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Atomic decomposition on Hardy–Sobolev spaces

by

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Abstract. As a natural extension of L^p Sobolev spaces, we consider Hardy–Sobolev spaces and establish an atomic decomposition theorem, analogous to the atomic decomposition characterization of Hardy spaces. As an application, we deduce several embedding results for Hardy–Sobolev spaces.

1. Introduction. Let H^p be the real-variable Hardy space on $\mathbb{R}^n$ for $p > 0$. Given a positive integer ℓ, the *homogeneous Hardy–Sobolev space* $\dot{H}^p_\ell$ consists of all tempered distributions f on $\mathbb{R}^n$ such that each weak derivative $\partial^\sigma f$ of order $|\sigma| = \ell$ belongs to H^p. We define

$$\|f\|_{\dot{H}^p_\ell} = \sum_{|\sigma|=\ell} \|\partial^\sigma f\|_{H^p},$$

a quasi-norm on $\dot{H}^p_\ell$ modulo polynomials of degree less than ℓ. For $p > 1$, $\dot{H}^p_\ell$ is identical to the homogeneous L^p Sobolev space $\dot{W}^{\ell,p}$. For $0 < p \le 1$, it is well known that the H^p provide an ideal alternative of the L^p and thus the $\dot{H}^p_\ell$ may be thought as a natural substitute of the $\dot{W}^{\ell,p}$. The *inhomogeneous Hardy–Sobolev space* H^p_ℓ is defined as $H^p_\ell = H^p \cap \dot{H}^p_\ell$ with the quasi-norm (1)

$$\|f\|_{H^p_\ell} = \|f\|_{H^p} + \|f\|_{\dot{H}^p_\ell} \approx \sum_{|\sigma|\le\ell} \|\partial^\sigma f\|_{H^p}.$$

The primary purpose of the present paper is to obtain an atomic decomposition theorem on Hardy–Sobolev spaces, analogous to the atomic decomposition characterization of Hardy spaces.

Briefly, the atomic decomposition theorem of Coifman [Co] and Latter [L] asserts that $f \in H^p$ for $0 < p \le 1$ if and only if there exist a sequence (a_k) of H^p atoms and a sequence (λ_k) of scalars with $\sum |\lambda_k|^p < \infty$ such that $f =$

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(1) Throughout this article, the symbol $\approx$ stands for quasi-norm equivalence.

$\sum \lambda_k a_k$ in the sense of distributions. An H^p *atom* is any bounded function a with support in a cube Q, with $\|a\|_\infty \leq |Q|^{-1/p}$, and with vanishing moments at least up to order $n(1/p-1)$, that is,

$$\int x^\sigma a(x)dx = 0 \quad \text{for all } |\sigma| \leq n_p,$$

where n_p denotes the smallest integer greater than or equal to $n(1/p-1)$.

For each positive integer m and a point $z \in \mathbb{R}^n$, we denote by Δ_z^m the mth forward difference operator defined inductively as

$$\Delta_z f(x) = f(x+z) - f(x), \qquad \Delta_z^m f(x) = \Delta_z[\Delta_z^{m-1} f](x) \qquad (m \geq 2)$$

for each function f on $\mathbb{R}^n$. We recall from [G] that a bounded continuous function f on $\mathbb{R}^n$ belongs to the *Lipschitz space* $\Lambda_\alpha(\mathbb{R}^n)$ of order $\alpha > 0$ if

$$|\Delta_z^{[\alpha]+1} f(x)| \leq C|z|^\alpha \qquad (x, z \in \mathbb{R}^n)$$

for some constant $C > 0$. When $\alpha = k + \delta$ for some $k \in \mathbb{Z}_+$, $0 < \delta < 1$, it is the space of C^k functions f with bounded derivatives up to order k and

$$|\Delta_z[\partial^\sigma f](x)| \leq C|z|^\delta \qquad (x, z \in \mathbb{R}^n)$$

for all $|\sigma| = k$ and for some constant $C > 0$. If α is an integer, then it is the space of $C^{\alpha-1}$ functions f with bounded derivatives up to order $\alpha - 1$ and

$$|\Delta_z^2[\partial^\sigma f](x)| \leq C|z| \qquad (x, z \in \mathbb{R}^n)$$

for all $|\sigma| = \alpha - 1$ and for some constant $C > 0$.

The notion of atoms relevant to Hardy–Sobolev spaces is given as follows.

Definition 1.1. A function $b \in \Lambda_\ell(\mathbb{R}^n)$ is called an *atom* for $\dot{H}_\ell^p$ if it has the following properties.

- (i) It has support in a cube Q.
- (ii) $\|b\|_\infty \leq |Q|^{-1/p+\ell/n}$, $|\Delta_z^{\ell+1} b(x)| \leq |Q|^{-1/p}|z|^\ell$ $(x, z \in \mathbb{R}^n)$.
- (iii) For $0 < p \leq n/(n+\ell)$, it has vanishing moments at least up to order $n(1/p - \ell/n - 1)$.

Owing to the method of our proofs and the embedding nature of the statements, it turns out that the hypotheses and the convergence behavior of atomic decompositions on Hardy–Sobolev spaces vary with the range of p's. Accordingly, we put our results into two separate statements.

In the range $0 < p \leq n/\ell$, we have the following result that resembles the aforementioned theorem of Coifman and Latter.

Theorem A. *Let ℓ be a positive integer.*

(a) *If $f \in \dot{H}_\ell^p$ for $0 < p < n/\ell$ or $f \in H_\ell^p$ for $p = n/\ell$, then there exist a sequence (μ_k) of scalars and a sequence (b_k) of $\dot{H}_\ell^p$ atoms such that $f = \sum \mu_k b_k$ in the sense of distributions and*

$$\sum |\mu_k|^p \leq C\|f\|_{\dot{H}_\ell^p}^p.$$

Moreover, the series $\sum \mu_k b_k$ *converges absolutely almost everywhere and represents a locally integrable function when* $p \geq n/(n+\ell)$.

(b) *For* $0 < p \leq \min(n/\ell, 1)$, *suppose that* $f = \sum \mu_k b_k$ *in the sense of distributions, where* (μ_k) *is a sequence of scalars with* $\sum |\mu_k|^p < \infty$ *and* (b_k) *is a sequence of* $\dot{H}^p_\ell$ *atoms. Then* $f \in \dot{H}^p_\ell$ *and*

$$\|f\|^p_{\dot{H}^p_\ell} \leq C \sum |\mu_k|^p.$$

As a particular consequence, Theorem A gives an atomic decomposition characterization of $\dot{H}^p_\ell$ for $0 < p \leq 1$ when $1 \leq \ell < n$.

In the range $p > n/\ell$, it turns out that each distribution of H^p_ℓ coincides with a bounded continuous function satisfying a certain Lipschitz estimate and has an atomic decomposition with absolute convergence.

Theorem B. *Suppose that* $f \in H^p_\ell$ *for* $n/\ell < p < \infty$. *Then* f *coincides with a function in the class* $\Lambda_{\ell - n/p}$ *having the following properties*:

(i) $\|f\|_\infty \leq C\|f\|_{H^p_\ell}$ *and* $|\Delta^m_z f(x)| \leq C\|f\|_{\dot{H}^p_\ell} |z|^{\ell - n/p}$ $(x, z \in \mathbb{R}^n)$ *for any integer* $m > \ell - n/p$.

(ii) *There exist a sequence* (μ_k) *of scalars and a sequence* (b_k) *of* $\dot{H}^p_\ell$ *atoms such that* $f = \sum \mu_k b_k$ *with absolute convergence and*

$$\sum |\mu_k|^p \leq C\|f\|^p_{\dot{H}^p_\ell}.$$

It is the size and Lipschitz conditions on atoms that provide a basic insight into the structure of H^p Sobolev spaces. For instance, each atom b for $\dot{H}^p_\ell$ is an H^q atom with $1/q = 1/p - \ell/n$ when $q \leq 1$, which indicates how the Sobolev exponent arises. More extensively, we shall derive several Sobolev embedding inequalities as an application of our results.

It is possible to characterize $\dot{H}^p_\ell$ by certain variants of non-tangential maximal functions. The proofs of our results rely on such a characterization and the standard method of decomposing distributions via Calderón's reproducing formula and Whitney's decomposition lemma as in the article [Ca] of Calderón and the book [GR] of García-Cuerva and Rubio de Francia.

We finally remark that there are other atomic decomposition results different from ours as well as alternative definitions of Hardy–Sobolev spaces. We shall state some of those results in Section 5 and compare them with our results.

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2. A characterization. For a tempered distribution f and a Schwartz function φ on $\mathbb{R}^n$, let $\varphi_t(x) = t^{-n}\varphi(x/t)$ for each $t > 0$ and $u(x,t) = (f * \varphi_t)(x)$. One of the several equivalent characterizing means of H^p is the *non-tangential maximal function* defined by

$$u_\delta^*(x) = \sup_{|y-x|<\delta t} |u(y,t)| \quad (\delta > 0).$$

According to the fundamental work of Fefferman and Stein [FS], $f \in H^p$ for $0 < p \leq \infty$ if and only if $u_\delta^* \in L^p$ for any choice of φ satisfying $\widehat{\varphi}(0) \neq 0$ and $\delta > 0$. Set $\|f\|_{H^p} = \|u_\delta^*\|_p$. As usual, $\widehat{\varphi}$ stands for the Fourier transform of φ given by

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\varphi(x)\,dx \quad (\xi \in \mathbb{R}^n).$$

Under certain admissibility conditions on f and φ, it turns out that a variant of the non-tangential maximal functions characterizes $\dot{H}_\ell^p$.

Definition 2.1. Given a positive integer ℓ, define

$$\mathcal{M}_\delta^\ell(x,u) = \sup_{|y-x|<\delta t} t^{-\ell}|u(y,t)| \quad (\delta > 0).$$

Lemma 2.2 ([CT], Lemma 4.1). *For a Schwartz function φ on $\mathbb{R}^n$, the following statements are equivalent.*

(a) $\sup_{t>0} |\widehat{\varphi}(t\xi)| > 0$ *for each* $\xi \neq 0$.
(b) *There exists a Schwartz function ζ such that $\widehat{\zeta}$ has compact support away from the origin and*

$$\int_0^\infty \widehat{\varphi}(t\xi)\widehat{\zeta}(t\xi)\,\frac{dt}{t} = 1 \quad (\xi \neq 0).$$

This lemma leads instantly to the so-called Calderón's reproducing formula that plays a crucial role in establishing our result. Let $\mathcal{O}_\ell$ be the class of all Schwartz functions φ on $\mathbb{R}^n$ with vanishing moments up to order $\ell - 1$ and $\sup_{t>0} |\widehat{\varphi}(t\xi)| > 0$ for $\xi \neq 0$. (See [BPT] and [Ch2] for the detailed properties of $\mathcal{O}_\ell$.)

Theorem 2.3 ([Ch1, Theorem I]). *Let f be a tempered distribution on $\mathbb{R}^n$ satisfying the following* admissibity condition: *either $\widehat{f}$ coincides with a locally integrable function away from the origin or $f \in L^q$ for some $1 \leq q \leq \infty$. Then $f \in \dot{H}_\ell^p$ for $0 < p \leq \infty$ if and only if $\mathcal{M}_\delta^\ell(x,u) \in L^p$ for any $\delta > 0$ and $\varphi \in \mathcal{O}_\ell$ with*

$$\|f\|_{\dot{H}_\ell^p} \approx \|\mathcal{M}_\delta^\ell(x,u)\|_p.$$

Moreover, different choices of $\delta > 0$ or $\varphi \in \mathcal{O}_\ell$ yield equivalent norms.

Remark 2.4. It can be shown that statement (a) or (b) in Lemma 2.2 is equivalent to

$$\int_0^\infty |\widehat{\varphi}(t\xi)|^2 \frac{dt}{t} \geq c > 0 \quad (\xi \neq 0)$$

(see [Ch2, Lemma A2]). Regarding the admissibility condition on f in Theorem 2.3, we observe the following facts that will be used later on.

(i) Let $f \in H^p$. If $p \leq 1$, then it is known ([CT, Theorem 4.4]) that $\widehat{f}$ is a continuous function satisfying

$$|\widehat{f}(\xi)| \leq C\|f\|_{H^p}|\xi|^{n(1/p-1)} \quad (\xi \in \mathbb{R}^n).$$

Thus f satisfies the admissibility condition for any $p > 0$.

(ii) Let $f \in \dot{H}^p_\ell$. Since $(\partial^\sigma f)\widehat{\ }(\xi) = (i\xi)^\sigma \widehat{f}(\xi)$, the preceding remark shows that $\widehat{f}$ is a locally integrable function away from the origin when $p \leq 1$. In the case $1 < p < n/\ell$, $f \in L^q$ for $1/q = 1/p - \ell/n$ by the usual L^p Sobolev embedding theorem (see [A]). Thus f satisfies the admissibility condition automatically when $0 < p < n/\ell$ or $0 < p \leq 1$.

3. Preliminary lemmas. The purpose of this section is to state and prove a few preliminary results in preparation for the proofs of Theorems A and B. To begin with, consider the difference operators Δ_z^m. In addition to the well known identity

$$\Delta_z^m f(x) = \sum_{k=0}^m (-1)^{m-k}\binom{m}{k} f(x+kz), \tag{1}$$

we shall need the following form of the mean value theorem.

Lemma 3.1. *If $f \in C^m(\mathbb{R}^n)$ for $m \geq 1$, then for any $x, z \in \mathbb{R}^n$,*

$$\Delta_z^m f(x) = m! \sum_{|\sigma|=m} \frac{z^\sigma}{\sigma!} \int_0^1 \cdots \int_0^1 (\partial^\sigma f)[x + (\theta_1 + \cdots + \theta_m)z]\, d\theta_1 \cdots d\theta_m.$$

Proof. The fundamental theorem of calculus shows that

$$\Delta_z f(x) = (z_1\partial_1 + \cdots + z_n\partial_n)\int_0^1 f(x+\theta z)\, d\theta.$$

It follows that Δ_z^m operates on C^m functions by the formula

$$\Delta_z^m f(x) = (z_1\partial_1 + \cdots + z_n\partial_n)^m \Big\{\int_0^1 \cdots \int_0^1 f[x + (\theta_1 + \cdots + \theta_m)z]\, d\theta_1 \cdots d\theta_m\Big\}.$$

Applying the multinomial theorem and changing the order of integrations, we obtain the stated result immediately. ∎

We next derive an elementary combinatorial identity.

LEMMA 3.2. *For a positive integer* m,

$$\sum_{k=1}^{m}(-1)^k\binom{m}{k}k^j=\begin{cases}0 & \text{for } 1\le j<m,\\ (-1)^m m! & \text{for } j=m.\end{cases}$$

Proof. Apply the differential operator $D=td/dt$ to the identity

$$(1+t)^m=\sum_{k=0}^{m}\binom{m}{k}t^k$$

repeatedly and then evaluate at $t=-1$. ∎

LEMMA 3.3. *If b is an $\dot{H}^p_\ell$ atom for $0<p<\infty$, then $b\in\dot{H}^p_\ell$ with*

$$\|b\|_{\dot{H}^p_\ell}\le C_{p,\ell}$$

where the constant $C_{p,\ell}$ depends on p,ℓ,n but not on b.

Proof. Let b be an atom supported in the cube Q. In view of translation invariance, we may assume that the center of Q lies at the origin. Choose a non-zero radial function $\zeta\in C_c^\infty(B(0,1))$ with $\widehat{\zeta}(0)=0$ and let

$$\Phi(x)=\sum_{k=1}^{\ell+1}(-1)^{\ell+1-k}\binom{\ell+1}{k}\frac{1}{k^n}\,\zeta\Big(-\frac{x}{k}\Big).$$

Then $\widehat{\Phi}(t\xi)$ does not vanish identically as a function of $t>0$ for $\xi\ne0$, and

$$(\partial^\sigma\widehat{\Phi})(0)=(\partial^\sigma\widehat{\zeta})(0)\sum_{k=1}^{\ell+1}(-1)^{\ell+1-k}\binom{\ell+1}{k}(-k)^{|\sigma|}=0$$

for all σ with $|\sigma|\le\ell$ in view of Lemma 3.2. Thus $\Phi\in\mathcal{O}_\ell$. Put now $U(y,t)=(b*\Phi_t)(y)$. Since $b\in L^1$, Theorem 2.3 is applicable and so it suffices to verify $\|\mathcal{M}_1^\ell(x,U)\|_p\le C_{p,\ell}$.

On account of the identity (1) and $\widehat{\zeta}(0)=0$, we have

$$U(y,t)=\int\limits_{|z|<t}(\Delta_z^{\ell+1}b)(y)\zeta_t(z)\,dz.$$

By the Lipschitz condition on b in Definition 1.1(ii),

$$|U(y,t)|\le\|\zeta\|_1|Q|^{-1/p}t^\ell$$

for any $y\in\mathbb{R}^n$ and $t>0$. Thus $\mathcal{M}_1^\ell(x,U)\le\|\zeta\|_1|Q|^{-1/p}$ and

$$\int\limits_{3\sqrt{n}Q}[\mathcal{M}_1^\ell(x,U)]^p dx\le C.$$

For $x \in (3\sqrt{n}\,Q)^{\mathrm{c}}$, we have $|x| > 3\sqrt{n}\,l(Q)/2$ and $|z| < |x|/3$ for all $z \in Q$. Since Φ is supported in the unit ball, this implies that $U(y,t) = 0$ for $|y-x| < t \le |x|/3$. Assume $t > |x|/3$. If $p > n/(n+\ell)$, then we estimate

$$t^{-\ell}|U(y,t)| \le \|\Phi\|_\infty |Q|^{-1/p+\ell/n+1}|x|^{-(n+\ell)}.$$

As $\mathcal{M}_1^\ell(x,U)$ satisfies the same estimate, we have

$$\int\limits_{(3\sqrt{n}Q)^{\mathrm{c}}} [\mathcal{M}_1^\ell(x,U)]^p\,dx \le C.$$

In the case $0 < p \le n/(n+\ell)$, let N_p denote the smallest integer greater than or equal to $n(1/p - \ell/n - 1)$. Using the moment cancelation property of b, we can write

$$U(y,t) = \int\limits_Q b(z)\biggl\{\Phi_t(y-z) - \sum_{|\sigma|\le N_p} \frac{(\partial^\sigma \Phi_t)(y)}{\sigma!}\,(-z)^\sigma\biggr\}\,dz$$

and so Taylor's formula gives the estimate

$$t^{-\ell}|U(y,t)| \le C|Q|^{-1/p+\ell/n+1+(1/n)(N_p+1)}|x|^{-(n+\ell+N_p+1)}.$$

Since $\mathcal{M}_1^\ell(x,U)$ satisfies the same estimate and $-(n+\ell+N_p+1)p + n < 0$, we obtain the same conclusion as above. ∎

Lemma 3.4. *Let $\varphi \in C_{\mathrm{c}}^\infty(B(0,1))$ and let ζ be a Schwartz function on $\mathbb{R}^n$ such that $\widehat{\zeta}$ has compact support away from the origin. For a tempered distribution f on $\mathbb{R}^n$, define*

$$g(x) = \int\limits_0^\infty (f * \zeta_t * \varphi_t)(x)\,\frac{dt}{t} \tag{2}$$

if the integral converges.

(a) *Let $f \in \dot{H}_\ell^p$ for $0 < p < n/\ell$ and $u(y,t) = (f * \zeta_t)(y)$. Then the integral* (2) *converges absolutely almost everywhere and*

$$|g(x)| \le C\|f\|_{\dot{H}_\ell^p}^{p\ell/n}[\mathcal{M}_1^\ell(x,u)]^{p(1/p-\ell/n)}.$$

If $f \in H_\ell^p$ for $p = n/\ell$ and $\widehat{\varphi}(0) \ne 0$, then the integral also converges and

$$|g(x)| \le C\|f\|_{H^p}^{1/2}[\mathcal{M}_1^\ell(x,u)]^{1/2}.$$

(b) *Let $f \in H_\ell^p$ for $n/\ell < p < \infty$ and $\widehat{\varphi}(0) \ne 0$. Then the integral* (2) *converges absolutely everywhere, $g \in \Lambda_{\ell-n/p}$ and*

$$\|g\|_\infty \le C\|f\|_{H_\ell^p}, \qquad |\Delta_z^m g(x)| \le C\|f\|_{\dot{H}_\ell^p}|z|^{\ell-n/p} \qquad (x, z \in \mathbb{R}^n)$$

for any integer $m > \ell - n/p$.

Proof. To prove (a), consider first the case $0 < p < n/\ell$. Write the integral in (2) as

$$\int_0^\infty \int_{B(x,t)} u(y,t)\varphi_t(x-y)\,dy\,\frac{dt}{t}.$$

By Theorem 2.3 and Remark 2.4, $\mathcal{M}_1^\ell(x,u) \in L^p$ and

$$\|\mathcal{M}_1^\ell(x,u)\|_p \approx \|f\|_{\dot{H}_\ell^p}.$$

Fix a point $y \in \mathbb{R}^n$. For any $z \in B(y,t)$, we have by definition

$$|u(y,t)| \le t^\ell \mathcal{M}_1^\ell(z,u). \tag{3}$$

Raising this inequality to the power p and then integrating over $B(y,t)$ with respect to dz, we get

$$|u(y,t)| \le Ct^{\ell-n/p}\|\mathcal{M}_1^\ell(z,u)\|_p. \tag{4}$$

It follows readily from the estimates (3) and (4) that the integral

$$\int_0^\infty \int_{B(x,t)} |u(y,t)|\,|\varphi_t(x-y)|\,dy\,\frac{dt}{t}$$

is bounded by

$$\|\varphi\|_1\left\{\mathcal{M}_1^\ell(x,u)\int_0^A t^\ell\,\frac{dt}{t} + C\|\mathcal{M}_1^\ell(z,u)\|_p\int_A^\infty t^{\ell-n/p}\,\frac{dt}{t}\right\}$$
$$\le C\{\mathcal{M}_1^\ell(x,u)A^\ell + \|\mathcal{M}_1^\ell(z,u)\|_p A^{\ell-n/p}\}$$

for any $A > 0$. Choosing A so that

$$\mathcal{M}_1^\ell(x,u)A^\ell = \|\mathcal{M}_1^\ell(z,u)\|_p A^{\ell-n/p},$$

we obtain the desired estimate.

In order to prove the remaining case $p = n/\ell$ and (b), assume $f \in H_\ell^p$ and put $v(y,t) = (f*\varphi_t)(y)$. Since $\widehat{\varphi}(0) \neq 0$, the non-tangential maximal function $v^*(x) = \sup_{|y-x|<t}|v(y,t)|$ is in L^p with $\|v^*\|_p \approx \|f\|_{H^p}$. Proceeding as above, we have the well known estimate

$$|v(y,t)| \le Ct^{-n/p}\|f\|_{H^p} \tag{5}$$

valid for all $y \in \mathbb{R}^n$. In the case $p = n/\ell$, we use the estimates (3) and (5) as before to obtain the stated property. In the case $p > n/\ell$, if we set

$$G(t) = \begin{cases} \|f\|_{\dot{H}_\ell^p} t^{\ell-n/p} & \text{for } 0 < t \le 1,\\ \|f\|_{H^p} t^{-n/p} & \text{for } t > 1,\end{cases}$$

then $|(f*\zeta_t*\varphi_t)(x)| \le CG(t)$ for all $x \in \mathbb{R}^n$ and $t > 0$ because of the estimates (4) and (5). Since G is integrable on $(0,\infty)$ with respect to dt/t, the integral in (2) converges absolutely for every $x \in \mathbb{R}^n$ with

$$|g(x)| \le C \int_0^\infty G(t)\,\frac{dt}{t} \le C\|f\|_{H^p_\ell} \qquad (x \in \mathbb{R}^n).$$

Moreover, as the function $f * \zeta_t * \varphi_t$ is smooth, the continuity of g follows from the dominated convergence theorem. To verify the Lipschitz estimate, note that

$$\Delta_z^m g(x) = \int_0^\infty \int_{B(x,t)} u(y,t)[\Delta_z^m \varphi_t(x-y)]\,dy\,\frac{dt}{t}.$$

Using Lemma 3.1 and the identity (1), it is plain to observe

$$\int_{\mathbb{R}^n} |\Delta_z^m \varphi_t(x-y)|\,dy \le C\min\{1, |z/t|^m\}.$$

By the estimate (4), we obtain in turn

$$\begin{aligned}|\Delta_z^m g(x)| &\le C\|f\|_{\dot{H}^p_\ell}\Big\{\int_0^{|z|} t^{\ell-n/p}\,\frac{dt}{t} + |z|^m \int_{|z|}^\infty t^{\ell-m-n/p}\,\frac{dt}{t}\Big\}\\ &\le C\|f\|_{\dot{H}^p_\ell}|z|^{\ell-n/p}. \ \blacksquare\end{aligned}$$

REMARK 3.5. The integral in (2) converges if the double limit

$$\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N (f * \zeta_t * \varphi_t)(x)\,\frac{dt}{t}$$

exists. Assume that $f \in \dot{H}^p_\ell$ for $p > 0$.

(i) Owing to the estimate (3), the limit

$$\lim_{\varepsilon\to 0} \int_\varepsilon^1 (f * \zeta_t * \varphi_t)(x)\,\frac{dt}{t}$$

exists for almost every x for any $p > 0$.

(ii) If $0 < p < n/\ell$, the estimate (4) ensures that the limit

$$\lim_{N\to\infty} \int_1^N (f * \zeta_t * \varphi_t)(x)\,\frac{dt}{t}$$

exists for every x. However, in the case $p \ge n/\ell$, it is necessary to impose an extra condition on f such as $f \in H^p$ in order to obtain the convergence of this limit.

4. Proofs of Theorems A and B

Proof of Theorem A. To prove (a), assume first $f \in \dot{H}^p_\ell$ for $0 < p < n/\ell$. For the sake of clarity, we divide our proof into several stages.

1. Take a function $\varphi \in C_c^\infty(B(0,1))$ with vanishing moments up to order N_p, the smallest integer greater than or equal to $n(1/p - \ell/n - 1)$, in the case $p \le n/(n+\ell)$ and $\sup_{t>0} |\widehat{\varphi}(t\xi)| > 0$ for $\xi \ne 0$. By Lemma 2.2, there exists a Schwartz function ζ such that $\widehat{\zeta}$ has compact support away from the origin and

$$\int_0^\infty \widehat{\varphi}(t\xi)\widehat{\zeta}(t\xi)\,\frac{dt}{t} = 1 \qquad (\xi \ne 0).$$

Let $u(y,t) = (f * \zeta_t)(y)$. By Theorem 2.3 and Remark 2.4, $\mathcal{M}_\delta^\ell(x,u) \in L^p$ for any $\delta > 0$ and $\|\mathcal{M}_\delta^\ell(x,u)\|_p \approx \|f\|_{\dot{H}_\ell^p}$. According to Lemma 3.4(a), the integral

$$g(x) = \int_0^\infty (f * \zeta_t * \varphi_t)(x)\,\frac{dt}{t} = \int_0^\infty \int_{\mathbb{R}^n} u(y,t)\varphi_t(x-y)\,dy\,\frac{dt}{t}$$

converges absolutely for almost every $x \in \mathbb{R}^n$. Upon taking Fourier transforms, it is plain to see that

$$\begin{aligned}(f,\eta) &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbb{R}^n} \left[\int_\varepsilon^N (f * \zeta_t * \varphi_t)(x)\,\frac{dt}{t}\right]\eta(x)\,dx \\ &= \int_{\mathbb{R}^n} \left[\int_0^\infty (f * \zeta_t * \varphi_t)(x)\,\frac{dt}{t}\right]\eta(x)\,dx = \int_{\mathbb{R}^n} g(x)\eta(x)\,dx\end{aligned}$$

for every Schwartz function η (see [Ca] and [GR, pp. 260–266] for details) ([2]).

2. Fix δ with $\delta \ge 6\sqrt{n}$. For each $k \in \mathbb{Z}$, put

$$\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}_\delta^\ell(x,u) > 2^k\}.$$

Since the maximal function $\mathcal{M}_\delta^\ell(x,u)$ is lower semicontinuous, the Ω_k is a decreasing sequence of open sets. We use Whitney's decomposition lemma to write $\Omega_k = \bigcup_{j\ge 1} Q_j^k$, where (Q_j^k) is a sequence of non-overlapping cubes satisfying

$$\operatorname{diam}(Q_j^k) \le \operatorname{dist}(Q_j^k, \Omega_k^{\mathrm{c}}) \le 4\operatorname{diam}(Q_j^k).$$

For any cube $Q \subset \mathbb{R}^n$ with side length $l(Q)$, let $\widehat{Q}$ denote the square tent $\widehat{Q} = Q \times (0, l(Q)]$ in the upper-half space $\mathbb{R}_+^{n+1}$. Setting

$$\widehat{\Omega}_k = \bigcup_{j\ge 1} \widehat{Q_j^k}, \quad T_j^k = \widehat{Q_j^k} - \widehat{\Omega}_{k+1},$$

we have the following decomposition:

$$g = \sum_{k\in\mathbb{Z}} \sum_{j\ge 1} a_j^k, \qquad a_j^k(x) = \iint_{T_j^k} u(y,t)\varphi_t(x-y)\,dy\,\frac{dt}{t}. \tag{6}$$

([2]) One cannot simply claim $f = g$ unless g itself defines a tempered distribution.

3. Let us investigate the properties of a_j^k:

(i) *It has support in* $\widetilde{Q_j^k} = 3Q_j^k$, *the concentric triple of* Q_j^k.

(ii) *It is bounded and continuous with* $\|a_j^k\|_\infty \le A_\ell 2^k |\widetilde{Q_j^k}|^{\ell/n}$.

As to the boundedness, we claim that

$$|u(y,t)| \le 2^{k+1} t^\ell \quad \text{for each } (y,t) \in T_j^k. \tag{7}$$

To see this, take any $(y,t) \in T_j^k$. If $y \in Q_j^k - \Omega_{k+1}$, then it is obvious. If $y \in Q_j^k \cap \Omega_{k+1}$, then $y \in Q_m^{k+1}$ for some m. By the definition of T_j^k, we have $t > l(Q_m^{k+1})$. Since $\operatorname{dist}(Q_m^{k+1}, \Omega_{k+1}^{\mathrm{c}}) \le 4 \operatorname{diam}(Q_m^{k+1})$, we can find a point $z \in \Omega_{k+1}^{\mathrm{c}}$ satisfying $\operatorname{dist}(z, Q_m^{k+1}) \le 5 \operatorname{diam}(Q_m^{k+1})$. Hence

$$|y - z| \le 6 \operatorname{diam}(Q_m^{k+1}) < 6\sqrt{n}\, t \le \delta t.$$

It follows that

$$|u(y,t)| \le t^\ell \mathcal{M}_\delta^\ell(z,u) \le 2^{k+1} t^\ell,$$

proving the claim. Using the estimate (7), we have

$$\begin{aligned} |a_j^k(x)| &\le 2^{k+1} \iint_{T_j^k} |\varphi_t(x-y)| t^\ell \, dy \, \frac{dt}{t} \\ &\le 2^{k+1} \|\varphi\|_1 \int_0^{l(Q_j^k)} t^\ell \, \frac{dt}{t} = A_\ell 2^k |\widetilde{Q_j^k}|^{\ell/n}. \end{aligned}$$

The continuity of a_j^k follows plainly from the dominated convergence theorem on account of the estimate (7).

(iii) *For all* $x, z \in \mathbb{R}^n$, *it satisfies* $|(\Delta_z^{\ell+1} a_j^k)(x)| \le B_\ell 2^k |z|^\ell$.

To verify this, use the estimate (7) to observe

$$|(\Delta_z^{\ell+1} a_j^k)(x)| \le 2^{k+1} \iint_{T_j^k} |\Delta_z^{\ell+1} \varphi_t(x-y)| t^\ell \, dy \, \frac{dt}{t}.$$

An application of Lemma 3.1 and the identity (1) yield the estimate

$$\int_{\mathbb{R}^n} |\Delta_z^{\ell+1} \varphi_t(x-y)| dy \le C \min\{1, |z/t|^{\ell+1}\}.$$

Consequently, we have

$$|(\Delta_z^{\ell+1} a_j^k)(x)| \le C 2^{k+1} \left(\int_0^{|z|} t^\ell \, \frac{dt}{t} + |z|^{\ell+1} \int_{|z|}^\infty t^{-1} \, \frac{dt}{t} \right),$$

which gives the desired inequality upon integrating.

(iv) *It has vanishing moments up to order* N_p *when* $p \le n/(n+\ell)$.

4. To finish our proof, we let $C_\ell = \max(A_\ell, B_\ell)$ and

$$\mu_j^k = C_\ell 2^k |\widetilde{Q_j^k}|^{1/p}, \qquad b_j^k(x) = \frac{a_j^k(x)}{C_\ell 2^k |\widetilde{Q_j^k}|^{1/p}}$$

so that $g = \sum \mu_j^k b_j^k$. Because of properties (i) to (iv) in the preceding stage, each b_j^k is an atom for $\dot{H}_\ell^p$. Enumerating suitably, we now have the desired atomic decomposition of f.

It remains to prove the size estimate for $\sum |\mu_j^k|^p$. Let h be any nonnegative measurable function on $\mathbb{R}^n$. For $p > 0$, observe

$$\int_{\mathbb{R}^n} [h(x)]^p\, dx = p \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} s^{p-1} |\{h > s\}|\, ds \ge (2^p - 1) 2^{-p} \sum_{k\in\mathbb{Z}} 2^{kp} |\{h > 2^k\}|.$$

Rewriting, we have

$$\sum_{k\in\mathbb{Z}} 2^{kp} |\{h > 2^k\}| \le \frac{2^p}{2^p - 1} \|h\|_p^p. \tag{8}$$

Exploiting this inequality, we proceed to estimate

$$\begin{aligned}
\sum_{k,j} |\mu_j^k|^p &= C \sum_{k\in\mathbb{Z}} 2^{kp} \Big(\sum_{j=1}^{\infty} |Q_j^k| \Big) = C \sum_{k\in\mathbb{Z}} 2^{kp} |\Omega_k| \\
&= C \sum_{k\in\mathbb{Z}} 2^{kp} |\{\mathcal{M}_\delta^\ell(x,u) > 2^k\}| \\
&\le C \|\mathcal{M}_\delta^\ell(x,u)\|_p^p \approx C \|f\|_{\dot{H}_\ell^p}^p.
\end{aligned}$$

The proof of (a) in the case $0 < p < n/\ell$ is now complete.

As for the proof of the case $p = n/\ell$, we take $\varphi \in C_c^\infty(B(0,1))$ with $\widehat{\varphi}(0) \ne 0$ in the first stage. Then Lemma 3.4 shows that the integral

$$g(x) = \int_0^\infty (f * \zeta_t * \varphi_t)(x)\, \frac{dt}{t}$$

converges absolutely for almost every $x \in \mathbb{R}^n$ and the rest of the proof is the same as above.

Regarding (b), let $0 < p \le \min(n/\ell, 1)$ and suppose that $f = \sum \mu_k b_k$ where (μ_k) is a sequence of scalars with $\sum |\mu_k|^p < \infty$ and (b_k) is a sequence of $\dot{H}_\ell^p$ atoms. Choose any $\psi \in \mathcal{O}_\ell$ and let

$$u(y,t) = (f * \psi_t)(y), \qquad u_k(y,t) = (b_k * \psi_t)(y).$$

For any $\delta > 0$, we have $\mathcal{M}^\ell_\delta(x,u) \le \sum |\mu_k| \mathcal{M}^\ell_\delta(x,u_k)$. Since $p \le 1$,

$$\int_{\mathbb{R}^n} [\mathcal{M}^\ell_\delta(x,u)]^p\,dx \le \sum |\mu_k|^p \int_{\mathbb{R}^n} [\mathcal{M}^\ell_\delta(x,u_k)]^p\,dx$$
$$\le C \sum |\mu_k|^p \|b_k\|^p_{\dot{H}^p_\ell} \le C \sum |\mu_k|^p$$

where the last inequality results from Lemma 3.3. Thus the desired conclusion is a consequence of Theorem 2.3 once we verify the admissibility condition that either $\widehat{f}$ coincides with a locally integrable function away from the origin or $f \in L^q$ for some $1 \le q \le \infty$.

First, consider the case $p \le n/(n+\ell)$. By definition, each $\dot{H}^p_\ell$ atom b is an H^q atom with $1/q = 1/p - \ell/n$. It follows that

$$\|f\|^q_{H^q} \le \sum |\mu_k|^q \|b_k\|^q_{H^q} \le C \sum |\mu_k|^q \le C\Big(\sum |\mu_k|^p\Big)^{q/p}$$

since $p < q \le 1$. Thus $f \in H^q$ and f satisfies the admissibility condition in view of Remark 2.4. Next consider the case $n/(n+\ell) < p < n/\ell$ and $p \le 1$. With q determined by $1/q = 1/p - \ell/n < 1$, we observe that each $\dot{H}^p_\ell$ atom b is an L^q function with $\|b\|_q \le 1$. Therefore $f \in L^q$ because

$$\|f\|_q \le \sum |\mu_k|\,\|b_k\|_q \le \sum |\mu_k| \le \Big(\sum |\mu_k|^p\Big)^{1/p}.$$

In the last case $p = n/\ell \le 1$, each $\dot{H}^p_\ell$ atom b satisfies $\|b\|_\infty \le 1$. Thus

$$\|f\|_\infty \le \sum |\mu_k| \le \Big(\sum |\mu_k|^p\Big)^{1/p}$$

so the admissibility condition on f is satisfied. ∎

Proof of Theorem B. Suppose that $f \in H^p_\ell$ for $n/\ell < p < \infty$. Choose $\varphi \in C^\infty_{\mathrm{c}}(B(0,1))$ satisfying $\widehat{\varphi}(0) \ne 0$ and let ζ be as in Lemma 2.2. According to Lemma 3.4(b), the integral

$$g(x) = \int_0^\infty (f * \zeta_t * \varphi_t)(x)\,\frac{dt}{t}$$

converges absolutely for every $x \in \mathbb{R}^n$ and represents a bounded continuous function. Hence g defines a tempered distribution on $\mathbb{R}^n$ and $(f,\eta) = (g,\eta)$ for each Schwartz function η. Therefore f is actually a function coinciding with g. The proof concerning the atomic decomposition of f is the same as that of Theorem A. The Lipschitz property of f is contained in Lemma 3.4. ∎

5. Alternatives of atoms and other results. Inspecting the proof of Theorem A, it is simple to observe that atoms may have additional properties or may be definable in terms of different conditions. In this section we list three noteworthy alternatives of atoms.

(A1) *The Lipschitz condition* (ii) *of Definition* 1.1 *may be replaced by*

$$|\varDelta_z^m b(x)| \le |Q|^{-1/p}|z|^\ell \qquad (x, z \in \mathbb{R}^n)$$

for any given integer $m > \ell$.

This follows from the estimate $|(\varDelta_z^m a_j^k)(x)| \le B_m 2^k |z|^\ell$ instead of estimate (iii) in stage 3 of the proof of Theorem A by using

$$\int_{\mathbb{R}^n} |\varDelta_z^m \varphi_t(x-y)|\, dy \le C \min\{1, |z/t|^m\}.$$

The necessary modifications in the proof of Lemma 3.3 are trivial.

(A2) *An atom for* $\dot{H}_\ell^p$ *may be defined as a function* $b \in C^{\ell-1}(\mathbb{R}^n)$ *with properties* (i), (iii) *of Definition* 1.1 *and*

(ii)′ $\quad \|\partial^\sigma b\|_\infty \le |Q|^{-1/p+(\ell-|\sigma|)/n}, \quad |\varDelta_z^m[\partial^\sigma b](x)| \le |Q|^{-1/p}|z|^{\ell-|\sigma|}$

for all $x, z \in \mathbb{R}^n$ *and all* σ *with* $|\sigma| \le \ell - 1$, *where* $m + |\sigma| > \ell$.

This alternative can be obtained from the properties of a_j^k in the proof of Theorem A with easy modifications.

Given a positive integer m and a point $z \in \mathbb{R}^n$, consider the mth symmetric difference operator S_z^m defined inductively as

$$S_z f(x) = f(x+z) - 2f(x) + f(x-z), \qquad S_z^m f(x) = S_z[S_z^{m-1} f](x).$$

In view of the identity

$$S_z^m f(x) = [\varDelta_z^{2m} f](x - my) = \sum_{j=0}^{2m} (-1)^{2m-j} \binom{2m}{j} f(x + (j-m)z),$$

we have another alternative of atoms in the following form :

(A3) *An atom for* $\dot{H}_\ell^p$ *may be defined as a function* $b \in \varLambda_\ell$ *with properties* (i), (iii) *of Definition* 1.1 *and*

(ii)″ $\qquad \|b\|_\infty \le |Q|^{-1/p+\ell/n}, \quad |S_z^m b(x)| \le |Q|^{-1/p}|z|^\ell$

for all $x, z \in \mathbb{R}^n$, *where* m *is a given integer with* $m > \ell/2$.

As mentioned earlier, there are other atomic decompositions and it would be interesting to compare those results with ours. We shall take into account the following two results only.

Result of Strichartz. For $\alpha > 0$, let $I_\alpha(H^p)$ be the image of H^p under the Riesz potential operator I_α. As $I_\ell(H^p) = \dot{H}_\ell^p$ for each integer ℓ, these spaces give an extension of the homogeneous Hardy–Sobolev spaces to the case of fractional order. In [Sz, Theorem 5.2], Strichartz proved that $f \in I_\alpha(H^p)$ for $n/(n+\alpha) < p \le 1$ if and only if there exist a sequence (λ_k) of scalars and a sequence (b_k) of $I_\alpha(H^p)$ atoms such that

$$f = \sum \lambda_k b_k \quad \text{with} \quad \Big(\sum |\lambda_k|^p\Big)^{1/p} \approx \|f\|_{I_\alpha(H^p)}.$$

Here an $I_\alpha(H^p)$ *atom* means a function $b \in I_\alpha(L^q)$ for some $q \geq 2$ with support in a cube Q and $\|b\|_{I_\alpha(L^q)} \leq |Q|^{-1/p}$. In comparison with our result, the atoms in Strichartz's result seem to be more complicated while the decomposition forms are the same.

Result of Frazier and Jawerth. Denoting by $\dot{F}^\alpha_{p,q}$ the homogeneous Triebel–Lizorkin spaces, it is well known that $\dot{F}^\ell_{p,2} \simeq \dot{H}^p_\ell$. In [FJ, Theorem 7.4], Frazier and Jawerth obtained atomic decomposition theorems for general $\dot{F}^\alpha_{p,q}$ spaces. Let $\mathcal{D}$ denote the set of all dyadic cubes in $\mathbb{R}^n$. Given $Q \in \mathcal{D}$ and an integer N with $N \geq [n(1/p - \alpha/n - 1)]$, a function $a_Q \in C^\infty_{\mathrm{c}}(3Q)$ is called a *smooth N-atom* if it has vanishing moments up to order N and

$$\|\partial^\sigma a_Q\|_\infty \leq |Q|^{-|\sigma|/n-1/2} \quad \text{for all multi-indices } \sigma \in \mathbb{Z}^n_+.$$

Let us call A an $\dot{F}^\alpha_{p,q}$ *atom* if it can be decomposed as $A = \sum_{Q\in\mathcal{D}} r_Q a_Q$ where (a_Q) is a sequence of smooth N-atoms and (r_Q) is a sequence of scalars such that there exists $Q_0 \in \mathcal{D}$ with $r_Q = 0$ if Q is not contained in Q_0 and

$$\Big\|\Big[\sum_{Q\in\mathcal{D}} (|Q|^{-\ell/n-1/2}|r_Q|\chi_Q)^q\Big]^{1/q}\Big\|_\infty \leq |Q_0|^{-1/p}.$$

In a simplified version, their results state that a tempered distribution f on $\mathbb{R}^n$ belongs to $\dot{F}^\alpha_{p,q}$ for $0 < p \leq 1$, $p \leq q < \infty$ if and only if there exist a sequence (λ_k) of scalars and a sequence (A_k) of $\dot{F}^\alpha_{p,q}$ atoms such that

$$f = \sum \lambda_k A_k \quad \text{with} \quad \Big(\sum |\lambda_k|^p\Big)^{1/p} \approx \|f\|_{\dot{F}^\alpha_{p,q}}.$$

(See also [G, pp. 487–496] for more detailed expositions.) Comparing with Theorem A, it seems apparent that our atoms have more explicit smoothness properties and simpler forms than those of Frazier and Jawerth, which perhaps results from different characterizations.

6. Sobolev embedding theorem. One of the principal issues of interest in the theory of Sobolev spaces is the various inclusion inequalities known as the Sobolev embedding theorem. As our atomic decomposition results deal with the basic structure of H^p Sobolev spaces, they are intimately connected with the Sobolev embedding theorem.

The purpose of this section is to obtain several embedding results as well as some interesting weighted inequalities for H^p Sobolev spaces by applying our atomic decomposition theorems and Lemma 3.4. For $f \in \Lambda_\alpha$, we put

$$\|f\|_{\Lambda_\alpha} = \|f\|_\infty + \inf\{C > 0 : |\Delta^{[\alpha]+1}_z f(x)| \leq C|z|^\alpha \text{ for all } x, z \in \mathbb{R}^n\}.$$

THEOREM 6.1. *Let ℓ be a positive integer.*

(a) *If $f \in \dot{H}^p_\ell$ for $0 < p < n/\ell$, then $f \in H^q$ for $1/q = 1/p - \ell/n$ with*

$$\|f\|_{H^q} \le C_{p,\ell}\|f\|_{\dot{H}^p_\ell}.$$

(b) *If $f \in H^p_\ell$ for $p = n/\ell$ and $p \ge 1/2$, then $f \in L^{2p}$ with*

$$\|f\|_{2p} \le C_p\|f\|_{H^p}^{1/2}\|f\|_{\dot{H}^p_\ell}^{1/2}. \tag{9}$$

If in addition $p \le 1$, then $f \in L^\infty$ with $\|f\|_\infty \le C_p\|f\|_{\dot{H}^p_\ell}$.

(c) *If $f \in H^p_\ell$ for $p > n/\ell$, then $f \in \Lambda_{\ell-n/p}$ with*

$$\|f\|_{\Lambda_{\ell-n/p}} \le C_{p,\ell}\|f\|_{H^p_\ell}.$$

Proof. To prove (a), suppose that $f \in \dot{H}^p_\ell$ for $p < n/\ell$. Consider first the case $p \ge n/(n+\ell)$, that is, $q \ge 1$. With ζ and φ as in the proof of Theorem A and Lemma 3.4, we have

$$(f,\eta) = \int_{\mathbb{R}^n} g(x)\eta(x)\,dx, \quad g(x) = \int_0^\infty (f * \zeta_t * \varphi_t)(x)\,\frac{dt}{t}$$

for every Schwartz function η. According to Lemma 3.4,

$$|g(x)| \le C\|\mathcal{M}_1^\ell(x,u)\|_p^{p\ell/n}[\mathcal{M}_1^\ell(x,u)]^{p(1/p-\ell/n)},$$

which shows that $g \in L^q$ with $\|g\|_q \le C\|f\|_{\dot{H}^p_\ell}$. Consequently, g defines a tempered distribution on $\mathbb{R}^n$ and f coincides with g.

In the case $0 < p \le n/(n+\ell)$, we have $q \le 1$ and each atom for $\dot{H}^p_\ell$ is an H^q atom. Writing $f = \sum \mu_k b_k$ in accordance with Theorem A, we have

$$\|f\|_{H^q}^q \le \sum |\mu_k|^q\|b_k\|_{H^q}^q \le C\sum|\mu_k|^q \le C\Big(\sum|\mu_k|^p\Big)^{q/p}$$

because $0 < p < q \le 1$. This completes the proof of (a).

To prove (b), suppose that $f \in H^p_\ell$ for $p = n/\ell$. Under the same setting as above, Lemma 3.4 states that

$$|g(x)| \le C\|f\|_{H^p}^{1/2}[\mathcal{M}_1^\ell(x,u)]^{1/2}.$$

When $2p \ge 1$, the right side represents an L^{2p} function so that g defines a tempered distribution and $f = g$ satisfies the inequality (9). In the case $1/2 \le p \le 1$, it is straightforward to verify $f \in L^\infty$ once we decompose f in terms of atoms and notice $\|b\|_\infty \le 1$ for each atom b.

Finally, assertion (c) is part of Theorem B. ∎

REMARK 6.2. A special case $\ell = 1$ and $p = n = 2$ in (9) leads to

$$\|f\|_4 \le C\|f\|_2^{1/2}\|\nabla f\|_2^{1/2} \quad (f \in W^{1,2}(\mathbb{R}^2)),$$

known as Ladyzhenskaya's inequality (see [SS]). The space $\dot{H}^p_\ell$ for $p = n/\ell$ is known to be continuously embedded in the space BMO (see [K] and [Ch1]).

THEOREM 6.3. *Let $n/(n+\ell) \le p \le 1$ and $p < n/\ell$. Suppose that ω is a nonnegative measurable function on $\mathbb{R}^n$ satisfying*

$$\|\omega\|_* = \sup_Q |Q|^{-1/p+\ell/n} \int_Q \omega(x)\,dx < \infty,$$

where the supremum is taken over all finite cubes Q in $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} |f(x)|\omega(x)\,dx \le C_{p,\ell}\|\omega\|_*\|f\|_{\dot{H}^p_\ell}, \tag{10}$$

$$\int_{\mathbb{R}^n} |f(x)|^p|x|^{-\ell p}\,dx \le C_{p,\ell}\|f\|^p_{\dot{H}^p_\ell}. \tag{11}$$

Proof. By Theorem 6.1(a), note that f is actually an L^q function with $1/q = 1/p - \ell/n$ and so both inequalities make sense. Upon decomposing f into atoms as in Theorem A, it is trivial to verify both inequalities. ∎

REMARK 6.4. An example for (10) is given by $\omega(x) = |x|^{n(1/p-\ell/n-1)}$. The inequality (11) is motivated by Hardy's inequality ([SS]) which states originally that if $f \in W^{1,p}$ for $1 \le p < n$, then

$$\|f/|x|\|_p \le C\|f\|_{W^{1,p}}.$$

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