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On the Heyde theorem for discrete Abelian groups

by

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Abstract. Let X be a countable discrete Abelian group, $\mathrm{Aut}(X)$ the set of automorphisms of X, and $I(X)$ the set of idempotent distributions on X. Assume that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{Aut}(X)$ satisfy $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2. We prove that the symmetry of the conditional distribution of $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ implies that $\mu_1, \mu_2 \in I(X)$ if and only if the group X contains no elements of order two. This theorem can be considered as an analogue for discrete Abelian groups of the well-known Heyde theorem where the Gaussian distribution on the real line is characterized by the symmetry of the conditional distribution of one linear form given another.

1. Introduction. The well-known Skitovich–Darmois theorem asserts that a Gaussian distribution on the real line is characterized by the independence of two linear forms of independent random variables. A similar result of Heyde characterizes a Gaussian distribution by the symmetry of the conditional distribution of one linear form given another.

Theorem A (C. C. Heyde [6], see also [7, §13.4]). *Let $\xi_1, \ldots, \xi_n$, $n \geq 2$, be independent random variables, and let α_j, β_j be nonzero constants such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \neq 0$ for all $i \neq j$. If the conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then all random variables ξ_j are Gaussian.*

Let X be a locally compact Abelian separable metric group. Denote by $Y = X^*$ the character group of X. Let (x, y) be the value of a character $y \in Y$ on an element $x \in X$. Denote by $M^1(X)$ the convolution semigroup of probability distributions on X. For $\mu \in M^1(X)$ denote by $\widehat{\mu}$ its characteristic function,

$$\widehat{\mu}(y) = \int_X (x, y)\, d\mu(x).$$

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A distribution $\mu \in M^1(X)$ is called *Gaussian* ([9, Ch. IV]) if its characteristic function can be represented in the form

$$\widehat{\mu}(y) = (x, y) \exp\{-\varphi(y)\}, \qquad y \in Y,$$

where $x \in X$ and φ is a continuous nonnegative function on Y satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y.$$

Let $\mathrm{Aut}(X)$ be the set of topological automorphisms of X, and let $\xi_1, \ldots, \xi_n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j. Consider the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$, where $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ satisfy the condition $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Let us formulate the following general problem.

PROBLEM 1. Describe locally compact Abelian separable metric groups X for which the symmetry of the conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ implies that all distributions μ_j are either Gaussian or belong to a class of distributions that can be considered as a natural analogue of the class of Gaussian distributions.

Problem 1 was solved in the class of finite Abelian groups ([3]) and in the class of all locally compact Abelian separable metric groups under the additional assumption that the characteristic functions of the distributions μ_j do not vanish ([4]). The aim of the article is to give the solution of Problem 1 in the class of countable discrete Abelian groups. We will also study some similar problems.

We shall first fix some notation. If G is a subgroup of X, then denote by $A(Y, G) = \{y \in Y : (x, y) = 1 \text{ for all } x \in G\}$ its annihilator. For $\alpha \in \mathrm{Aut}(X)$ we define the conjugate automorphism $\widetilde{\alpha} \in \mathrm{Aut}(Y)$ by the formula $(x, \widetilde{\alpha}y) = (\alpha x, y)$ for all $x \in X$, $y \in Y$. Denote by I the identity automorphism of a group. Let $f_2 : X \to X$ be the homomorphism $f_2x = 2x$ and put $X_{(2)} = \mathrm{Ker}\, f_2$, $X^{(2)} = \mathrm{Im}\, f_2$. Denote by $\mathbb{T}$ the circle group (the one-dimensional torus) and by $\mathbb{Z}$ the group of integers. If A and B are subsets of Y, denote by $A + B = \{y \in Y : y = u + v,\ u \in A,\ v \in B\}$ their arithmetic sum. Let ψ be an arbitrary function on Y and let $h \in Y$. Denote by $\varDelta_h$ the finite difference operator

$$\varDelta_h\psi(y) = \psi(y + h) - \psi(y), \qquad y \in Y.$$

A continuous function ψ on Y is called a *polynomial* if for some nonnegative integer m,

$$\varDelta_h^{m+1}\psi(y) = 0 \qquad \text{for all } y, h \in Y.$$

If ξ is a random variable with values in X and with distribution μ, then $\widehat{\mu}(y) = \mathbf{E}[(\xi, y)]$. For $\mu \in M^1(X)$ we define $\overline{\mu} \in M^1(X)$ by $\overline{\mu}(E) = \mu(-E)$ for all Borel sets $E \subset X$. Observe that $\widehat{\overline{\mu}}(y) = \overline{\widehat{\mu}(y)}$. Denote by E_x the degenerate distribution concentrated at a point $x \in X$, and by $\sigma(\mu)$ the support of $\mu \in M^1(X)$. Let $I(X)$ be the set of idempotent distributions on X, i.e. the set of shifts of the Haar distributions m_K of compact subgroups K of X. Note that

$$\widehat{m}_K(y) = \begin{cases} 1, & y \in A(Y,K), \\ 0, & y \notin A(Y,K). \end{cases}$$

Observe that the Gaussian distributions on a discrete Abelian group X are degenerate, and the class $I(X)$ can be regarded as a natural analogue of the class of Gaussian distributions for discrete Abelian groups. We remark that if H is a closed subgroup of Y and $\widehat{\mu}(y) = 1$ for $y \in H$, then $\widehat{\mu}$ is H-invariant, i.e. $\widehat{\mu}(y+h) = \widehat{\mu}(y)$ for all $y \in Y$, $h \in H$, and $\sigma(\mu) \subset A(X,H)$. We will use the well-known facts concerning the structure of locally compact Abelian groups and the duality theory (see [5]). We now formulate the main result of the article.

THEOREM 1. *Let X be a countable discrete Abelian group. Assume that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{Aut}(X)$ satisfy $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2. The symmetry of the conditional distribution of $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ implies that $\mu_1, \mu_2 \in I(X)$ if and only if $X_{(2)} = \{0\}$, i.e. the group X contains no elements of order two.*

First we study the case when X is a discrete torsion-free Abelian group.

2. The Heyde theorem for discrete torsion-free Abelian groups. We will prove the group analogue of the Heyde theorem for discrete torsion-free Abelian groups and use this result to prove Theorem 1. We need some lemmas.

LEMMA 1 ([4]). *Let X be a locally compact Abelian separable metric group. Let $\xi_1, \ldots, \xi_n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j. Assume that $\alpha_j, \beta_j \in \mathrm{Aut}(X)$. The conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric if and only if*

$$\prod_{j=1}^{n} \widehat{\mu}_j(\widetilde{\alpha}_j u + \widetilde{\beta}_j v) = \prod_{j=1}^{n} \widehat{\mu}_j(\widetilde{\alpha}_j u - \widetilde{\beta}_j v), \qquad u, v \in Y. \tag{1}$$

LEMMA 2 ([1], see also [2, Appendix 1]). *Let Y be a compact Abelian group and $\psi(y)$ be a polynomial on Y. Then $\psi(y) = \mathrm{const}$.*

LEMMA 3 ([8, Ch. 6, §1]). *Let $F(t)$, $t \in \mathbb{R}^k$, be a characteristic function, and let $\Phi(t)$, $t \in \mathbb{R}^k$, be the restriction to $\mathbb{R}^k$ of an entire function $\Phi(z)$, $z \in \mathbb{C}^k$. Assume that*

$$F(t) = \Phi(t), \qquad t \in U, \tag{2}$$

where U is a neighbourhood of zero in $\mathbb{R}^k$. Then $F(t)$ can be extended onto $\mathbb{C}^k$ as an entire function and (2) holds for all $\mathbb{R}^k$.

We can now prove the main result of this section.

PROPOSITION 1. *Let X be a countable discrete torsion-free Abelian group. Assume that $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ satisfy $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Let $\xi_1, \dots, \xi_n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j. If the conditional distribution of $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ is symmetric, then all μ_j are degenerate distributions.*

Proof. By Lemma 1, the symmetry of the conditional distribution of L_2 given L_1 implies that the characteristic functions $\widehat{\mu}_j$ satisfy (1). We note that $Y = X^*$ is a connected compact Abelian group. Passing to the random variables $\xi'_j = \alpha_j\xi_j$ we can assume without loss of generality that $L_1 = \xi_1 + \dots + \xi_n$ and $L_2 = \delta_1\xi_1 + \dots + \delta_n\xi_n$, where the automorphisms $\delta_j \in \mathrm{Aut}(X)$ satisfy $\delta_i \pm \delta_j \in \mathrm{Aut}(X)$ for all $i \neq j$. Then equation (1) is transformed into

$$\prod_{j=1}^{n} \widehat{\mu}_j(u + \widetilde{\delta}_j v) = \prod_{j=1}^{n} \widehat{\mu}_j(u - \widetilde{\delta}_j v), \qquad u, v \in Y, \tag{3}$$

where $\widetilde{\delta}_i \pm \widetilde{\delta}_j \in \mathrm{Aut}(Y)$ for all $i \neq j$. It is clear that the characteristic functions of the distributions $\nu_j = \mu_j * \overline{\mu}_j$ also satisfy (3). Set $f_j(y) = \widehat{\nu}_j(y)$, $\varepsilon_j = \widetilde{\delta}_j$ and rewrite equation (3) using the new notation:

$$\prod_{j=1}^{n} f_j(u + \varepsilon_j v) = \prod_{j=1}^{n} f_j(u - \varepsilon_j v), \qquad u, v \in Y.$$

We will prove that $f_j(y) = 1$ for all $y \in Y$ and j. It is obvious that $f_j(y) = |\widehat{\mu}_j(y)|^2 \geq 0$. Choose a neighbourhood U of zero in Y such that $f_j(y) > 0$ for all $y \in U$ and j. Set $\psi_j(y) = -\ln f_j(y)$, $y \in U$. Take a symmetric neighbourhood U_1 of zero in Y such that $U_1 + \varepsilon_j(U_1) \subset U$, $j = 1, \dots, n$. The functions ψ_j satisfy

$$\sum_{j=1}^{n} \psi_j(u + \varepsilon_j v) = \sum_{j=1}^{n} \psi_j(u - \varepsilon_j v), \qquad u, v \in U_1.$$

In order to solve this equation we apply the finite difference method. We restrict ourselves to the case $n = 2$. Let V be a symmetric neighbourhood of zero in Y such that

$$\sum_{j=1}^{8} \lambda_j(V) \subset U$$

for any $\lambda_j \in \{I, \varepsilon_1, \varepsilon_2\}$. Then

$$\psi_1(u + \varepsilon_1 v) + \psi_2(u + \varepsilon_2 v) - \psi_1(u - \varepsilon_1 v) - \psi_2(u - \varepsilon_2 v) = 0, \quad u, v \in V. \tag{4}$$

Let $k_1 \in V$. Put $h_1 = \varepsilon_2 k_1$ and hence $h_1 - \varepsilon_2 k_1 = 0$. Give u and v in (4) the increments h_1 and k_1 respectively. Subtracting (4) from the resulting equation we find

$$(5) \qquad \Delta_{l_{11}}\psi_1(u+\varepsilon_1 v) + \Delta_{l_{12}}\psi_2(u+\varepsilon_2 v) - \Delta_{l_{13}}\psi_1(u-\varepsilon_1 v) = 0, \quad u,v \in V,$$

where $l_{11} = (\varepsilon_2 + \varepsilon_1)k_1$, $l_{12} = 2\varepsilon_2 k_1$, $l_{13} = (\varepsilon_2 - \varepsilon_1)k_1$. Let $k_2 \in V$. Put $h_2 = \varepsilon_1 k_2$ and hence $h_2 - \varepsilon_1 k_2 = 0$. Give u and v in (5) the increments h_2 and k_2 respectively. Subtracting (5) from the resulting equation we arrive at

$$(6) \qquad \Delta_{l_{21}}\Delta_{l_{11}}\psi_1(u+\varepsilon_1 v) + \Delta_{l_{22}}\Delta_{l_{12}}\psi_2(u+\varepsilon_2 v) = 0, \quad u,v \in V,$$

where $l_{21} = 2\varepsilon_1 k_2,\ l_{22} = (\varepsilon_1 + \varepsilon_2)k_2$. Let $k_3 \in V$. Put $h_3 = -\varepsilon_2 k_3$ and hence $h_3 + \varepsilon_2 k_3 = 0$. Give u and v in (6) the increments h_3 and k_3 respectively. Subtracting (6) from the resulting equation we find

$$(7) \qquad \Delta_{l_{31}}\Delta_{l_{21}}\Delta_{l_{11}}\psi_1(u+\varepsilon_1 v) = 0, \quad u,v \in V,$$

where $l_{31} = (\varepsilon_1 - \varepsilon_2)k_3$. Substituting $v = 0$ in (7) we infer that

$$(8) \qquad \Delta_{l_{31}}\Delta_{l_{21}}\Delta_{l_{11}}\psi_1(u) = 0, \quad u \in V.$$

Since Y is a connected Abelian group, we have $Y^{(2)} = Y$. Hence, $f_2 : Y \to Y$ is an open homomorphism. The condition $\varepsilon_1 \pm \varepsilon_2 \in \mathrm{Aut}(Y)$, the expressions for l_{11}, l_{21}, l_{31} and equation (8) imply that there is a neighbourhood W of zero in Y such that

$$(9) \qquad \Delta_h^3\psi_1(y) = 0, \quad h,y \in W.$$

Since Y is a connected compact Abelian group, this implies that there exists a compact subgroup $H \subset W$ such that $Y/H \approx \mathbb{T}^k$ ([5, §24.7]). Consider the restriction of equation (9) to H. As by Lemma 2, all polynomials on a compact Abelian group are constants and $\psi_1(0) = 0$, we have $\psi_1(y) = 0$, $y \in H$. Hence, $f_1(y) = 1$, $y \in H$. It follows that $f_1(y+h) = f_1(y)$, $y \in Y$, $h \in H$. Let $p_1 : Y \to Y/H$ be the natural homomorphism, and $p_2 : Y/H \to \mathbb{T}^k$ be the above mentioned isomorphism. Consider the composition $p = p_2 p_1 : Y \to \mathbb{T}^k$. Since p is an open homomorphism, $p(W)$ is a neighbourhood of zero in $\mathbb{T}^k$. Denote elements of $\mathbb{T}^k$ by $t = (t_1, \ldots, t_k)$, where $-\pi \le t_j < \pi$. The group operation in $\mathbb{T}^k$ is coordinatewise addition modulo 2π. The function f_1 induces a positive definite function $\widetilde{f}_1$ on $\mathbb{T}^k$ by the formula $\widetilde{f}_1(t) = f_1(y)$, $t = py$. By the Bochner theorem, there is a distribution $\lambda_1 \in M^1(\mathbb{Z}^k)$ such that $\widehat{\lambda}_1(t) = \widetilde{f}_1(t)$, $t \in \mathbb{T}^k$. Moreover it follows from (9) that in the neighbourhood $p(W)$ of zero in $\mathbb{T}^k$ we have the representation

$$(10) \qquad \widetilde{f}_1(t) = e^{-\widetilde{\psi}_1(t)}, \quad t \in p(W),$$

where $\widetilde{\psi}_1(t) = \psi_1(y)$, $t = py$. It is clear that $\widetilde{\psi}_1(t)$ is an ordinary polynomial of k variables. Since $\mathbb{Z}^k \subset \mathbb{R}^k$, we can consider λ_1 as a distribution on $\mathbb{R}^k$

with support in $\mathbb{Z}^k$, i.e. we can assume that the function $\widetilde{f}_1(t)$ is defined on $\mathbb{R}^k$ and is 2π-periodic in each variable. Note that the right hand side of (10) can be extended to $\mathbb{C}^k$ as an entire function. By Lemma 3, the same holds for the left-hand side of (10), and (10) holds for any $t \in \mathbb{R}^k$. Since the polynomial $\widetilde{\psi}_1(t)$ is 2π-periodic in each variable, we infer that $\widetilde{\psi}_1(t) = 0$, $t \in \mathbb{R}^k$. This implies that $\widetilde{f}_1(t) = 1$, $t \in \mathbb{R}^k$, and hence $f_1(y) = 1$, $y \in Y$. We proved that ν_1 is a degenerate distribution, so that the same is true for μ_1. Reasoning similarly we prove that μ_2 is also a degenerate distribution. The proof for arbitrary n uses the same scheme. Proposition 1 is proved.

It is well known that any locally compact Abelian group is topologically isomorphic to a group of the form $\mathbb{R}^m \times G$, where $m \geq 0$ and G contains a compact open subgroup. Proposition 1 implies the following statement.

COROLLARY 1. *Assume that a locally compact Abelian separable metric group X is of the form $X = \mathbb{R}^m \times G$, where $m \geq 0$ and the group G contains a compact open subgroup. Assume that $\alpha_j, \beta_j \in \mathrm{Aut}(X)$. Let $\xi_1, \dots, \xi_n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j. If the conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then for some shifts ξ_j' of the random variables ξ_j the conditional distribution of $L_2' = \beta_1\xi_1' + \cdots + \beta_n\xi_n'$ given $L_1' = \alpha_1\xi_1' + \cdots + \alpha_n\xi_n'$ is symmetric and $\sigma(\mu_j') \subset \mathbb{R}^m \times G_0$ for all j, where μ_j' is the distribution of ξ_j' and the subgroup G_0 consists of all compact elements of G.*

Proof. Put $Y = X^*$ and denote by C_Y the connected component of zero in Y. By the structure theorem for connected locally compact Abelian groups, $C_Y = M \times L$, where $M \approx \mathbb{R}^m$, and L is a connected compact Abelian group. By Lemma 1, the symmetry of the conditional distribution of L_2 given L_1 is equivalent to equation (1). It is easily seen that $c(L) = L$ for any $c \in \mathrm{Aut}(Y)$. Hence, we can restrict equation (1) to the subgroup L. Since L is a connected compact Abelian separable metric group, it is the character group of a countable discrete torsion-free Abelian group. Proposition 1 implies that the restrictions of the characteristic functions $\widehat{\mu}_j$ to L are characters of the subgroup L. Extending them to characters of Y we find that there are $x_j \in X$ such that

$$\widehat{\mu}_j(y) = (x_j, y), \qquad y \in L,\ j = 1, \dots, n. \tag{11}$$

Substitute (11) into (1) and consider the resulting equation on L. We infer that

$$2\sum_{j=1}^{n} \beta_j x_j \in A(X, L) = \mathbb{R}^m \times G_0.$$

It follows from $L^{(2)} = L$ that

$$x_0 = \sum_{j=1}^{n} \beta_j x_j \in \mathbb{R}^m \times G_0.$$

It is obvious that $\delta(\mathbb{R}^m \times G_0) = \mathbb{R}^m \times G_0$ for any $\delta \in \mathrm{Aut}(X)$. So $\beta_1^{-1} x_0 \in \mathbb{R}^m \times G_0$. Put $x_1' = x_1 - \beta_1^{-1} x_0$, $x_j' = x_j$, $j = 2, \ldots, n$. Then

$$\widehat{\mu}_j(y) = (x_j', y), \quad y \in L, \ j = 1, \ldots, n. \tag{12}$$

Moreover,

$$\sum_{j=1}^{n} \beta_j x_j' = 0. \tag{13}$$

Put $\mu_j' = E_{-x_j'} * \mu_j$. Equality (13) implies that the characteristic functions $\widehat{\mu}_j'(y) = (-x_j', y)\widehat{\mu}_j(y)$ satisfy (1). By Lemma 1, if ξ_j' are independent random variables with values in X and distributions μ_j', then the conditional distribution of $L_2' = \beta_1 \xi_1' + \cdots + \beta_n \xi_n'$ given $L_1' = \alpha_1 \xi_1' + \cdots + \alpha_n \xi_n'$ is symmetric. It follows from (12) that

$$\widehat{\mu}_j'(y) = 1, \quad y \in L, \, j = 1, \ldots, n.$$

Hence, $\sigma(\mu_j') \subset A(X, L) = \mathbb{R}^m \times G_0$. Corollary 1 is proved.

Remark 1. Corollary 1 implies the following statement (with the same notation). If the conditional distribution of $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$ given $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ is symmetric, then studying the possible distributions μ_j one can suppose without loss of generality that $G = G_0$, i.e. the group G itself consists of compact elements.

3. Proof of Theorem 1. To prove Theorem 1 we need some lemmas.

Lemma 4 ([3]). *Let X be a finite Abelian group with $X_{(2)} = \{0\}$. Assume that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{Aut}(X)$ satisfy $\beta_1 \alpha_1^{-1} \pm \beta_2 \alpha_2^{-1} \in \mathrm{Aut}(X)$. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2. Then the symmetry of the conditional distribution of $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$ given $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$ implies that $\mu_1, \mu_2 \in I(X)$.*

Lemma 5. *Let X be a locally compact Abelian group, K a compact subgroup of X, $L = A(Y, K)$, and $\alpha \in \mathrm{Aut}(X)$. Then the following statements are equivalent*:

(i) *$\alpha(K) \supset K$;*
(ii) *if $\widetilde{\alpha} y \in L$, then $y \in L$.*

Corollary 2. *Under the conditions of Lemma* 5 *the following statements are equivalent*:

(i) *$\alpha(K) = K$;*
(ii) *$\widetilde{\alpha}(L) = L$.*

The proofs of Lemma 5 and Corollary 2 are standard and we omit them.

Lemma 6. *Let X be a countable discrete torsion Abelian group such that $X_{(2)}=\{0\}$. Assume that $\alpha_j,\beta_j\in\mathrm{Aut}(X)$ satisfy $\beta_i\alpha_i^{-1}\pm\beta_j\alpha_j^{-1}\in\mathrm{Aut}(X)$ for all $i\neq j$. Let $\xi_1,\dots,\xi_n$, $n\geq 2$, be independent random variables with values in X and distributions μ_j such that $\widehat{\mu}_j(y)\geq 0$. If the conditional distribution of $L_2=\beta_1\xi_1+\cdots+\beta_n\xi_n$ given $L_1=\alpha_1\xi_1+\cdots+\alpha_n\xi_n$ is symmetric, then $\sigma(\mu_j)\subset F$ for all j, where F is a finite subgroup of X.*

Proof. Set $Y=X^*$ and note that Y is a totally disconnected compact Abelian group. The compactness of Y implies that $\overline{Y^{(2)}}=Y^{(2)}$. Since $X_{(2)}=\{0\}$, we have $\overline{Y^{(2)}}=Y^{(2)}=Y$, and hence $f_2:Y\to Y$ is an open homomorphism. Put $f_j(y)=\widehat{\mu}_j(y)$. We restrict ourselves to the case $n=2$. Reasoning as in the proof of Proposition 1 we come to equation (9) for the function $\psi_1(y)=-\ln f_1(y)$ in a neighbourhood W of zero in Y. Since Y is a totally disconnected compact group, there is an open subgroup H of Y such that $H\subset W$ ([5, §24.6]). Since H is an open subgroup, it is also closed and hence compact. By Lemma 2, $\psi_1(y)=0$ on H. This implies that $f_1(y)=1$ for $y\in H$. Thus, $\sigma(\mu_1)\subset A(X,H)=F_1$. Since H is an open subgroup its annihilator F_1 is compact, and as X is discrete, F_1 is finite. For μ_2 we reason similarly. Denote by F the subgroup of X generated by F_1 and F_2. Lemma 6 is proved.

Lemma 7. *Let X be a locally compact Abelian separable metric group, and let ξ_1, ξ_2 be independent random variables with values in X and distributions $\mu_1=m_{K_1}$, $\mu_2=m_{K_2}$, where K_1, K_2 are finite subgroups of X. If $f_2,\delta,I\pm\delta\in\mathrm{Aut}(X)$, then the symmetry of the conditional distribution of $L_2=\xi_1+\delta\xi_2$ given $L_1=\xi_1+\xi_2$ implies that $K_1=K_2=K$ and $\delta(K)=K$.*

Proof. Set $Y=X^*$, $f(y)=\widehat{m}_{K_1}(y)$, $g(y)=\widehat{m}_{K_2}(y)$, $\varepsilon=\widetilde{\delta}$, $a=I-\varepsilon$, $b=I+\varepsilon$, $c=ab^{-1}$. Then $c=\widetilde{\gamma}$, where $\gamma=(I+\delta)^{-1}(I-\delta)$. By Lemma 1, the symmetry of the conditional distribution of L_2 given L_1 implies that the characteristic functions $f(y)$ and $g(y)$ satisfy equation (1), which takes the form

$$f(u+v)g(u+\varepsilon v)=f(u-v)g(u-\varepsilon v),\qquad u,v\in Y. \tag{14}$$

Substituting $v=-u$ in (14) we obtain

$$g(au)=f(2u)g(bu),\qquad u\in Y.$$

This implies that

$$g(cy)=f(2b^{-1}y)g(y),\qquad y\in Y. \tag{15}$$

Put $H_j=A(Y,K_j)$, $j=1,2$. It follows from (15) that if $cy\in H_2$, then $y\in H_2$. By Lemma 5, this implies that $\gamma(K_2)\supset K_2$. Since K_2 is finite,

$$\gamma(K_2)=K_2. \tag{16}$$

We observe that $I+\gamma = 2(I+\delta)^{-1}$, $I-\gamma = 2\delta(I+\delta)^{-1}$. Inasmuch as $f_2 \in \mathrm{Aut}(X)$, we have $I \pm \gamma \in \mathrm{Aut}(X)$ and $\delta = (I-\gamma)(I+\gamma)^{-1}$. It follows from (16) that $\delta(K_2) = K_2$, and by Corollary 2, $\varepsilon(H_2) = H_2$. Consider the restriction of equation (14) to the subgroup H_2. We have

$$f(u+v) = f(u-v), \quad u, v \in H_2.$$

Hence,

$$f(2y) = 1, \quad y \in H_2. \tag{17}$$

Since $f_2 \in \mathrm{Aut}(X)$ and K_2 is a finite group, we conclude that $(K_2)^{(2)} = K_2$, and by Corollary 2, $(H_2)^{(2)} = H_2$. It follows from (17) that $f(y) = 1$ for $y \in H_2$, and hence $H_2 \subset H_1$. Reasoning similarly we deduce that (14) implies $\varepsilon(H_1) = H_1$ and

$$g(2\varepsilon y) = 1, \quad y \in H_1,$$

so that $H_1 \subset H_2$. Thus, $H_1 = H_2 = H$, $K_1 = K_2 = K$. Since $\varepsilon(H) = H$, by Corollary 2, $\delta(K) = K$. Lemma 7 is proved.

Now we can prove Theorem 1.

Proof of Theorem 1. Set $Y = X^*$. The necessity of the condition $X_{(2)} = \{0\}$ follows from the fact that if ξ_j are arbitrary independent random variables with values in $X_{(2)}$ and $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, then the conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric (see in more detail [3, Remark 1]). Let us prove the sufficiency.

Considering the new independent random variables $\xi_1' = \alpha_1\xi_1$, $\xi_2' = \alpha_2\xi_2$, we can assume from the beginning that $L_1 = \xi_1+\xi_2$, $L_2 = \delta_1\xi_1+\delta_2\xi_2$, where $\delta_1, \delta_2, \delta_1 \pm \delta_2 \in \mathrm{Aut}(X)$. By Lemma 1, the symmetry of the conditional distribution of L_2 given L_1 implies that the characteristic functions $\widehat{\mu}_j$ satisfy equation (3) which takes the form

$$\widehat{\mu}_1(u+\widetilde{\delta}_1 v)\widehat{\mu}_2(u+\widetilde{\delta}_2 v) = \widehat{\mu}_1(u-\widetilde{\delta}_1 v)\widehat{\mu}_2(u-\widetilde{\delta}_2 v), \quad u, v \in Y. \tag{18}$$

Put $\nu_j = \mu_j * \overline{\mu}_j$, $j = 1, 2$. Then $\widehat{\nu}_j(y) = |\widehat{\mu}_j(y)|^2 \geq 0$. Set $f(y) = \widehat{\nu}_1(y)$, $g(y) = \widehat{\nu}_2(y)$, $\delta = \delta_1^{-1}\delta_2$, $\varepsilon = \widetilde{\delta}$. In this notation equation (18) is transformed into (14). We will prove Theorem 1 if we verify that the functions $f(y)$ and $g(y)$ take on the values 0 and 1 only.

By Remark 1, we can suppose from the beginning that X is a torsion group. Put $L = \{y \in Y : f(y) = 1\}$, $H = \{y \in Y : g(y) = 1\}$, $K = A(X, L)$, $G = A(X, H)$. By Lemma 6, $\sigma(\nu_j) \subset F$, $j = 1, 2$, where F is a finite subgroup of X. It is obvious that K and G must also be finite subgroups because $K, G \subset F$. It follows from (14) that

$$f^n(u+v)g^n(u+\varepsilon v) = f^n(u-v)g^n(u-\varepsilon v), \quad u, v \in Y, \tag{19}$$

for any natural n. It is clear that the limits

$$\lim_{n\to\infty} f^n(y) = \widehat{m}_K(y), \qquad \lim_{n\to\infty} g^n(y) = \widehat{m}_G(y)$$

exist. Letting $n \to \infty$ in (19) we see that the functions $\widehat{f}(y) = \widehat{m}_K(y)$ and $\widehat{g}(y) = \widehat{m}_G(y)$ also satisfy (14). Since X is a torsion group and $X_{(2)} = \{0\}$, we have $f_2 \in \mathrm{Aut}(X)$ and so we can apply Lemma 7. We obtain $K = G$, $L = H$ and $\delta(K) = K$. By Corollary 2,

$$\varepsilon(L) = L. \tag{20}$$

This implies that the homomorphism induced by ε on Y/L is an automorphism. Moreover, it follows from $L = H$ that $f(y) = g(y) = 1$ for $y \in L$. Hence, f and g are L-invariant. Therefore (14) induces an equation on Y/L. Since $Y/L \approx K^*$ and K is a finite subgroup with $K_{(2)} = \{0\}$, we can apply Lemma 4 to complete the proof of Theorem 1.

We add to Theorem 1 the following statement.

Proposition 2. *Let X be a locally compact Abelian separable metric group, and let ξ_1, ξ_2 be independent random variables with values in X and distributions $\mu_1 = \mu_2 = m_K$, where K is a compact subgroup of X. Assume that $\delta, I\pm\delta \in \mathrm{Aut}(X)$. Set $\gamma = (I+\delta)^{-1}(I-\delta)$. Then the following statements are equivalent*:

- (i) *the conditional distribution of $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric*;
- (ii) $\gamma(K) \supset K$.

Proof. Set $\varepsilon = \widetilde{\delta}$, $a = I-\varepsilon$, $b = I+\varepsilon$, $c = ab^{-1}$. Then $c = \widetilde{\gamma}$. Assume that (i) holds. Put $L = A(Y,K)$, $f(y) = \widehat{m}_K(y)$. By Lemma 1, f satisfies (1), which takes the form

$$f(u+v)f(u+\varepsilon v) = f(u-v)f(u-\varepsilon v), \qquad u, v \in Y. \tag{21}$$

Substituting $v = -u$ we find

$$f(au) = f(2u)f(bu), \qquad u \in Y.$$

Hence,

$$f(cy) = f(2b^{-1}y)f(y), \qquad y \in Y. \tag{22}$$

Since

$$f(y) = \begin{cases} 1, & y \in L, \\ 0, & y \notin L, \end{cases}$$

equation (22) implies that if $cy \in L$, then $y \in L$. Now Lemma 5 yields (ii).

Conversely, assume that (ii) holds. We will verify that f satisfies (21), which, by Lemma 1, proves (i). Note that by Lemma 5, (ii) is equivalent to the statement: if $cy \in L$, then $y \in L$. Suppose that for some $u, v \in Y$ the left-hand side of (21) is equal to 1. Then

$$u + v, u + \varepsilon v \in L. \tag{23}$$

This implies that $av \in L$. Inasmuch as $av = cbv$, we have $cbv \in L$, and hence

$$bv = (I + \varepsilon)v \in L. \tag{24}$$

It follows from (23) and (24) that $u - v, u - \varepsilon v \in L$, i.e. the right-hand side of (21) is 1. We verify similarly that if the right-hand side of (21) is 1, then the same is true for the left-hand side. Proposition 2 is proved.

REMARK 2. It follows from the proof of Theorem 1 that if X is a countable discrete Abelian group such that $X_{(2)} = \{0\}$, $Y = X^*$ and $\varepsilon, I \pm \varepsilon \in \mathrm{Aut}(Y)$, then all solutions of (14) in the class of characteristic functions are of the form

$$f(y) = (x_1, y)\widehat{m}_K(y), \qquad g(y) = (x_2, y)\widehat{m}_K(y),$$

where $x_1, x_2 \in X$ and K is a finite subgroup of X with $\delta(K) = K$, $\delta = \widetilde{\varepsilon}$.

REMARK 3. Let X be a countable discrete Abelian group. Assume that $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ satisfy $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Let $\xi_1, \ldots, \xi_n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions. If the conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then $\sigma(\mu_j') \subset X_{(2)}$ for all j, for some shifts μ_j' of the distributions μ_j.

We shall restrict ourselves to the proof for the case $n = 2$. Clearly we can assume that $L_1 = \xi_1 + \xi_2$, $L_2 = \xi_1 + \delta\xi_2$, where $\delta, I \pm \delta \in \mathrm{Aut}(X)$. Put $\nu_j = \mu_j * \overline{\mu}_j$, $j = 1, 2$. It is easily seen that our statement will be proved if we verify that $\sigma(\nu_j) \subset X_{(2)}$. We note that $\widehat{\nu}_j(y) > 0$ for all Y. Put $f(y) = \widehat{\nu}_1(y)$, $g(y) = \widehat{\nu}_2(y)$. Reasoning as in the proof of Proposition 1 we arrive at equation (8) for the function $\psi_1(y) = -\ln\widehat{\nu}_1(y)$ for all $u, v \in Y$. It follows that ψ_1 satisfies (9) on the subgroup $Y^{(2)}$. Since $Y^{(2)}$ is compact, by Lemma 2, $\psi_1(y) = 0$ for $y \in Y^{(2)}$. This implies that $\sigma(\nu_1) \subset A(X, Y^{(2)}) = X_{(2)}$. Reasoning similarly we prove that $\sigma(\nu_2) \subset X_{(2)}$.

4. Heyde theorem for $X = \mathbb{R} \times G$, where G is a countable discrete Abelian group with $G_{(2)} = \{0\}$. We will use Theorem 1 to prove the following statement.

THEOREM 2. *Let $X = \mathbb{R} \times G$, where G is a countable discrete Abelian group such that $G_{(2)} = \{0\}$. Assume that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{Aut}(X)$ satisfy $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. Let ξ_1, ξ_2 be independent random variables with values in X and distributions μ_1, μ_2. If the conditional distribution of $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ is symmetric, then $\mu_j = \lambda_j * \pi_j$, where λ_j are Gaussian distributions on $\mathbb{R}$, and $\pi_j \in I(X)$, $j = 1, 2$.*

To prove Theorem 2 we need the following

LEMMA 8 ([3]). *The conclusion of Theorem* 2 *is true when G is a finite Abelian group.*

Proof of Theorem 2. We have $Y = X^* \approx \mathbb{R} \times H$, where $H = G^*$. To simplify notation we assume that $Y = \mathbb{R} \times H$. Reasoning as in the proof of Theorem 1 we reduce the proof to the case when $L_1 = \xi_1 + \xi_2$, $L_2 = \xi_1 + \delta\xi_2$ and $\delta, I \pm \delta \in \mathrm{Aut}(X)$. We need to solve equation (14), where $f(y) = \widehat{\mu}_1(y)$, $g(y) = \widehat{\mu}_2(y)$, $\varepsilon = \widetilde{\delta}$. By Remark 1 we can assume that G is a torsion group. This implies that H is a totally disconnected compact Abelian group. Denote elements of Y by (s, h), $s \in \mathbb{R}$, $h \in H$. It is obvious that if $d \in \mathrm{Aut}(Y)$, then $d(\mathbb{R}) = \mathbb{R}$ and $d(H) = H$. We will retain notation d for the restrictions of d to $\mathbb{R}$ and to H. For this reason we write $d(s, h) = (ds, dh)$, $(s, h) \in Y$. In this notation equation (14) takes the form

$$\begin{aligned} (25) \qquad & f(s + s', h + h')g(s + \varepsilon s', h + \varepsilon h') \\ &= f(s - s', h - h')g(s - \varepsilon s', h - \varepsilon h'), \qquad (s, h), (s', h') \in Y. \end{aligned}$$

Substituting $s = s' = 0$ in (25) we come to the equation

$$(26) \qquad f(0, h + h')g(0, h + \varepsilon h') = f(0, h - h')g(0, h - \varepsilon h'), \qquad h, h' \in H.$$

It follows from Remark 2 that all solutions of (26) are of the form

$$(27) \qquad f(0, h) = (g_1, h)\widehat{m}_K(h), \qquad g(0, h) = (g_2, h)\widehat{m}_K(h), \qquad h \in H,$$

where K is a finite subgroup of $G, g_1, g_2 \in G$ and $\delta(K) = K$. Put $B = A(H, K)$. Substitute (27) into (26) and consider the resulting equation on B. We obtain

$$(28) \qquad 2(g_1 + \delta g_2) \in K.$$

Since K is a finite group and $G_{(2)} = \{0\}$, this implies that

$$(29) \qquad g_1 + \delta g_2 \in K.$$

Put $\mu_1' = E_{\delta g_2} * \mu_1$, $\mu_2' = E_{-g_2} * \mu_2$. It follows from (29) that the characteristic functions $\widetilde{\mu}_1'(s, h)$ and $\widetilde{\mu}_2'(s, h)$ satisfy (25). Note that

$$\widetilde{\mu}_1'(0, h) = \widetilde{\mu}_2'(0, h) = \begin{cases} 1, & h \in B, \\ 0, & h \notin B. \end{cases}$$

It follows that the characteristic functions $\widetilde{\mu}_1'(s, h)$ and $\widetilde{\mu}_2'(s, h)$ are B-invariant. Hence, $\sigma(\mu_j) \subset A(X, B) = \mathbb{R} \times K$. Since $\delta(\mathbb{R} \times K) = \mathbb{R} \times K$, we can apply Lemma 8 and complete the proof of Theorem 2.

REMARK 4. The assertion of Theorem 2 is also valid for the group $X = \mathbb{R}^m \times G$, where $m > 1$, and G is a countable discrete Abelian group such that $G_{(2)} = \{0\}$. To prove this we reason as in the proof of Theorem 2 and reduce the proof to the case when G is a finite group. The proof in [3] for the case of G finite is based on Theorem A. This proof remains valid when $m > 1$, but instead of Theorem A we need the following statement: Let $X = \mathbb{R}^m$, where $m > 1$. Assume that $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ satisfy $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Let $\xi_1, \dots, \xi_n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j. If the conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then all μ_j are Gaussian. To prove this, we reason as in the proof of Proposition 1. We retain the same notation and restrict ourselves to the case $n = 2$. We arrive at equation (9) in a neighbourhood W of zero in $Y = \mathbb{R}^m$. Hence, $\psi_1(s)$ is an ordinary polynomial in W. By Lemma 3, the representation

$$f_1(s) = e^{-\psi_1(s)}, \qquad s \in W,$$

can be extended from W to $\mathbb{R}^m$. Standard arguments involving the Marcinkiewicz theorem ([8, Ch. 1, §5]) and the Cramer theorem on decomposition of a Gaussian distribution in $\mathbb{R}^m$ show that μ_1 is Gaussian. For μ_2 we reason similarly.

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