Approximate amenability for Banach sequence algebras

by

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Abstract. We consider when certain Banach sequence algebras $A$ on the set $\mathbb{N}$ are approximately amenable. Some general results are obtained, and we resolve the special cases where $A = \ell^p$ for $1 \leq p < \infty$, showing that these algebras are not approximately amenable. The same result holds for the weighted algebras $\ell^p(\omega)$.

1. Introduction. The concept of amenability for a Banach algebra $A$, introduced by Johnson in 1972 [7], has proved to be of enormous importance in Banach algebra theory (see [1], for example). In [3] several modifications of this notion were introduced; in this paper we shall focus on one of these, that of approximate amenability. We recall the definition in Definition 1.1 below.

Let $A$ be an algebra, and let $X$ be an $A$-bimodule. A derivation is a linear map $D : A \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A).$$

For $x \in X$, set $\text{ad}_x : a \mapsto a \cdot x - x \cdot a$, $A \to X$. Then $\text{ad}_x$ is a derivation; these are the inner derivations.

Let $A$ be a Banach algebra, and let $X$ be a Banach $A$-bimodule. A continuous derivation $D : A \to X$ is approximately inner if there is a net $(x_\alpha)$ in $X$ such that

$$D(a) = \lim_{\alpha} (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in A),$$

so that $D = \lim_{\alpha} \text{ad}_{x_\alpha}$ in the strong-operator topology of $B(A)$.

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The dual of a Banach space $X$ is denoted by $X'$; in the case where $X$ is a Banach $A$-bimodule, $X'$ is also a Banach $A$-bimodule. For the standard dual module definitions, see [1].

**Definition 1.1 ([3])**. Let $A$ be a Banach algebra. Then $A$ is *approximately amenable* if, for each Banach $A$-bimodule $X$, every continuous derivation $D : A \to X'$ is approximately inner.

The qualifier *sequential* prefixed to the above definition specifies that there is a sequence of inner derivations approximating the given continuous derivation.

We remark that, in [3], the notion of uniform approximate amenability was also introduced: a Banach algebra $A$ is said to be *uniformly approximately amenable* if, for each Banach $A$-bimodule $X$, each continuous derivation $D : A \to X'$ is the limit of a sequence of inner derivations in the norm topology of $B(A, X')$. In fact, it has recently been shown independently by Pirkovskii [10] and Ghahramani [4] that a uniformly approximately amenable Banach algebra is already amenable.

Of course, each amenable Banach algebra is approximately amenable. Some approximately amenable Banach algebras which are not amenable are constructed in [3]. For example, let $(A_n)$ be a sequence of unital, amenable Banach algebras. Then the sum $c_0(A_n)$ is always approximately amenable, but is not necessarily amenable [3, Example 6.1]. Further, it has been shown by Ghahramani and Stokke [5] that the Fourier algebra $A(G)$ is approximately amenable for each amenable, discrete group $G$, but it is known that $A(G)$ is not always amenable for an amenable group $G$ [9]. Examples of semigroup algebras of the form $\ell^1(S)$ that are approximately amenable but not amenable are given in [2]. Nevertheless there is something of a shortage of “natural” examples of approximately amenable Banach algebras which are not amenable.

In this paper, we shall consider when certain Banach sequence algebras on $\mathbb{N}$ are approximately amenable, a question left open in [3]. In particular, we shall consider the standard Banach sequence algebras $\ell^p = \ell^p(\omega)$, where $1 \leq p < \infty$ and $\omega$ is a weight on $\mathbb{N}$.

**2. Basic constructions.** When determining whether or not our Banach algebras are approximately amenable, we shall work from a characterization of approximately amenable Banach algebras which is a modification of that given in [3].

Let $A$ be Banach algebra. The projective tensor product $A \hat{\otimes} A$ is a Banach $A$-bimodule under the operations defined by

$$c \cdot a \otimes b = ca \otimes b, \quad a \otimes b \cdot c = a \otimes bc \quad (a, b, c \in A),$$
and there is a continuous linear $A$-bimodule homomorphism $\pi : A \hat{\otimes} A \to A$ such that $\pi(a \otimes b) = ab$ $(a, b \in A)$ (see [1]).

**Proposition 2.1.** Let $A$ be a Banach algebra. Then $A$ is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset $S$ of $A$, there exist $F \in A \otimes A$ and $u, v \in A$ such that $\pi(F) = u + v$ and, for each $a \in S$:

(i) $\|a \cdot F - F \cdot a + u \otimes a - a \otimes v\| < \varepsilon$;

(ii) $\|a - au\| < \varepsilon$ and $\|a - va\| < \varepsilon$.

**Proof.** Suppose that $A$ is approximately amenable. Then by [3, Corollary 2.2] there are nets $(M_\alpha)$ in $(A \hat{\otimes} A )''$ and $(U_\alpha)$ and $(V_\alpha)$ in $A''$ such that, for each $a \in A$:

(i) $a \cdot M_\alpha - M_\alpha \cdot a + U_\alpha \otimes a - a \otimes V_\alpha \to 0$;

(ii) $a - a \cdot U_\alpha \to 0$ and $a - V_\alpha \cdot a \to 0$;

(iii) $\pi''(M_\alpha) - U_\alpha - V_\alpha \to 0$.

(This corrects a typographical error in [3].) In each case convergence is in the $\| \cdot \|$-topology.

Let $Y$ denote the Banach space $(A \hat{\otimes} A ) \oplus A \oplus A \oplus A$. For each $a \in A$, define a convex set in $Y$ by setting

$$K_a := \{ (a \cdot m - m \cdot a + u \otimes a - a \otimes v, a - au, a - va, \pi(m) - u - v) : m \in A \hat{\otimes} A, u, v \in A \}.$$ 

For the specified finite subset $S$ of $A$,

$$K := \prod\{ K_a : a \in S \}$$

is a convex set in the Banach space $Y^S$. The conditions above show that the weak closure of $K$ in $Y^S$ contains the zero element $0$ of $Y^S$. By Mazur's theorem, it follows that $0$ belongs to the $\| \cdot \|$-closure of $K$ in $Y^S$. Thus, with $\varepsilon > 0$ as specified, there exist $F \in A \otimes A$ and $u, v \in A$ such that clauses (i) and (ii) of the proposition are satisfied and, further, such that $\| \pi(F) - u - v\| < \varepsilon$. By modifying $F$ and $u$ slightly, we may suppose, further, that $F \in A \otimes A$ and that $\pi(F) = u + v$.

Conversely, suppose that the condition in the proposition is satisfied. Consider the set $D := (0, 1) \times \mathcal{F}(A)$, where $\mathcal{F}(A)$ is the family of finite subsets of $A$, and order $D$ by setting

$$(\varepsilon_1, S_1) \leq (\varepsilon_2, S_2) \quad \text{whenever} \quad \varepsilon_1 \geq \varepsilon_2 \text{ and } S_1 \subseteq S_2.$$ 

Then $(D, \leq)$ is a directed set. The conditions show that there exist nets $(F_\alpha)$ in $A \hat{\otimes} A$ and $(u_\alpha), (v_\alpha)$ in $A$, each indexed by $(D, \leq)$, such that $\pi(F_\alpha) = u_\alpha + v_\alpha$ and such that, for each $a \in A$, we have:

$$a \cdot F_\alpha - F_\alpha \cdot a + u_\alpha \otimes a - a \otimes v_\alpha \to 0;$$

$$a - au_\alpha \to 0, \quad a - v_\alpha a \to 0.$$
Thus we have satisfied the conditions of [3, Corollary 2.2], and so $A$ is approximately amenable. 

**Corollary 2.2.** Let $A$ be a Banach algebra with identity $e$. Then $A$ is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset $S$ of $A$, there exists $G \in A \otimes A$ with $\pi(G) = e$ and such that

$$\|a \cdot G - G \cdot a\| < \varepsilon \quad (a \in S).$$

**Proof.** Suppose that such a $G$ exists, and set $u = v = e$ and $F = G + e \otimes e$. Then $\pi(F) = u + v$, and $F, u, v$ satisfy the conditions of Proposition 2.1.

Conversely, suppose that $F, u, v$ satisfy the above condition for a finite subset $S$ and with $\varepsilon/3\|e\|$ replacing $\varepsilon$, and set $G = F - u \otimes e - e \otimes v + e \otimes e$.

Then $\pi(G) = e$, and

$$\|a \cdot G - G \cdot a\| \leq \|a \cdot F - F \cdot a + u \otimes a - a \otimes v\| + \|a - au\| + \|a - va\| < \varepsilon,$$

and so $A$ is approximately amenable by Proposition 2.1. 

For comparison, we recall [1], [8] that a Banach algebra $A$ is amenable if and only if there is a constant $C > 0$ such that, for each $\varepsilon > 0$ and each finite subset $S$ of $A$, there exists $F \in A \otimes A$ with $\|F\| \leq C$ such that, for each $a \in S$, we have:

(i) $\|a \cdot F - F \cdot a\| < \varepsilon$;

(ii) $\|a - a\pi(F)\| < \varepsilon$.

We remark that (ii) of Proposition 2.1 is exactly the condition that $A$ has both left and right approximate units [1, Definition 2.9.10]. We do not know whether or not an approximately amenable Banach algebra necessarily has (two-sided) approximate units.

We now give a variation of Proposition 2.1 in the case where $A$ is commutative. For each Banach algebra $A$, there is an isometry $\iota : A \hat{\otimes} A \to A \hat{\otimes} A$ such that $\iota(a \otimes b) = b \otimes a$ ($a, b \in A$).

**Proposition 2.3.** Let $A$ be a commutative Banach algebra. Then $A$ is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset $S$ of $A$, there exist $F \in A \otimes A$ with $\iota(F) = F$ and $u \in A$ such that $\pi(F) = 2u$, and, for each $a \in S$:

(i) $\|a \cdot F - F \cdot a + u \otimes a - a \otimes u\| < \varepsilon$;

(ii) $\|a - au\| < \varepsilon$.

**Proof.** Since $A$ is commutative,

$$\iota(a \cdot F) = \iota(F) \cdot a \quad (a \in A, F \in A \hat{\otimes} A).$$

Suppose that $A$ is approximately amenable, and take $\varepsilon > 0$ and a finite subset $S$ of $A$. By Proposition 2.1, there exist $F, u, v$ satisfying
conditions (i) and (ii) of that result. For each \( a \in S \), we have
\[
\| \iota(F) \cdot a - a \cdot \iota(F) + a \otimes u - v \otimes a \| < \varepsilon.
\]
Set \( G = (F + \iota(F))/2 \) and \( w = (u + v)/2 \). Then \( \iota(G) = G \) and \( \pi(G) = 2w \).
Further,
\[
\|a \cdot G - G \cdot a + w \otimes a - a \otimes w\| < \varepsilon \quad \text{and} \quad \|a - aw\| < \varepsilon.
\]
Thus the specified conditions are satisfied (with \( w \) for \( u \)).

The converse is immediate. \( \blacksquare \)

3. Banach sequence algebras. We now introduce the specific algebras that will be considered in this paper. As usual \( c_{00} \) will be the subalgebra of \( \mathbb{C}^\mathbb{N} \) consisting of the sequences having finite support.

**Definition 3.1.** A Banach sequence algebra on \( \mathbb{N} \) is a Banach algebra \( A \) which is a subalgebra of \( \mathbb{C}^\mathbb{N} \) such that \( c_{00} \subset A \).

For example, \( c_0 = c_0(\mathbb{N}) \) and \( \ell^p = \ell^p(\mathbb{N}) \) for \( 1 \leq p \leq \infty \) are Banach sequence algebras on \( \mathbb{N} \).

Let \( (A, \| \cdot \|) \) be a Banach sequence algebra on \( \mathbb{N} \). Then
\[
\|a\| \geq |a|_N \quad (a \in A),
\]
where \( | \cdot |_N \) denotes the uniform norm on \( \mathbb{N} \). In the case where \( c_{00} \) is dense in \( A \), the algebra \( A \) is natural on \( \mathbb{N} \) [1, Proposition 4.1.35].

Throughout we write \( \delta_i \) for the characteristic function of \( \{i\} \) for \( i \in \mathbb{N} \), and set
\[
e_n = \sum_{i=1}^{n} \delta_i \quad (n \in \mathbb{N}),
\]
so that \( (e_n) \subset c_{00} \subset A \). When convenient we identify \( a \in A \) both as the sequence \( (a_i) \) and as the formal sum \( \sum_i a_i \delta_i \). We shall also identify \( A \otimes A \) with a space of functions on \( \mathbb{N} \times \mathbb{N} \) by setting
\[
(a \otimes b)(i, j) = a_i b_j \quad (a, b \in A, i, j \in \mathbb{N});
\]
in particular, \( \delta_i \otimes \delta_j = \delta_{(i,j)} \), the characteristic function of \( \{(i, j)\} \), for \( i, j \in \mathbb{N} \). We shall also sometimes write \( F = \sum_{i,j} F(i, j) \delta_{(i,j)} \) for \( F \in c_{00} \otimes c_{00} \). Note that
\[
(a \cdot F)(i, j) = a_i F(i, j), \quad (F \cdot a)(i, j) = a_j F(i, j) \quad (i, j \in \mathbb{N}),
\]
and that \( \pi(F) = \sum_i F(i, i) \delta_i \).

**Definition 3.2.** Let \( A \) be a Banach sequence algebra on \( \mathbb{N} \), and let \( a \in A \). For \( F \in c_{00} \otimes c_{00} \), set
\[
\Delta_a(F) = a \cdot F - F \cdot a + \pi(F) \otimes a - a \otimes \pi(F).
\]
Clearly \( \Delta_a(F) \in c_{00} \otimes c_{00} \) whenever \( a \in c_{00} \).
Proposition 3.3. Let $A$ be a Banach sequence algebra with $c_{00}$ dense in $A$. Then $A$ is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset $S$ of $A$, there exists $F \in c_{00} \otimes c_{00}$ with $\iota(F) = F$ such that, for each $a \in S$:

(i) $\|\Delta_a(F)\| < \varepsilon$;
(ii) $\|a - a\pi(F)\| < \varepsilon$.

Proof. Suppose that $A$ is approximately amenable, and take $\varepsilon > 0$ and a finite subset $S$ of $A$. Let $F$ and $u$ be given by Proposition 2.3. Since $c_{00}$ is dense in $A$, the space $c_{00} \otimes c_{00}$ is dense in $A \otimes A$, and so we can replace $F$ by an element $G \in c_{00} \otimes c_{00}$ such that (i) and (ii) of that proposition remain true, with $v = \pi(G)/2$ replacing $u$. Now replace $G$ by

$$H = G + \sum_i (u_i - \pi(G)_i) \delta_i \otimes \delta_i,$$

noting that the number of non-zero summands in the above sum is finite. This does not affect clauses (i) or (ii) of Proposition 2.3, and now $\pi(H) = v$. Thus conditions (i) and (ii) of the current proposition are satisfied.

The converse is similar. ■

We shall later consider only Banach sequence algebras $A$ which are self-adjoint. In such a situation the map $a \mapsto \overline{a}$ is necessarily continuous on $A$. It follows that we may replace $F$ by $F + F$, and so take $F$ to be real-valued. Similarly, we may also suppose that the elements of the “test sets” $S$ are real-valued.

Proposition 3.4. Let $A$ be a Banach sequence algebra. Suppose that there is $\eta > 0$ such that, for each $\varepsilon > 0$ and each finite subset $S$ of $A$, there exists $u \in c_{00}$ with

$$\|u\| \geq \eta \quad \text{and} \quad \|a - au\| \cdot \|u\| < \varepsilon.$$  

Then $A$ is approximately amenable.

Proof. Take $u$ as given by (3.1), with $\varepsilon$ replaced by $\varepsilon\eta/2$. Set

$$F = u \otimes u + \sum_i (u_i - u_i^2) \delta_i \otimes \delta_i.$$

Then $\pi(F) = u$ and, for each $a \in S$, we have

$$\|a \cdot F - F \cdot a - a \otimes u + u \otimes a\| = \|au \otimes u - a \otimes u + u \otimes a - u \otimes au\| < \varepsilon$$
and $\|a - au\| < \varepsilon$. By Proposition 3.3, $A$ is approximately amenable.

The converse is immediate. ■

More general forms of this result for Banach function algebras on discrete spaces can be shown by the same sort of argument; see, for example, [5, Proposition 3.16].
We make the conjecture that the sufficient condition in Proposition 3.4 is in fact also necessary for $A$ to be approximately amenable. Indeed, we do not know an example of a Banach sequence algebra which is approximately amenable, but which does not have a bounded approximate identity. It is also conceivable that each Banach sequence algebra $A$ such that $c_{00}$ is dense in $A$ and $A = A^2$ is approximately amenable.

**Corollary 3.5.** Let $A$ be a Banach sequence algebra such that $A$ has a bounded approximate identity contained in $c_{00}$. Then $A$ is sequentially approximately amenable.

**Proof.** It is standard that $A$ has a sequential bounded approximate identity, say $(u_n)$, in $c_{00}$ [1, Corollary 2.9.18], and satisfying $\inf_n \|u_n\| \geq 1$. Let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of $A$. Then, for each $n \in \mathbb{N}$, there exists $i = i(n)$ such that $\|x_j - x_j u_{i(n)}\| < 1/n$ for $1 \leq j \leq n$. Following Proposition 3.4, we set

$$F_n = u_{i(n)} \otimes u_{i(n)} + \sum_{j \in \mathbb{N}} (u_{i(n),j} - u_{i(n),j}^2) \delta_j \otimes \delta_j.$$ 

Then, for each $a \in A$ and $\varepsilon > 0$, we have

$$\|a \cdot F_n - F_n : a - \otimes u_{i(n)} + u_{i(n)} \otimes a\| = 2\|au_{i(n)} - a\| \cdot \|u_{i(n)}\| < \varepsilon$$

for $n$ sufficiently large. Thus $(F_n, u_{i(n)})$ gives a sequence with the required properties of [3, Corollary 2.2]. The sequential variant of [3, Theorem 2.1] holds (with the same argument), and so $A$ is sequentially approximately amenable.

Special cases of the above corollary have been shown in [4], where it is also shown that the converse holds for certain Banach sequence algebras.

We wish to stress that the function $F$ specified in Proposition 3.3 must satisfy conditions (i) and (ii) for each finite collection $S$ of elements. The following shows that, for many Banach sequence algebras $A$, we can find $F$ to satisfy these conditions for each single element $a \in A$. Indeed, the Banach sequence algebra $\ell^1$ satisfies the conditions of Proposition 3.6 below, but we shall see that it is not approximately amenable. To determine whether or not such an algebra $A$ is approximately amenable, we must look at sets $S$ with at least two elements.

We introduce the following notation. Let $A$ be a Banach sequence algebra on $\mathbb{N}$. For each $a \in A$ and each finite or cofinite subset $T$ of $\mathbb{N}$, set

$$P_T : a \mapsto \sum_i \{a_i \delta_i : i \in T\}, \quad A \to A.$$ 

We also write $P_n = P_{\{1, \ldots, n\}}$ and $Q_n = I - P_n$ for $n \in \mathbb{N}$. The family $C$ of cofinite subsets of $\mathbb{N}$ will be directed by reverse set inclusion.
Proposition 3.6. Let $A$ be a Banach sequence algebra, and let $a \in A$. Suppose that
\begin{equation}
\lim\{\|P_{C}a\| : C \in C\} = 0.
\end{equation}
Then, for each $\varepsilon > 0$, there exists $F \in c_{00} \otimes c_{00}$ such that
\begin{equation}
\|\Delta_{a}(F)\| < \varepsilon \quad \text{and} \quad \|a - a\pi(F)\| < \varepsilon.
\end{equation}

Proof. Let $\{B_{i} : i \in \mathbb{Z}^{+}\}$ be the partition of $\mathbb{N}$ such that $a$ takes the constant value $a_{i}$ on $B_{i}$ for $i \in \mathbb{N}$, the value 0 on $B_{0}$, and $a_{i} \neq a_{j}$ whenever $i, j \in \mathbb{Z}^{+}$ and $i \neq j$. Note that, by (3.2), each $B_{i}$ for $i \in \mathbb{N}$ is finite. For $n \in \mathbb{N}$, set
\begin{equation*}
D_{n} = \bigcup_{i=1}^{n} B_{i} \quad \text{and} \quad E_{n} = \bigcup_{i=n+1}^{\infty} B_{i},
\end{equation*}
and set $\mu(n) = \min E_{n}$, so that $\mu(n) \to \infty$ as $n \to \infty$.

Fix $\varepsilon > 0$, and take $n_{0} \in \mathbb{N}$ such that $\|P_{C}a\| < \varepsilon$ for each cofinite subset $C$ of $\mathbb{N}$ with $\min C \geq n_{0}$. Next choose $n_{1} \in \mathbb{N}$ such that $\mu(n_{1}) \geq n_{0}$. Set $C = E_{n_{1}} \cup (B_{0} \cap [n_{0}, \infty))$, so that $D_{n_{1}}$ is finite and $C$ is cofinite with $\min C \geq n_{0}$. Take $u$ to be the characteristic function of $D_{n_{1}}$, so that
\begin{equation*}
a - au = a\chi_{\mathbb{N}\setminus D_{n_{1}}} = P_{C}a,
\end{equation*}
and hence
\begin{equation*}
\|a - au\| = \|P_{C}a\| < \varepsilon.
\end{equation*}

By (3.2), we may choose $m_{0} \in \mathbb{N}$ with $m_{0} > n_{1}$ and such that
\begin{equation}
|D_{n_{1}}| \cdot \|Q_{m_{0}}a\| < \varepsilon/2.
\end{equation}

Now define $F$ as follows:
\begin{enumerate}
\item[(a)] For $j, k \leq n_{1}$, set $F = 1$ on $B_{j} \times B_{k}$;
\item[(b)] for $j \leq n_{1}$ and $n_{1} < k \leq m_{0}$, set $F = \frac{-a_{k}}{a_{j} - a_{k}}$ on $B_{j} \times B_{k}$;
\item[(c)] by symmetry for $k \leq n_{1}$ and $n_{1} < j \leq m_{0}$; and
\item[(d)] at remaining points, $F = 0$.
\end{enumerate}

Note that $u \in c_{00}$, $F \in c_{00} \otimes c_{00}$, and $\pi(F) = u$. Set $\Delta_{a} = \Delta_{a}(F)$.

Clearly $\Delta_{a}$ is zero except on the sets $(B_{j} \times B_{k}) \cup (B_{k} \times B_{j})$ where $j \leq n_{1}$ and $k > m_{0}$. On the set
\begin{equation*}
\left( \bigcup_{j \leq n_{0}} B_{j} \right) \times \left( \bigcup_{k > m_{0}} B_{k} \right),
\end{equation*}
we see that $a \cdot F - F \cdot a$ and $a \otimes u$ are zero, and that $u \otimes a = u \otimes Q_{m_{0}}a$.

A similar formula holds when $j$ and $k$ are interchanged. Note that $Q_{m_{0}}a \otimes u$
and $u \otimes Q_{m_0} a$ have disjoint supports in $N \times N$. Thus

$$\|\Delta a\| = 2\|Q_{m_0} a \otimes u\| = 2\left\|\sum_{r \in D_{n_3}} Q_{m_0} a \otimes \delta_r\right\| \leq 2|D_{n_1}| \cdot \|Q_{m_0} a\| < \varepsilon$$

by (3.4). This establishes (3.3). 

Note the explicit dependence of $F$ on the element $a$ in clause (b) above. One is tempted to try the “more obvious” definition

$$F_{i,j} = \begin{cases} 1 & (i, j \leq n), \\ 0 & \text{(otherwise)}, \end{cases}$$

for suitably large $n \in \mathbb{N}$, so that $\pi(F) = e_n$. In this case, $F$ is independent of $a$. Suppose that $S$ is a finite subset of $c_00$ (rather than $A$). Then our function $F$ satisfies (i) and (ii) of Proposition 3.3 for each $a \in S$ (for sufficiently large $n \in \mathbb{N}$). However, this choice of $F$ does not work for all $a \in A$. For example, take $A = \ell^1$, and set $a = \sum_{j=1}^{n+1} j^{-3/2} \delta_j \in A$. Then

$$\|\Delta a\| = \sum_{j=n+1}^{\infty} j^{-3/2} \|\delta_j \otimes e_n - e_n \otimes \delta_j\| = 2n \sum_{j=n+1}^{\infty} j^{-3/2} \geq 4$$

for each $n \in \mathbb{N}$.

In fact, let $A = \ell^1$, and let $S$ be a finite subset of $A^2$. Then we claim that, for each $\varepsilon > 0$, there exists $F \in A \otimes A$ such that (3.3) holds for each $a \in S$. This may add some credence to our conjecture that $A^2 = A$ for an approximately amenable Banach sequence algebra.

To prove this claim, we first recall Pringsheim’s theorem: for a decreasing sequence $(a_i) \in A$, one has $\lim i a_i = 0$.

Now take $a = (a_i) \in A$ with $0 \leq a_i \leq 1$ ($i \in \mathbb{N}$). Certainly $a_i \rightarrow 0$, and so there is a permutation $\sigma$ of $\mathbb{N}$ such that $a_{\sigma(j)} \leq a_{\sigma(i)}$ for $j \geq i$ in $\mathbb{N}$. Thus $i a_{\sigma(i)} \rightarrow 0$. Fix $\varepsilon \in (0, 1)$, and take $n \in \mathbb{N}$ such that $ja_{\sigma(j)} < \varepsilon/2$ for $j \geq n$ and also $\sum_{j=n+1}^{\infty} a_j < \varepsilon$. Set $B = \sigma^{-1}(\mathbb{N}_n) \cup \mathbb{N}_n$, where $\mathbb{N}_n = \{1, \ldots, n\}$. Then set $u = \chi_B$, the characteristic function of $B$,

$$F_{i,j} = \begin{cases} 1 & (i, j \in B), \\ 0 & \text{(otherwise)}, \end{cases}$$

and $F = \sum_{i,j} F_{i,j} \delta_{(i,j)}$, so that $\pi(F) = u$. We see that

$$s := |B| \sum_{i \in \mathbb{N} \setminus B} a_i^2 \leq 2n \sum_{j=n+1}^{\infty} a_{\sigma(j)}^2 \leq \frac{n\varepsilon^2}{2} \sum_{j=n+1}^{\infty} j^{-2} < \varepsilon.$$ 

Thus

$$\|a^2 \cdot F - F \cdot a^2 + u \otimes a^2 - a^2 \otimes u\| = 2\|Q_B a^2\| \|u\| = s < \varepsilon,$$
and we have built in the fact that $\|a^2 - ua^2\| < \varepsilon$. It follows that the conditions of (3.3) are satisfied for $a^2$.

For finitely many elements in $A^2$, it suffices to consider the case where each of them is real-valued, and hence we need only consider differences of finitely many squares of non-negative elements of $A$, say the elements are $a^{(1)}, \ldots, a^{(k)}$. We then have finitely many permutations $\sigma_1, \ldots, \sigma_k$ of $\mathbb{N}$ that respectively render each of these latter sequences decreasing. We argue as above, with $n \in \mathbb{N}$ chosen so that, for each $1 \leq i \leq k$, we have $ja^{(i)}_{\sigma_i(j)} < \varepsilon/2k$ for $j \geq n$ and also $\sum_{j=n+1}^{\infty} a^{(i)}_j < \varepsilon$. Finally, we set

$$B = N_n \cup \bigcup_{i=1}^k \sigma_i^{-1}(N_n).$$

The above claim now follows.

4. **Approximate amenability for $\ell^p$.** Take $1 \leq p < \infty$. Then $\ell^p$ is a Banach sequence algebra, and $c_00$ is dense in $\ell^p$. These algebras are discussed in [1, Example 4.1.42].

It is well known that $\ell^p$ is weakly amenable, but not amenable. Clearly the sequence $(e_n)$ is an approximate identity for $\ell^p$ such that $\|e_n\|_p = n^{1/p}$ ($n \in \mathbb{N}$). Certainly each $a \in \ell^p$ satisfies equation (3.2) above.

It is shown in [3, Example 6.3] that $\ell^p$ is not sequentially approximately amenable. In this section we show that $\ell^p$ is not approximately amenable.

To this end, some preliminaries and further notations are needed.

First, note that the map

$$T : \ell^p \times \ell^p \to \ell^p(\mathbb{N} \times \mathbb{N}), \quad T(x, y)(i, j) = x_iy_j,$$

is bilinear with $\|T\| = 1$, and so there is a map

$$\tilde{T} : \ell^p \hat{\otimes} \ell^p \to \ell^p(\mathbb{N} \times \mathbb{N})$$

with $\tilde{T}(x \otimes y) = T(x, y) (x, y \in \ell^p)$ and $\|\tilde{T}\| = 1$. Let $H \in c_00 \otimes c_00$. Then

$$(4.1) \quad \sum_{i,j} |H(i, j)|^p \leq \|H\|^p,$$

where $\|H\|$ denotes the norm of $H$ in $\ell^p \hat{\otimes} \ell^p$. (Of course, equality holds in the case where $p = 1$.)

Fix throughout $\gamma_j = 1/j(j + 1)$ and set $\gamma = (\gamma_j)$. Note that $\gamma$ is positive, decreasing, and satisfies

$$(4.2) \quad k\gamma_k \leq \sum_{j=k+1}^{\infty} \gamma_j.$$
Now let $\eta = (\eta_j) \in \ell^1$ be positive and decreasing, and define elements $a, b$ in $\ell^p$ by

$$a = \sum_{j=1}^{\infty} \eta_j^{1/p} \delta_{2j-1}, \quad b = \sum_{j=1}^{\infty} \eta_j^{1/p} \delta_{2j}.$$  

We show that, for a suitable choice of $\eta$ and for a certain $\varepsilon > 0$, there is no element $F \in c_{00} \otimes c_{00}$ such that both the following inequalities are true:

\[(4.3) \quad \|\Delta_a(F)\| + \|\Delta_b(F)\| < \varepsilon;\]

\[(4.4) \quad \|a - \pi(F)a\| + \|b - \pi(F)b\| < \varepsilon.\]

It would then follow from Proposition 3.3 that $\ell^p$ is not approximately amenable.

Throughout, we set $u = \pi(F)$. As we remarked earlier, we may suppose that $F$ (and $u$) are real-valued.

We first make a small reduction. We may suppose that $\varepsilon < \eta_1^{1/p}$. Now assume that $F$ satisfies (4.4), with $\varepsilon$ replaced by $\varepsilon/2$. Then $\eta_1^{1/p} (1 - u_1) < \eta_1^{1/p}/2$, and so $u_1 > 1/2$. By replacing $u$ and $F$ by $u/u_1$ and $F/u_1$, respectively, we find new elements $F \in c_{00} \otimes c_{00}$ and $u \in c_{00}$ such that $\pi(F) = u$ and

$$u_1 = 1 \quad \text{and} \quad \|\Delta_a(F)\| + \|\Delta_b(F)\| < \varepsilon.$$  

Thus we may always suppose that $u_1 = 1$.

We shall need to estimate $\|\Delta_x\| = \|\Delta_x(F)\|$ for $x = a, b$, and for this we shall use (4.1). Thus we require lower bounds for $|\Delta_x(m, n)|$ for $m, n \in \mathbb{N}$.

First consider the points $(2i - 1, 2j)$, where $i, j \in \mathbb{N}$. For convenience, define $s = F_{2i-1,2j}$. We calculate the values

$$\Delta_a(2i - 1, 2j) = \eta_i^{1/p} (s - u_{2j}),$$

$$\Delta_b(2i - 1, 2j) = \eta_j^{1/p} (u_{2i-1} - s).$$

In the case where $i \leq j$, so that $\eta_i \geq \eta_j$, geometrical considerations show that

$$|s - u_{2j}|^p \eta_i + |u_{2i-1} - s|^p \eta_j \geq \eta_j (|u_{2i-1} - u_{2j}|/2)^p.$$  

In a similar manner, the points $(2i, 2j - 1)$ taken with $i \leq j - 1$ and $j \geq 2$, so that $\eta_i \geq \eta_j$, lead to the estimate

$$|t - u_{2j-1}|^p \eta_i + |u_{2i} - t|^p \eta_j \geq \eta_j (|u_{2i} - u_{2j-1}|/2)^p,$$

where $t = F_{2i,2j-1}$. 


[At the points \((2i - 1, 2j - 1)\) and \((2i, 2j)\), where \(i, j \in \mathbb{N}\), we have

\[
\Delta_a(2i - 1, 2j - 1) = (\eta_i^{1/p} - \eta_j^{1/p}) F_{2i-1,2j-1} - \eta_i^{1/p} u_{2i-1} + \eta_j^{1/p} u_{2j-1},
\]

\[
\Delta_b(2i, 2j) = (\eta_i^{1/p} - \eta_j^{1/p}) F_{2i,2j} - \eta_i^{1/p} u_{2j} + \eta_j^{1/p} u_{2i},
\]

\[
\Delta_a(2i, 2j) = \Delta_b(2i - 1, 2j - 1) = 0.
\]

Since \(\eta_i \neq \eta_j\) for \(i \neq j\), there are choices of the values of \(F\) at the points \((2i - 1, 2j - 1)\) and \((2i, 2j)\) giving zero values to both \(\Delta_a\) and \(\Delta_b\) at all these points. We shall not use this fact.]

For \(u = (u_i) \in c_{00}\), set

\[
(4.5) \quad \Phi_p(\eta, u) = \sum_{j=1}^{\infty} \sum_{i=1}^{j} |u_{2i-1} - u_{2j}|^p + \sum_{j=2}^{\infty} \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}|^p.
\]

It follows from \((4.1)\), the above estimates, and the simple inequality

\[
(\|\alpha\| + \|\beta\|)^p \geq \|\alpha\|^p + \|\beta\|^p \quad (\alpha, \beta \in \mathbb{C}),
\]

that

\[
2^p(\|\Delta_a(F)\| + \|\Delta_b(F)\|)^p \geq \Phi_p(\eta, u).
\]

Set

\[
\theta_p(\eta) = \inf\{\Phi_p(\eta, u) : u \in c_{00}, u_1 = 1\}.
\]

We seek to show that, for suitable choice of \(\eta\), we have \(\theta_p(\eta) > 0\), for then 
\((4.3)\) fails for any \(\varepsilon\) with \(0 < \varepsilon < \min\{\theta_p(\eta)^{1/p}, \eta_1^{1/p}\}/2\), and so \(\ell^p\) is not approximately amenable.

We note that \(\Phi_p(\eta, u)\) is reduced if every value of \(u_i\) outside \([0, 1]\) is replaced by its nearest neighbour in \([0, 1]\). Thus we may suppose throughout that

\[
0 \leq u_i \leq 1 \quad (i \in \mathbb{N}).
\]

For \(d \geq 2\), consider the set

\[
S_d = \{u \in c_{00} : u_1 = 1, u_i \in [0, 1] \ (i = 1, \ldots, d), u_i = 0 \ (i > d)\}.
\]

Certainly

\[
\alpha_d = \min\{\Phi_p(\eta, u) : u \in S_d\} > 0,
\]

and this minimum is attained. The question is whether or not

\[
\lim_{d \to \infty} \alpha_d > 0.
\]

Suppose for the moment that \(p = 1\), and take \(\eta = \gamma\). Thus, in this case, \(\Phi_1(\eta, u)\) from \((4.5)\) becomes

\[
\Phi_1(\gamma, u) = \sum_{j=1}^{\infty} \gamma_j \sum_{i=1}^{j} |u_{2i-1} - u_{2j}| + \sum_{j=2}^{\infty} \gamma_j \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}|.
\]
Consider the values of $\Phi_1(\gamma, u)$ for sequences $u \in S_d$, where $d \geq 2$. Indeed, take such a point $u$ with $u_d > 0$. We claim that, by setting $u_d = 0$, the value of $\Phi_1(\gamma, u)$ is reduced.

To establish this claim, first suppose that $d = 2k + 1$ for some $k \in \mathbb{N}$. By the change specified, we first increase each term in the summand

$$\sum_{i=1}^{k} |u_{2i} - u_{2k+1}|$$

by at most $u_{2k+1} \gamma_{k+1}$, and so the sum itself increases by at most $ku_{2k+1} \gamma_{k+1}$. On the other hand, we decrease the term

$$\sum_{j=k+1}^{\infty} \gamma_j |u_{2k+1} - u_{2j}| = \left( \sum_{j=k+1}^{\infty} \gamma_j \right) u_{2k+1}$$

by $u_{2k+1}$ times the sum $\sum_{j=k+1}^{\infty} \gamma_j$ of the tail. Other terms are not affected. However, for each $k \in \mathbb{N}$, we have

$$k \gamma_{k+1} \leq k \gamma_k \leq \sum_{j=k+1}^{\infty} \gamma_j$$

by (4.2), and so, in total, the value of $\Phi_1(\gamma, u)$ has been decreased.

Now suppose that $d = 2k$ for some $k \in \mathbb{N}$. By the change specified, we firstly increase each term in the summand

$$\gamma_k \sum_{i=1}^{k} |u_{2i-1} - u_2|$$

by at most $u_2 \gamma_k$, and so the sum itself increases by at most $ku_2 \gamma_k$. On the other hand, we decrease the term

$$\sum_{j=k+1}^{\infty} \gamma_j |u_k - u_{2j-1}|$$

by $u_k$ times the sum $\sum_{j=k+1}^{\infty} \gamma_j$ of the tail. Other terms are not affected. Once again, (4.2) ensures that the value of $\Phi_1(\gamma, u)$ has been decreased.

By continuing, we see that, subject to the constraints we have imposed, and in particular that $u \in c_{00}$ and $u_1 = 1$, the minimum value of $\Phi_1(\gamma, u)$ is attained at the point $v = (1, 0, 0, \ldots)$, and so

$$\theta_1 = \Phi_1(\gamma, v) = \sum_{j=1}^{\infty} \gamma_j = 1.$$

Hence we obtain the required contradiction, at least in the case where $p = 1$.

Now consider the case where $p > 1$. Again we should like to show that $\theta_p(\eta) > 0$ for suitable $\eta$. The above method for the case that $p = 1$ does not now work; indeed, the minimum value $\min \{ \Phi_p(\eta, u) : u \in S_d \}$ need not
occur at the point \( u = (1, 0, 0, \ldots) \), and in fact, perhaps surprisingly, it does not necessarily occur at a decreasing sequence \( u \) of \( S_d \). In fact we cannot explicitly calculate \( \theta_p(\eta) \), but we obtain a lower bound by the use of Hölder’s inequality.

With \( \frac{1}{p} + \frac{1}{q} = 1 \), choose \( \alpha > 0 \) so small that \( 1 - \frac{p\alpha}{q} > 1/2 \). Then we have

\[
\delta = \sum_{j=1}^{\infty} j \gamma_j^{1+\alpha} < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \gamma_j^{1-p\alpha/q} < \infty,
\]

and so, in particular, the formula \( \eta_j = \gamma_j^{1-p\alpha/q} (j \in \mathbb{N}) \) defines a sequence \( \eta \in \ell^1 \) which is positive and decreasing.

Note that

\[
\frac{1 + \alpha}{q} + \left(1 - \frac{p\alpha}{q}\right) \frac{1}{p} = \frac{p + q}{pq} = 1
\]

and that \( \gamma_j = \eta_j^{1/p} \cdot \gamma_j^{(1+\alpha)/q} \). For each \( u \in c_{00} \) with \( u_1 = 1 \) we apply Hölder’s inequality to the sequence \( (x_r y_r) \), where \( (x_r) \) has generic term \( \eta_j^{1/p}|u_{2i-1} - u_{2j}| \) or \( \eta_j^{1/p}|u_{2i} - u_{2j-1}| \), and \( (y_r) \) has the corresponding generic term \( \gamma_j^{(1+\alpha)/q} \). Thus we obtain

\[
1 \leq \Phi_1(\gamma, u) = \sum_{j=1}^{\infty} \gamma_j \sum_{i=1}^{j} |u_{2i-1} - u_{2j}| + \sum_{j=2}^{\infty} \gamma_j \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}|
\]

\[
\leq \left( \sum_{j=1}^{\infty} \sum_{i=1}^{j} \gamma_j^{1+\alpha} + \sum_{j=2}^{\infty} \sum_{i=1}^{j-1} \gamma_j^{1+\alpha} \right)^{1/q} \Phi_p(\eta, u)^{1/p}
\]

\[
\leq (2\delta)^{1/q} \Phi_p(\eta, u)^{1/p}.
\]

It follows that \( \theta_p(\eta) \geq (2\delta)^{-p/q} > 0 \), as required.

Thus we have the following result.

**Theorem 4.1.** The Banach sequence algebras \( \ell^p(\mathbb{N}) \), \( 1 \leq p < \infty \), are not approximately amenable.

It is immediate that \( \ell^p(S) \) is not approximately amenable for any infinite set \( S \), since there is a continuous epimorphism \( \ell^p(S) \to \ell^p(\mathbb{N}) \).

Take \( 1 \leq p < \infty \). In [3, Corollary 7.1] it was shown that the Banach algebras \( \ell^p \) are essentially amenable, that is, any derivation into the dual of a neo-unital bimodule is inner. From Theorem 4.1 we conclude that essential amenability does not imply approximate amenability. It also follows by the Plancherel theorem that \( L^2(\mathbb{T}) \) fails to be approximately amenable, though by [6, Theorem 4.5] it is pseudo-contractible, that is, it admits a central (unbounded) approximate diagonal.

We finally consider a weighted variant of the \( \ell^p \) algebras.
Let $\omega \in [1, \infty)^N$. For $p \geq 1$, we consider
\[ \ell^p(\omega) = \{ f \in \mathbb{C}^N : f \cdot \omega \in \ell^p \}, \]
where $f \cdot \omega$ denotes the sequence with the $i$th coordinate $(f \cdot \omega)(i) = f_i \omega_i$ ($i \in \mathbb{N}$). With the norm
\[ \| f \|_{p,\omega} = \| f \cdot \omega \|_p \quad (f \in \ell^p(\omega)), \]
$\ell^p(\omega)$ is a Banach algebra under pointwise operations. As previously, the map $T: \ell^p(\omega) \times \ell^p(\omega) \to \ell^p(\omega \otimes \omega) = \ell^p(\omega \otimes \omega, \mathbb{N} \times \mathbb{N})$ given by
\[ T(x, y)(i, j) = x_i y_j \quad (x, y \in \ell^p(\omega), i, j \in \mathbb{N}) \]
defines a contractive operator $\hat{T}: \ell^p(\omega) \hat{\otimes} \ell^p(\omega) \to \ell^p(\omega \otimes \omega)$, where $\omega \otimes \omega$ denotes the weight on $\mathbb{N} \times \mathbb{N}$ such that $\omega \otimes \omega(i, j) = \omega_i \omega_j$ ($i, j \in \mathbb{N}$). As for the case of $\ell^p$, we aim to show that for some $\varepsilon > 0$ and elements $a, b \in \ell^p(\omega)$, there is no $F \in c_{00} \otimes c_{00}$ such that both the following inequalities are true:
\[ \| \Delta_a(F) \|_{p,\omega \otimes \omega} + \| \Delta_b(F) \|_{p,\omega \otimes \omega} < \varepsilon; \]
\[ \| a - \pi(F)a \|_{p,\omega} + \| b - \pi(F)b \|_{p,\omega} < \varepsilon. \]

We take $\gamma = (\gamma_i)$ and $\eta = (\eta_i)$ the same as in the proof of Theorem 4.1. Set
\[ \eta_j' = \frac{\eta_j}{\omega^p_{2j-1}}, \quad \eta_j'' = \frac{\eta_j}{\omega^p_{2j}}, \quad (j \in \mathbb{N}), \]
and define
\[ a = \sum_{j=1}^{\infty} (\eta_j')^{1/p} \delta_{2j-1}, \quad b = \sum_{j=1}^{\infty} (\eta_j'')^{1/p} \delta_{2j}, \]
so that $a, b \in \ell^p(\omega)$. Now for $F \in c_{00} \otimes c_{00}$ and $u = \pi(F)$, following the same argument as in the proof of Theorem 4.1, we find that
\[ 2^p(\| \Delta_a(F) \|_{p,\omega \otimes \omega} + \| \Delta_b(F) \|_{p,\omega \otimes \omega})^p \geq \Phi_p(\eta, u), \]
where $\Phi_p(\eta, u)$ is given by equation (4.5). This finally shows that the value of $\| \Delta_a(F) \|_{p,\omega \otimes \omega} + \| \Delta_b(F) \|_{p,\omega \otimes \omega}$ is bounded away from 0 as a function of $F \in c_{00} \otimes c_{00}$. We therefore conclude with the following theorem.

**Theorem 4.2.** The Banach sequence algebras $\ell^p(\omega)$, $1 \leq p < \infty$, are not approximately amenable for any weight $\omega$. \[ \blacksquare \]

**References**


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