N$ and $|\widehat{\mu}(\gamma) - \varphi(\gamma)| < \varepsilon$ for all $\gamma \in E$.*
(4) *There exists $0 < \varepsilon < 1$ (equivalently, for all $0 < \varepsilon < 1$) and N such that for all Hermitian $\phi \in B(l^\infty(E))$ there is some $\mu \in D^{\mathrm{r}}(N, U)$ (resp. $D^{+}(N, U)$) with $|\widehat{\mu}(\gamma) - \phi(\gamma)| < \varepsilon$ for all $\gamma \in E$.*
(5) *There exists $0 < \varepsilon < 1$ (equivalently, for all $0 < \varepsilon < 1$) such that for all Hermitian $\varphi \in B(\ell^\infty(E))$ there exists $\mu \in M_{\mathrm{d}}^{\mathrm{r}}(U)$ (resp. $M_{\mathrm{d}}^{+}(U)$) with $|\widehat{\mu}(\gamma) - \varphi(\gamma)| < \varepsilon$ for all $\gamma \in E$.*

Proof. (2)⇒(1)⇒(5) and (4)⇒(3) are trivial. The implication (3)⇒(2) is an iteration argument which is essentially shown in [7]. A finite iteration argument shows the equivalences of “there exists” and “for all” in (3)–(5). We remark that for $RI_0(U)$ sets, the implication (1)⇒(2) is a standard application of the open mapping theorem. It remains only to show (5)⇒(4). We give the proof for the $FZI_0(U)$ case.

There is no loss of generality in assuming E is asymmetric since a Hermitian function has a unique Hermitian extension from E to $E \cup E^{-1}$.

As U is σ-compact we can write $U = \bigcup_{n=1}^\infty V_n$, where V_n are compact, nested sets. Let $\mathbb{D}_\gamma = [-1, 1]$ if $\gamma = \gamma^{-1}$ and $\mathbb{D}_\gamma = \{t \in \mathbb{C} : |t| \le 1\}$ if $\gamma \ne \gamma^{-1}$. Let $\mathbb{D}_E = \prod_{\gamma \in E} \mathbb{D}_\gamma$. Set

$$W_{n,k} = \{\phi \in \mathbb{D}_E : \exists \mu \in D^{+}(n, V_k) \text{ with } |\widehat{\mu}(\gamma) - \phi(\gamma)| \le \varepsilon/2 \ \forall \gamma \in E\}.$$

Given $\mu = \sum_{j=1}^n a_j \delta_{x_j}$ with $x_j \in U$, there exists k such that $x_j \in V_k$ for all $j = 1, \dots, n$. Hence

$$D^{+}(n, U) = \bigcup_{k=1}^{\infty} D^{+}(n, V_k).$$

Using (5) with suitably small $\varepsilon > 0$, we have $\bigcup_{n,k} W_{n,k} = \mathbb{D}_E$.

The closure of V_k ensures that $W_{n,k}$ is closed and hence the Baire category theorem implies that some $W_{n,k}$ has non-empty interior. If we let $W_n = \bigcup_k W_{n,k}$, it follows that some W_n has non-empty interior and therefore there is a finite set $F \subseteq E$ and a point $(z_1, \dots, z_{|F|})$ such that $(z_1, \dots, z_{|F|}) \times \mathbb{D}_{E \setminus F} \subseteq W_n$.

Consider the subset S of $l^\infty(E)$ consisting of the Hermitian elements which vanish off F. As F is finite, S is a finite-dimensional, real subspace. Take a basis of S, say $e_1, \dots, e_l$, where $e_j \in B(l^\infty(E))$. Since all norms are comparable on a finite-dimensional space, there is some $c > 0$ such that

$$\Big\| \sum b_j e_j \Big\|_{l^\infty} \geq c \sum |b_j|.$$

Each $\pm e_j$ is Hermitian, so again by (5) we can obtain $\mu_j, \nu_j \in M_{\rm d}^+(U)$ such that

$$|\widehat{\mu}_j(\gamma) - e_j(\gamma)| < c\varepsilon/4n, \qquad |\widehat{\nu}_j(\gamma) - (-e_j)(\gamma)| < c\varepsilon/4n$$

for all $\gamma \in E$. By taking suitably large partial sums we can assume there exists some m such that $\mu_j, \nu_j \in D^+(m, U)$ for all j.

Let $\phi \in B(l^\infty(E))$ be Hermitian. Since ϕ coincides on $E \setminus F$ with an element of W_n, we can find $\mu \in D^+(n, U)$ such that $|\widehat{\mu}(\gamma) - \phi(\gamma)| \leq \varepsilon/2$ for all $\gamma \in E \setminus F$. As μ is a positive measure and E is asymmetric, $(\phi - \widehat{\mu})|_F$ (extended by 0 off F) belongs to S and therefore equals $\sum b_j e_j$ for some b_j real. Write $b_j = b_j^+ - b_j^-$ where $b_j^\pm \geq 0$. Notice

$$c \sum |b_j| \leq \|(\phi - \widehat{\mu})|_F\|_{l^\infty} \leq 1 + \|\mu\|_{M(U)} \leq 2n.$$

For $\gamma \in E$,

$$\begin{aligned}
&\Big|\phi - \widehat{\mu} - \Big(\sum (b_j^+ \widehat{\mu}_j + b_j^- \widehat{\nu}_j)\Big)(\gamma)\Big| \\
&\quad = \Big|(\phi - \widehat{\mu})|_{E\setminus F} + \sum (b_j^+(e_j - \widehat{\mu}_j) + b_j^-(-e_j - \widehat{\nu}_j))\Big| \\
&\quad \leq \sup_{\gamma \in E\setminus F} |(\phi - \widehat{\mu})(\gamma)| + \sup_{\gamma \in E} \Big|\Big(\sum (b_j^+(e_j - \widehat{\mu}_j) + b_j^-(-e_j - \widehat{\nu}_j))\Big)(\gamma)\Big| \\
&\quad \leq \frac{\varepsilon}{2} + \frac{c\varepsilon}{4n} \sum |b_j| \leq \varepsilon.
\end{aligned}$$

Finally, we note that

$$\mu + \sum b_j^+ \mu_j + \sum b_j^- \nu_j \in D^+(N, U),$$

with $N = n + m \dim F$, and as N is independent of the choice of ϕ this completes the proof. ■

Remark 2.2. A similar Baire category theorem argument will show that $E \subseteq \Gamma$ is $I_0(U)$ if there exists some $\varepsilon > 0$ such that for every bounded ϕ on E there is a discrete measure μ, concentrated on U, such that $|\widehat{\mu}(\gamma) - \varphi(\gamma)| < \varepsilon$ for all $\gamma \in E$. This improves [7] where it was shown that such a set E is $I_0(U^2)$.

Remark 2.3. Proposition 2.1(3) implies that ε-Kronecker sets with $\varepsilon < 1$ are FZI_0. We will show in Theorem 3.1 that this is true whenever $\varepsilon < \sqrt{2}$ and that such sets are even $FZI_0(U)$ with bounded constants.

It is a classical result that I_0 sets are characterized by having the property that ± 1-valued E-functions have continuous extensions to $\overline{\Gamma}$. A similar result holds for $RI_0(U)$ and $FZI_0(U)$ sets, as we show in Proposition 2.4 below.

Note that a set is $I_0(U)$, $RI_0(U)$ or $FZI_0(U)$, if and only if the same is true for any translate of U, so there is no loss of generality in assuming U is a neighbourhood of the identity.

PROPOSITION 2.4. *Let* $0 < \varepsilon < 1$, $E \subset \Gamma$ *be asymmetric, and* $U \subset G$ *a symmetric, compact neighbourhood of the identity. Let* $E_2 = \{\chi \in E : \chi^2 = \mathbf{1}\}$. *Suppose that for each pair of Hermitian functions* $r : E \to \{\pm 1\}$ *and* $s : E \setminus E_2 \to \{\pm i\}$ *there are measures* $\mu_1, \mu_2 \in M_{\mathrm{d}}^{\mathrm{r}}(U)$ *(resp.* $\in M_{\mathrm{d}}^{+}(U)$*) such that* $|\widehat{\mu}_1(\chi) - r(\chi)| < \varepsilon$ *for all* $\chi \in E$ *and* $|\widehat{\mu}_2(\chi) - s(\chi)| < \varepsilon$ *for all* $\chi \in E \setminus E_2$. *Then* E *is* $RI_0(U^2)$ *(resp.* $FZI_0(U^2)$*).*

Proof. We give the proof in the $FZI_0(U)$ case. Let $0 < \varepsilon_0 < 1/2$. First, suppose $\varphi \in B(\ell^\infty(E \setminus E_2))$ is imaginary and Hermitian. Put

$$s(\chi) = \begin{cases} i & \text{if } \Im\varphi(\chi) \geq 0, \\ -i & \text{if } \Im\varphi(\chi) < 0. \end{cases}$$

Since E is asymmetric, s is Hermitian. Thus, we can obtain μ_2 as in the hypothesis. Then

$$\left| \frac{\widehat{\mu_2}(\chi)}{2} - \varphi(\chi) \right| < \frac{1+\varepsilon}{2} \qquad \text{for all } \chi \in E \setminus E_2.$$

Replacing φ with $\varphi - i\Im\widehat{\mu}_2/2$, and iterating this process, we obtain $\nu_2 \in M_{\mathrm{d}}^{+}(U)$ such that

$$|\widehat{\nu}_2 - \varphi| < \varepsilon_0 \qquad \text{on } E \setminus E_2.$$

Now consider the general case, $\varphi(\chi) = a_\chi + b_\chi i$ on E. We find ν_2 as above for $i\Im\varphi$ on $E \setminus E_2$. We note that $\varphi - \widehat{\nu}_2$ is Hermitian on E since ν_2 is a real measure. We set

$$r(\chi) = \begin{cases} 1 & \text{if } \Re\varphi(\chi) - \Re\widehat{\nu}_2(\chi) \geq 0, \\ -1 & \text{if } \Re\varphi(\chi) - \Re\widehat{\nu}_2(\chi) < 0, \end{cases}$$

for $\chi \in E$. Obtain μ_1 as in the hypothesis. Then $\mu_1 + \widetilde{\mu}_1 \in M_{\mathrm{d}}^{+}(U)$ ([4]). Also $\widehat{\mu}_1(\chi) + (\widetilde{\mu}_1)^\wedge(\chi) = 2\Re\widehat{\mu}_1(\chi)$. Hence

$$\left| \frac{\widehat{\mu}_1(\chi) + (\widetilde{\mu}_1)^\wedge(\chi)}{4} - (\Re\varphi(\chi) - \Re\widehat{\nu}_2(\chi)) \right| < \frac{1+\varepsilon}{2} \qquad \text{for all } \chi \in E.$$

As $\widehat{\mu}_1 + (\widetilde{\mu}_1)^\wedge$ is real-valued, we iterate as before, and thus we find a $\nu_1 \in M_{\mathrm{d}}^{+}(U)$ with a real-valued Fourier transform and satisfying

$$|\widehat{\nu}_1(\chi) - (\Re\varphi(\chi) - \Re\widehat{\nu}_2(\chi))| < \varepsilon_0.$$

Then $|\widehat{\nu}_1(\chi) + \widehat{\nu}_2(\chi) - \varphi(\chi)| < 2\varepsilon_0 < 1$.

We appeal to Proposition 2.1 to complete the argument. ■

([4]) $\widetilde{\mu}(X) = \overline{\mu(X^{-1})}$ for a Borel set X.

Of course, E is $RI_0(U)$ (or $FZI_0(U)$) if and only if $E \cup E^{-1}$ is $RI_0(U)$ ($FZI_0(U)$), as the transform of a real measure is Hermitian. More is true, however.

THEOREM 2.5. *Let $E \subset \Gamma$ and $U \subset G$ be a symmetric neighbourhood of the identity. Then E is $RI_0(U)$ if and only if $E \cup E^{-1}$ is $I_0(U)$.*

Proof. Suppose that E is $RI_0(U)$; we must show that $E \cup E^{-1}$ is $I_0(U)$. The argument is an adaptation of [7, proof of 2.11]. The novelty in the adaptation is to use "anti-Hermitian" functions: $\nu(\gamma) = -\nu(\gamma^{-1})$ for all relevant γ and to decompose each bounded real-valued $\varphi : E \cup E^{-1} \to \mathbb{C}$ into the sum of Hermitian and anti-Hermitian functions, which will be the transform of measures of the form $\frac{1}{2}(\mu + \widetilde{\mu})$ and $\frac{1}{2}i(\nu - \widetilde{\nu})$, respectively, where $\mu, \nu \in M_{\mathrm{d}}^{\mathrm{r}}(U)$. We omit further details.

Now suppose that $E \cup E^{-1}$ is $I_0(U)$. Let $\varphi : E \to \mathbb{C}$ be the bounded Hermitian function to be interpolated on E by an element of $M_{\mathrm{d}}^{\mathrm{r}}(U)$. Extend φ to $E^{-1} \setminus E$ by

$$\varphi(\gamma^{-1}) = \overline{\varphi(\gamma)}, \qquad \gamma \in E \setminus E^{-1}.$$

Let $\mu \in M_{\mathrm{d}}(U)$ be such that $\widehat{\mu}(\gamma) = \varphi(\gamma)$ for $\gamma \in E \cup E^{-1}$. Such a μ exists because $E \cup E^{-1}$ is $I_0(U)$. Let $\nu = \frac{1}{2}(\mu + \overline{\mu})$. Then for $\gamma \in E$, $\widehat{\nu}(\gamma) = \frac{1}{2}(\widehat{\mu}(\gamma) + \overline{\widehat{\mu}(\gamma^{-1})}) = \varphi(\gamma)$, so E is indeed $RI_0(U)$. ∎

COROLLARY 2.6. *Suppose E, F are RI_0 sets and that $E \cup E^{-1}$ and $F \cup F^{-1}$ have disjoint closures in $\overline{\Gamma}$. Then $E \cup F$ is RI_0.*

Proof. $E \cup E^{-1}$ and $F \cup F^{-1}$ are I_0 sets with disjoint closures. Hence their union is I_0 by [6, Lem. 2.1]. ∎

COROLLARY 2.7. *Let G be a connected group and suppose E is $RI_0(U)$. If $\lambda \in \Gamma$, then for any neighbourhood V of the identity of G, $E \cup \{\lambda\}$ is $RI_0(U \cdot V)$.*

Proof. This follows since the union of an $I_0(U)$ set and a finite set is $I_0(U \cdot V)$ [7, 2.7]. ∎

The characterization of RI_0 given in the theorem can be used to prove the class of RI_0 sets is strictly smaller than the I_0 sets:

EXAMPLE 2.8 (A set in $\mathbb{Z}$ that is $I_0(U)$ for all open U, but not RI_0). Let $E_1 = \{10^j + 5j + 1 : j \geq 1\}$ and $E_2 = \{10^j + 1 : j \geq 1\}$. Then $E_1 \cup E_2$ is not I_0 [17, p. 178]. Let $E = E_1 \cup -E_2$. Then $E \cup -E$ is not I_0 and so E is not RI_0 by the theorem.

Now suppose $U \supseteq (-4\pi/10^N, 4\pi/10^N)$ and put $b = \pi/10^N$. Except for a finite number of $n \in E$ (say, for all $n \in E \setminus \Delta$),

$$\widehat{\delta_b}(n) = \begin{cases} e^{-i\pi/10^N} & \text{if } n = -10^j - 1, \\ e^{i\pi(5j+1)/10^N} & \text{if } n = 10^j + 5j + 1. \end{cases}$$

Put $\nu = e^{i\pi/10^N}\delta_b - \delta_0$. Then $\widehat{\nu} = 0$ on $E_2 \setminus \Delta$, $|\widehat{\nu}| \geq |1 - e^{i\pi/10^N}| > 0$ on $E_1 \setminus \Delta$ and $\nu \in M_{\mathrm{d}}(-2\pi/10^N, 2\pi/10^N)$.

Let $\varphi \in \ell^\infty(E)$. As E_1 and E_2 are ε-Kronecker sets for some $\varepsilon < 1$, the tail of each is $I_0(-\pi/10^N, \pi/10^N)$ [7, Thm. 3.2]. Since $\widehat{\nu}$ is bounded away from 0 on $E_1 \setminus \Delta$ we can find $\omega_1, \omega_2 \in M_{\mathrm{d}}(-\pi/10^N, \pi/10^N)$ such that $\widehat{\omega}_1 = \varphi$ on $E_2 \setminus \Delta$ (Δ renamed as necessary) and

$$\widehat{\omega}_2 = (\varphi - \widehat{\omega}_1)/\widehat{\nu} \quad \text{on } E_1 \setminus \Delta.$$

Take $\omega = \omega_1 + \omega_2 * \nu$. Then $\omega \in M_{\mathrm{d}}(-3\pi/10^N, 3\pi/10^N)$ and interpolates φ on $E \setminus \Delta$. Now apply [7, 2.8] to conclude that E is $I_0(U)$.

Corollary 2.7 implies that a set is $RI_0(U)$ for all open sets U if it is $RI_0(U)$ with bounded constants (in the connected case). In contrast, *no* non-empty set is $FZI_0(U)$ for all open sets U since even singletons are not $FZI_0(U)$ for small neighbourhoods of the identity.

However, the class of FZI_0 sets is closed under the adjunction of finite sets.

PROPOSITION 2.9. *Suppose E is FZI_0 and F is a finite set not containing $\mathbf{1}$. Then $E \cup F$ is FZI_0.*

Proof. There is no loss in assuming F is a singleton $\{\gamma\}$ and $\gamma, \gamma^{-1} \notin E$. As $\overline{E} \cap \Gamma = E$ (see [18]), there is a (Bohr) closed neighbourhood V of $\mathbf{1}$ in $\overline{\Gamma}$ such that $E \cap V \cdot V^{-1} = \emptyset$ and $E \cap \{\gamma, \gamma^{-1}\} V \cdot V^{-1} = \emptyset$. Put

$$f = \frac{1}{2m(V)}\,(1_{\gamma V \cup V} * 1_{\gamma^{-1}V^{-1} \cup V^{-1}}).$$

Then $f(\gamma) \geq 1/2$ and $f = 0$ on E. As f is positive definite, there exists $\mu \in M_{\mathrm{d}}^+(G)$ such that $\widehat{\mu} = f$ on Γ. It follows easily that $E \cup \{\gamma\}$ is FZI_0. ■

3. $FZI_0(U)$ with bounded constants

3.1. *ε-Kronecker sets are $FZI_0(U)$ with bounded constants*

THEOREM 3.1. *Let $0 < \varepsilon < \sqrt{2}$ and let E be an ε-Kronecker subset of the discrete abelian group Γ. Let $U \subset G$ be open. Then E is $FZI_0(U)$ with bounded constants.*

Proof. Let U be a neighbourhood of the identity. Let $W \subset U$ be a symmetric neighbourhood such that $W^2 \subset U$. By [7, Thm. 3.2], for any (fixed) $\varepsilon' > \varepsilon$ there is a finite subset Δ such that $E \setminus \Delta$ is ε'-Kronecker(W). Choose $\delta = \delta(\varepsilon') \in (0,1)$ such that $|\langle \gamma, x_0 \rangle - 1| < \varepsilon'$ implies $\Re\langle \gamma, x_0\rangle \geq \delta$ and pick $b > 0$ such that $b + \sqrt{1-\delta^2} < 1$.

Let $\varphi \in B(\ell^\infty(E \setminus \Delta))$, say $\varphi(\gamma) = a_\gamma + ib_\gamma$, for a_γ, b_γ real and $\gamma \in E \setminus \Delta$. Let $x_0 \in W$ satisfy

$$\begin{aligned} |\langle \gamma, x_0\rangle - 1| &< \varepsilon' \quad \text{if } a_\gamma \in [b, 1], \\ |\langle \gamma, x_0\rangle + 1| &< \varepsilon' \quad \text{if } a_\gamma \in [-1, -b], \\ |\langle \gamma, x_0\rangle - i| &< \varepsilon' \quad \text{if } a_\gamma \in (-b, b). \end{aligned}$$

Then

$$|a_\gamma - \Re\langle \gamma, x_0\rangle| < \max(1-\delta, 1-b, b+\sqrt{1-\delta^2}) = \varepsilon_0 < 1.$$

By iterating we can interpolate any real sequence on $E \setminus \Delta$ with the Fourier–Stieltjes transform of some $\mu \in M_{\mathrm{d}}^+(W)$.

To interpolate $\{ib_\gamma\}_{\gamma \in E\setminus\Delta}$ we argue as follows: Let $x_1 \in W$ be such that

$$|\langle \gamma, x_1\rangle - r_\gamma i| < \varepsilon \quad \text{for all } \gamma \in E \setminus \Delta,$$

where $r_\gamma = 1$ if $b_\gamma \geq 0$ and $r_\gamma = -1$ if $b_\gamma < 0$. By the previous part of the proof there exists $\mu \in M_{\mathrm{d}}^+(W)$ such that $\widehat{\mu}(\gamma) = -\Re\langle \gamma, x_1\rangle$ for $\gamma \in E \setminus \Delta$. For such γ,

$$\left| \frac{\widehat{\mu + \delta_{x_1}}(\gamma)}{2} - ib_\gamma \right| = \left| \frac{\Im\langle \gamma, x_1\rangle}{2} - b_\gamma \right| < 1 - \frac{\delta}{2},$$

where δ is as above.

Consequently, if $\sigma \in M_{\mathrm{d}}^+(U)$ is a measure interpolating a_γ, then

$$\nu = \sigma + (\mu + \delta_{x_1})/2 \in M_{\mathrm{d}}^+(U)$$

satisfies $|\widehat{\nu}(\gamma) - \varphi(\gamma)| < 1 - \delta/2$. Now iterate. It is clear that the constant of interpolation depends only on ε. ∎

The following result should be contrasted with Example 5.1, which shows FZI_0 and RI_0 sets are not preserved under translation.

PROPOSITION 3.2. *Let E be an ε-Kronecker subset of the discrete abelian group Γ. Then for all γ and $\varepsilon' > \varepsilon$, there exists a finite set Δ such that $(E \setminus \Delta) \cdot \gamma$ is ε'-Kronecker and $FZI_0(U)$ with bounded constants.*

Proof. Choose a neighbourhood U of the identity such that $\gamma \approx 1$ on U. There exists Δ such that $E \setminus \Delta$ is ε'-Kronecker(U); thus we can approximate any $\varphi \in B(\ell^\infty((E \setminus \Delta) \cdot \gamma))$ with $\widehat{\delta_x}$, where $x \in U$. ∎

3.2. *Other $FZI_0(U)$ sets with bounded constants.* In this subsection we collect examples of $FZI_0(U)$ sets in non-classical abelian groups. These results will be used for the general existence theorems in Section 5.

PROPOSITION 3.3. *Let $\varepsilon > 0$ and let $E = \{\chi_n\}$ be an infinite sequence in the* (*discrete group of*) *rationals*, $\mathbb{Q}$. *Then E has an infinite ε-Kronecker subset and hence a subset that is $FZI_0(U)$ with bounded constants.*

Proof. Fix q such that $1 + \pi/\varepsilon < q$.

Case 1: $\{\chi_n\}$ is unbounded in $\mathbb{R}$. Then we can find a subsequence (not renamed) such that $\chi_{n+1}/\chi_n > q$ for all n. Such a set is ε-Kronecker by [15, 2.4(1)].

Case 2: The $\{\chi_n\}$ accumulate at $r \in \mathbb{R}$ in the usual topology of $\mathbb{R}$. Then we can find a subsequence (not renamed) such that $(\chi_{n+1} - r)/(\chi_n - r) < 1/3q$. Obviously all finite portions of such a set are $\varepsilon/3$-Kronecker. By taking elements in the dual of $\mathbb{R}_{\mathrm{d}}$, the Bohr compactification of $\mathbb{R}$, we see that $\{\chi_n - r : n \geq 1\}$ is $\varepsilon/2$-Kronecker. Given a 0-neighbourhood U there is some integer N such that $F = \{\chi_n\}_{n=N}^{\infty}$ is $2\varepsilon/3$-Kronecker(U). But then F is ε-Kronecker in $\mathbb{Q}$, and hence $FZI_0(U)$ with bounded constants by Theorem 3.1. ∎

For a prime p, $\mathcal{C}(p^{\infty})$ denotes the discrete group generated by all elements of the form $\chi = e^{2\pi ik/p^n} \in \mathbb{T}$, where $1 \leq k < p$ and $n \geq 1$, i.e., the infinite p-subgroup of $\mathbb{T}_d$ consisting of all elements whose orders are a power of p. Integers $\ell \in \mathbb{Z}$ give rise to characters on $\mathcal{C}(p^{\infty})$: $\langle \varrho, \ell\rangle = e^{2\pi ik\ell/p^n}$, where $\varrho = e^{2\pi ik/p^n}$. Of course, any element of the coset $\ell + p^n\mathbb{Z}$ gives rise to the same value.

Proposition 3.4. *Let $\varepsilon > 0$ and let $E = \{\gamma_j\}$ be an infinite sequence in $\mathcal{C}(p^{\infty})$. Then E has an infinite ε-Kronecker subset and hence a subset that is $FZI_0(U)$ with bounded constants.*

Proof. Let $1 \leq C < \infty$ be such that $|1 - e^{\pi i/p^C}| < \varepsilon/2$. By passing to a subsequence χ_j of the γ_j, we may assume $\chi_j = e^{2\pi ik_j/p^{n_j}}$ where $n_1 \geq C$ and $n_{j+1} - n_j \geq C$ for $j \geq 1$.

Let $\varphi : \{\chi_j\} \to \mathbb{T}$ be given. Then the possible values of a character at χ_1 are $e^{2\pi ik_1\ell/p^{n_1}}$. Since $1 \leq k_1 \leq p$, those values are spaced equidistantly on $\mathbb{T}$ with distance between them equal to $|1 - e^{2\pi ik_1\ell/p^{n_1}}| < \varepsilon/2$, so one of them is at most $|1 - e^{\pi i/p^{n_1}}| < \varepsilon/2$ from $\varphi(\chi_1)$. Choose $\ell_1 \in \mathbb{Z}$ such that

$$|e^{2\pi ik_1\ell_1/p^{n_1}} - \varphi(\chi_1)| < \varepsilon/2.$$

The last inequality does not change if we replace ℓ_1 by an element of the coset $\ell_1 + p^{n_1}\mathbb{Z}$.

Now consider $e^{2\pi ik_2(\ell_1+\ell_2p^{n_1})/p^{n_2}}$ for $\ell_2 \in \mathbb{Z}$. Those quantities are equally spaced on the unit circle at spacing of $|1 - e^{\pi ik_2/p^{n_2-n_1}}|$. Since $n_2 - n_1 \geq C$, the distance between those values is less than $\varepsilon/2$, so we can find ℓ_2 such that

$$|e^{2\pi ik_2(\ell_1+\ell_2p^{n_1})/p^{n_2}} - \varphi(\chi_2)| < \varepsilon/2.$$

Those values do not change if we replace ℓ_2 by an element of $\ell_2 + p^{n_2}\mathbb{Z}$.

Continuing in this manner, we obtain a sequence in $\mathbb{Z}$ which does the correct interpolation on each finite subset of E. So any accumulation point in the dual of $\mathcal{C}(p^\infty)$ will interpolate φ with error less than or equal to $\varepsilon/2$ and from this we may conclude that E is ε-Kronecker. ∎

We recall that a subset E of an abelian group L is called *independent* if $N \geq 1$, $x_1, \dots, x_N \in E$, $n_1, \dots, n_N \in \mathbb{Z}$ and $\sum_{j=1}^{N} x_j^{n_j} = \mathbf{1}_L$ imply $x_j^{n_j} = \mathbf{1}_L$ for $1 \leq j \leq N$.

Proposition 3.5. *Let $E \subset \Gamma$ be independent. Then E is $FZI_0(U)$ for all open U with bounded constants.*

Proof. Let Λ be the subgroup of Γ generated by E. For each $\chi \in E$, let Λ_χ be the cyclic subgroup of Γ generated by χ, and H_χ be the dual group of Λ_χ. The independence of E implies that $\Lambda = \bigoplus_{\chi\in E} \Lambda_\chi$ and that the dual group, H, of Λ is the direct product $H = \prod_{\chi\in E} H_\chi$. We may assume that $\Gamma = \Lambda$.

Let $U \subset G$ be any neighbourhood of the identity. Then there exists a finite set $\Delta \subset E$ such that $\prod_{\chi\in E\setminus\Delta} H_\chi \subset U$. Let $F \subset E \setminus \Delta$ be the elements of order 2 (if any). Without loss of generality, we may assume that $U = \{1\} \times U_0 \times U_1$, where $U_0 = \prod_{\chi\in F} H_\chi$ and $U_1 = \prod_{\chi\in E\setminus(\Delta\cup F)} H_\chi$.

Let $\varphi : E \setminus \Delta \to \mathbb{T}$ be Hermitian. If F is non-empty, then the independence of F and the fact that φ takes only the values ± 1 on F imply that for each $\chi \in F$ there exists $x_\chi \in H_\chi$ with $\langle \chi, x_\chi \rangle = \varphi(\chi)$. Let $x = \prod_{\chi\in F\setminus\Delta} x_\chi$ if F is non-empty, and let x be the identity of G otherwise.

For each $\chi \in E \setminus F$, there exists x_χ such that $|\langle \chi, x_\chi\rangle - \varphi(\chi)| \leq 1$, by the definition of H_χ. (The worst case occurs if χ has order 3, and otherwise the above difference is at most $\sqrt{2}/2$.) This shows that an independent set containing only elements of order greater than 2 is ε-Kronecker for all $\varepsilon > 1$.

By Theorem 3.1 applied to $E \setminus F$, there is a constant K (independent of U_1), a finite set $\Delta_1 \subset E \setminus F$ (depending only on U_1), and a measure $\nu_1 \in M_{\mathrm{d}}^{+}(U_1)$ such that

$$\widehat{\nu}_1(\chi) = \varphi(\chi) \quad \text{on } E \setminus (F \cup \Delta_1), \quad \|\nu_1\| \leq K.$$

Let $\nu = \delta_x \times \nu_1$. Then $\widehat{\nu}(\chi) = \varphi(\chi)$ for each $\chi \in E \setminus (\Delta \cup \Delta_1)$. Hence $E \setminus (\Delta \cup \Delta_1)$ is $FZI_0(U)$ with constant K. ∎

4. Existence theorems for FZI_0 and $FZI_0(U)$ sets

4.1. *Γ has large $FZI_0(U)$ sets with bounded constants*

Theorem 4.1. *Let Γ be an infinite discrete abelian group. Then Γ contains an $FZI_0(U)$ set with bounded constants and having the same cardinality as Γ.*

The following corollary is due to [13]; another proof is in [15].

Corollary 4.2. *Every discrete abelian group contains an I_0 set of the same cardinality.*

Before proving Theorem 4.1, we need a lemma.

LEMMA 4.3. *Let $E \subset \Gamma$. Suppose that E is uncountable. Then E contains an independent set A such that $\#A = \#E$.*

Proof of Lemma 4.3. Let $\kappa = \#E$. Let F be the torsion subgroup of the subgroup H of Γ that E generates. Suppose $\kappa = \#(H/F)$. Let A be a subset of E such that the set of cosets $A \cdot F$ is maximal independent in the quotient H/F. As E generates H, $E \cdot F$ generates H/F, and any maximal independent subset of $E \cdot F$ also generates H/F. Hence, the cardinalities of the four (types of) sets, $E \cdot F$, H/F, maximal independent subset of H/F, and maximal independent subset of $E \cdot F$, are all equal. Thus, $\#A = \kappa$ if $\kappa = \#(H/F)$. Of course, if $A \cdot F$ is independent in the torsion-free group H/F, then A must be independent in H.

If $\kappa > \#(H/F)$ then $\kappa = \#F$. We apply *verbatim* all but the first two paragraphs of the proof of [15, Lem. 3.7], from which we conclude that H contains an independent set A' with $\#A' = \kappa$. Of course that cannot happen unless E contains an independent set A with $\#A = \kappa$. ∎

Proof of Theorem 4.1. If Γ is uncountable, then the theorem follows from Proposition 3.5 and Lemma 4.3 (with $E = \Gamma$).

If Γ is countable and contains an infinite independent set, then the theorem follows from Proposition 3.5. We thus assume that E is countable and does not contain an infinite independent set.

If Γ has an element of infinite order, then Γ contains a copy of $\mathbb{Z}$, which contains Hadamard sets of ratio greater than 3. These are ε-Kronecker sets with $\varepsilon < \sqrt{2}$ [6, Prop. 2·3] and consequently $FZI_0(U)$ with bounded constants, by Theorem 3.1.

We thus assume that Γ is a countable torsion group with no infinite independent sets. By [2, 20.1], Γ can be identified with a subgroup of the direct sum $\bigoplus_\beta \mathcal{C}(p_\beta^\infty)$. If that direct sum is minimal with respect to containing Γ, then the direct sum must be finite, since otherwise induction would show that Γ contained an infinite independent set. Therefore the projection of Γ onto one of the summands must be infinite. Applying Proposition 3.4 gives the desired set in $\mathcal{C}(p_\beta^\infty)$. Taking the corresponding characters in Γ completes the proof. ∎

4.2. *E has large FZI_0 sets*

THEOREM 4.4. *Let E be an infinite subset of the discrete abelian group Γ. Then E has an infinite FZI_0 subset of the same cardinality as E.*

Proof. If E is uncountable, then E contains a maximal independent subset F of the same cardinality as E by Lemma 4.3, and we may apply Proposition 3.5.

We thus assume that $E = \{\chi_n\}$ is countable. By [2, 20.1], Γ can be identified with a subgroup of a divisible group Λ which is a direct sum

$$\bigoplus_{\alpha} \mathbb{Q}_\alpha \ \oplus \ \bigoplus_{\beta} \mathcal{C}(p_\beta^\infty),$$

where the $\mathbb{Q}_\alpha$ are copies of the rationals in $\mathbb{R}$. We may assume $\Gamma = \Lambda$. For convenience, let us write $\Gamma = \bigoplus_\ell \Gamma_\ell$ where each Γ_ℓ is one of $\mathbb{Q}$ or $\mathcal{C}(p^\infty)$, the direct sum is (at most) countable, and for each ℓ there is some character $\chi_n \in E$ whose projection onto Γ_ℓ is not trivial.

Let $I = \{\ell :$ there is some n such that the projection of χ_n onto Γ_ℓ is not trivial and not of order two$\}$.

Case 1: $\#I$ is infinite. As each χ_n has a non-trivial projection onto only finitely many factors Γ_i, it follows that there must be an infinite subsequence (not renamed) such that the projection onto Γ_{α_n} is of order ≥ 3, but the projections of χ_m onto Γ_{α_n} are trivial for $m < n$. As these χ_n are of order at least three, they form an ε-Kronecker set for any $\varepsilon > 1$, and hence even an $FZI_0(U)$ set with bounded constants.

Case 2: $\#I$ is finite. We distinguish between two subcases.

(a) There is some index ℓ such that the projection of χ_n onto Γ_ℓ is infinite. In this case apply Propositions 3.3 or 3.4 to find an infinite ε-Kronecker subset of these projections, and therefore of the original χ_n (for any $\varepsilon > 0$).

(b) Otherwise it follows that the projections of χ_n onto $\bigoplus_{\ell\in I} \Gamma_\ell$ form a finite set and hence infinitely many of them have the same projection, say γ. We restrict ourselves to this set of projections (again, not renamed). If $\ell \notin I$, then $\Gamma_\ell = \mathcal{C}(2^\infty)$ and the projection of χ_n onto Γ_ℓ is either trivial or is (the unique element) of order two. As the projections of these χ_n onto $\bigoplus_{\ell\in I} \Gamma_\ell$ coincide, their projections onto $\bigoplus_{\ell\in I^{\mathrm{c}}} \Gamma_\ell$ must be distinct.

Hence the complement I^{c} must be infinite and therefore we can obtain an infinite independent set as in Case 1, but consisting of elements of order 2, say $\{\gamma_j\}$. Such a set is $FZI_0(U)$ with bounded constants: we need only to interpolate real functions φ, since the characters are of order 2. The corresponding characters are of the form $\gamma \oplus \gamma_j$. If γ is of order one or two it is obvious that $\{\gamma \oplus \gamma_j\}$ is FZI_0. To see that $\{\gamma \oplus \gamma_j\}$ is FZI_0 otherwise, we apply Proposition 2.4: Given any choice of signs $\{r_j\}$ we can find $\mu \in M_{\mathrm{d}}^+(\bigoplus_{\ell\in I^{\mathrm{c}}} \Gamma_\ell)$ such that $\widehat{\mu}(\gamma_j) = r_j$. As $\gamma^2 \neq 1$ we can choose $\nu \in M_{\mathrm{d}}^+(\bigoplus_{\ell\in I^{\mathrm{c}}} \Gamma_\ell)$ such that $\widehat{\nu}(\gamma) = i$. Then $\widehat{\delta_1 \times \mu}(\gamma \oplus \gamma_j) = r_j$ and $\widehat{\nu \times \mu}(\gamma \oplus \gamma_j) = ir_j$. ■

Remark 4.5. E need not contain an $FZI_0(U)$ set for small U. Take, for instance, $G = \mathbb{Z}_3 \times \mathbb{D}_2$ and let $\varrho \in \widehat{\mathbb{Z}}_3$, $\varrho \neq \mathbf{1}$. Let $F \subset \widehat{\mathbb{D}}_2$ be an independent set. The set $E = \{\varrho\} \times F$ is FZI_0 (it is just a special case of Case 2(b) above) but no infinite subset is $FZI_0(\{0\} \times \mathbb{D}_2)$ as it is clearly only possible to interpolate real-valued sequences by positive measures supported on $\{0\} \times \mathbb{D}_2$.

4.3. *Existence of $FZI_0(U)$ subsets with bounded constants*

THEOREM 4.6. *Let G be a compact connected abelian group. Then every infinite $E \subset \Gamma$ contains an $FZI_0(U)$ set with bounded constants of the same cardinality.*

COROLLARY 4.7. *Γ contains a subset of cardinality $\#\Gamma$ that is $I_0(U)$ for all open U.*

Proof. If E is uncountable, then E has an independent set of the same cardinality and Proposition 3.5 completes the proof.

We may thus assume that E, and therefore Γ, is countable. Because Γ has no elements of finite order, Γ is contained in the countable direct sum of copies of $\mathbb{Q}$, and we may assume that Γ is such a sum.

Suppose that the projection of E on any one of the factors $\mathbb{Q}$ is infinite. Then by Proposition 3.3, we have an infinite $FZI_0(U)$ set with bounded constants.

Otherwise, E has non-zero projections on an infinite number of the factors. We again have an infinite independent set of elements whose order is infinite, so we have an $FZI_0(U)$ set with bounded constants by Proposition 3.5. ∎

5. Translation of FZI_0 sets. In contrast to the situation for I_0 sets, translation does not in general preserve RI_0 or FZI_0 sets.

EXAMPLE 5.1 (A translate of an FZI_0 set which is not RI_0). Let $E_1 = \{16^j + 4j : j \geq 1\}$, $E_2 = \{-16^j - 2 : j \geq 1\}$, and $E = E_1 \cup E_2$. The two sets E_1, E_2 are $FZI_0(U)$ with bounded constants, being ε-Kronecker for $\varepsilon < 1$. If we evaluate $\widehat{\delta_0} + \widehat{\delta_{\pi/2}}$ on E, we get 2 on E_1 and 0 on E_2. Standard arguments show that E is $FZI_0(U)$. But $E+1$ is not even RI_0 because $(E+1) \cup (-E-1)$ is not I_0.

One reason for the interest in sets that are $FZI_0(U)$ with bounded constants is that under this (additional) assumption FZI_0 *is* preserved under translation.

PROPOSITION 5.2. *Suppose E is an antisymmetric $FZI_0(U)$ set with bounded constants and suppose F is a finite, asymmetric set.*

(1) *Suppose there exists a neighbourhood $V \subset G$ such that F is $FZI_0(V)$. Then there is a finite set Δ such that $(E \setminus \Delta) \cdot F$ is $FZI_0(V)$.*
(2) *If $F^{-1} \cap E = \emptyset$ then $E \cdot F$ is FZI_0.*

Proof. (2) follows from (1), since a finite set is $FZI_0(G)$ and the union of a finite set with an FZI_0 set is FZI_0 (Prop. 2.9).

(1) Assume $F=\{\lambda_1,\ldots,\lambda_M\}$. For each $k=1,\ldots,M$ let $\mu_k=\sum_{j=1}^{\infty} a_{jk}\delta_{x_{jk}} \in M_{\mathrm{d}}^+(V)$ be such that

$$\widehat{\mu}_k(\lambda_i)=\begin{cases}1 & \text{if } i=k,\\ \widehat{\mu}_k(\lambda_i)=0 & \text{otherwise.}\end{cases}$$

(Here we use the fact that F is asymmetric, as well as $FZI_0(V)$.) Choose N_k such that $\|\mu_k-\sum_{j=1}^{N_k} a_{jk}\delta_{x_{jk}}\|<\varepsilon/M$.

Let K be as in the definition of $FZI_0(U)$ with bounded constants (Definition 3) and let $\varepsilon_{jk}=\varepsilon 2^{-j}/(KMa_{jk})$. Choose neighbourhoods $U_{jk}\subset V$ of x_{jk} such that $|\lambda_i(x_{jk})-\lambda_i(y)|<\varepsilon_{jk}$ for $y\in U_{jk}$, $i=1,\ldots,M$.

Let $\varphi\in\ell^\infty(E\cdot F)$ be a given Hermitian function of norm one. Select finite sets $\Delta_{jk}\subset\Gamma$ and measures $\nu_{jk}\in M_{\mathrm{d}}^+(U_{jk})$, of norm at most K, such that $\widehat{\nu}_{jk}(\chi)=\varphi(\chi\lambda_k)$ for $\chi\in E\setminus\Delta_{jk}$. Put $\Delta=\bigcup_{k=1}^{M}\bigcup_{j=1}^{N_k}\Delta_{jk}$ and let $\mu=\sum_{j,k}a_{jk}\nu_{jk}\in M_{\mathrm{d}}^+(V)$. From (5) we know that for $\chi\in E\setminus\Delta$ and $\lambda_i\in F$, $\widehat{\nu}_{jk}(\chi\lambda_i)=\lambda_i(x_{jk})\widehat{\nu}_{jk}(\chi)+E_{ijk}$ with error term E_{ijk} satisfying $|E_{ijk}|\le K\varepsilon_{jk}$. Thus

$$|\widehat{\mu}(\chi\lambda_i)-\varphi(\chi\lambda_i)|\le\Big|\sum_{k=1}^{M}\sum_{j=1}^{N_k}a_{jk}\lambda_i(x_{jk})\varphi(\chi\lambda_k)-\varphi(\chi\lambda_i)\Big|+\sum_{j,k}\frac{\varepsilon 2^{-j}a_{jk}}{a_{jk}M}.$$

As $|\sum_{j=1}^{N_k}a_{jk}\lambda_i(x_{jk})-\widehat{\mu}_k(\lambda_i)|\le\varepsilon/M$ and $\widehat{\mu}_k(\lambda_i)=1$ if $i=k$ and 0 else, it follows that $|\widehat{\mu}(\chi\lambda_i)-\varphi(\chi\lambda_i)|\le 2\varepsilon$. Thus $(E\setminus\Delta)\cdot F$ is $FZI_0(V)$. ■

A set can be $FZI_0(U)$ for all open U without bounded constants:

EXAMPLE 5.3. The set $\{9^j\}\cup\{9^j+3j+1\}$ is $FZI_0(U)$ for all open U, but not with bounded constants.

Proof. To see this assume $U=[-\pi/9^N,\pi/9^N]$. With $a=\pi/9^N$ we get $\widehat{\delta}_a(9^j)=-1$ for $j>N$ and $|\widehat{\delta}_a(9^j+3j+1)+1|\ge\varepsilon>0$. Since the transform of $\delta_0+\delta_a$ is 0 on $\{9^j\}$ and bounded away from zero on $\{9^j+3j+1\}$, it follows that $\{9^j\}_{j>N}\cup\{9^j+3j+1\}_{j>N}$ is $FZI_0(U)$. This set is not even $I_0(U)$ with bounded constants as $E\cup(E+1)$ is not I_0 [7, 3.1]. ■

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