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Maximal regularity of second-order evolution equations with infinite delay in Banach spaces

by

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Abstract. By using Fourier multiplier theorems we characterize the existence and uniqueness of periodic solutions for a class of second-order differential equations with infinite delay. We also establish maximal regularity results for the equations in various spaces. An example is provided to illustrate the applications of the results obtained.

1. Introduction. In this article, we are concerned with the maximal regularity and existence of periodic solutions for the following abstract second-order differential equation with infinite delay in a Banach space X:

$$u''(t) + Bu'(t) + Au(t) = Gu'_t + Fu_t + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where A and B are two closed linear operators defined on X with domains $D(A)$ and $D(B)$, respectively; $u(t)$ is the state function taking values in X, and the historical function $u_t : (-\infty, 0] \to X$, given as usual by $u_t(\theta) = u(t + \theta)$ for $\theta \le 0$, belongs to some abstract phase space $\mathscr{B}$ defined axiomatically below; F and G are bounded linear operators from $\mathscr{B}$ to X.

The theory of maximal regularity for linear evolution equations in abstract spaces is a very useful tool in the study of solutions to nonlinear partial differential equations. See, for instance, Poblete [P] and Denk, Hieber and Prüss [DHR] for more details. Initiated by Weis [W2] (see also [W1]), the operator-valued Fourier multiplier techniques have been successfully used in the investigation of maximal regularity for abstract differential equations. Some recent results on vector-valued Fourier multipliers in abstract spaces (Marcinkiewicz-type theorems), established by Arendt, Batty, Bu and Kim [AB1, AB2, BK], enable us to obtain characterizations of maximal regularity of solutions for abstract equations with periodic boundary conditions in L^p, Besov and Triebel–Lizorkin spaces. In fact, several results characterized the existence and uniqueness of solutions for several classes

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of equations in L^p spaces, Lebesgue spaces, Besov spaces, Triebel–Lizorkin spaces and Hölder spaces, have been obtained in the last years: see Keyntuo and Lizama [KL1, KL2, KL3], Lizama and Poblete [LP2] and Sforza [S] for integro-differential equations, and Chill and Srivastava [CS] for second-order differential equations without delay.

Motivated by the fact that abstract retarded functional differential equations (abbreviated ARFDE) with delay arise in many areas of applied mathematics, this type of equations has received much attention in recent years. In particular, the problem of existence of periodic, almost periodic and asymptotically almost periodic solutions has been considered by several authors. We refer to the books [HNMS, LNV] for information on this subject. The maximal regularity problem for delay differential equations has also been studied by many authors via the vector-valued Fourier multiplier approach. Lizama [L] has studied the first order finite delay equation

$$u'(t) = Au(t) + Fu_t + f(t), \qquad t \in \mathbb{R}; \tag{1.2}$$

he characterized the existence and uniqueness of periodic solutions of this inhomogeneous abstract equation and established maximal regularity results for strong solutions in L^p spaces. Some necessary and sufficient conditions for (1.2) to have C^α-maximal regularity on the real line have also been explored in Lizama and Poblete [LP1]. Moreover, Bu and Fang [BF2] have considered the problem in Besov spaces and Triebel–Lizorkin spaces. Poblete [P] characterized the well-posedness in Hölder spaces for equation (1.1) with finite delay under the condition that X is a B-convex space. In addition, Henríquez and Lizama [HL] have investigated the existence of periodic solutions for a class of abstract retarded equations with infinite delay.

The aim of this paper is to extend the results in [HL] and [P] to the second-order differential equation (1.1) with infinite delay. This equation is abstracted from many practical models. For example, some problems in viscoelasticity of materials and heat conduction with fading memory were described as partial functional differential equations with infinite delay which can be rewritten as (1.1) (see [BP, C] for more references on applications in physical problems). We are going to establish maximal regularity for (1.1) and obtain the existence of periodic solutions. We shall apply the operator-valued Fourier multiplier results established in [AB1] to study the existence of periodic solutions for (1.1) in L^p spaces. We will also discuss the maximal regularity of (1.1) with $B = \alpha I$, $\alpha \in \mathbb{C}$ in Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$. The main tools we will use are the operator-valued Fourier multiplier theorems on $B^s_{p,q}(\mathbb{T}, X)$ and $F^s_{p,q}(\mathbb{T}, X)$ obtained in [AB2] and [BK]. It is known that for $0 < \alpha < 1$, the periodic α-Hölder continuous function space $C^\alpha(\mathbb{T}, X)$ coincides with $B^s_{\infty,\infty}(\mathbb{T}, X)$.

Thus actually our result gives necessary and sufficient conditions for the problem (1.1) to have C^{α}-maximal regularity.

Clearly, our results extend those in [BF2], [L] and [P] to the case of infinite delay equations. We point out here that, other than the work in [HL], our discussion is not based on the existing corresponding results for equations with finite delay—we study the problem directly via operator-valued Fourier multiplier arguments by introducing a natural axiom (C_2) for the phase space $\mathscr{B}$ (see Section 2), which weakens the conditions on the operators F and G. It is also worth mentioning that the integro-differential equations considered in [BF1, KL1, KL2, LP2] are special cases of (1.1), hence our results extend the corresponding theorems in those references as well.

The paper is organized as follows. Section 2 collects some results about operator-valued Fourier multipliers, R-boundedness and the phase space $\mathscr{B}$. Section 3 is devoted to L^p-maximal regularity for (1.1). In Section 4 we consider $B^s_{p,q}$-maximal regularity and $F^s_{p,q}$-maximal regularity for (1.1) when $B = \alpha I$, $\alpha \in \mathbb{C}$. Finally, in Section 5, we present an example to illustrate the applications of the results obtained.

2. Preliminaries. We denote by $\mathbb{T}$ the group $\mathbb{R}/2\pi\mathbb{Z}$. There is an obvious identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$; the interval $[0, 2\pi]$ is a model for $\mathbb{T}$.

Given $1 \le p < \infty$, define $L^p(\mathbb{T}, X)$ as the space of all Bochner measurable vector-valued, p-integrable functions on $\mathbb{T}$. For a function $f \in L^1(\mathbb{T}, X)$, the kth Fourier coefficient of f is denoted by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t)\, dt$$

for all $k \in \mathbb{Z}$ and $t \in \mathbb{T}$. In what follows, for $f \in L^1(\mathbb{T}, X)$, it will always be understood that f is periodically extended to the left onto the interval $(-\infty, 0]$. In this way, for the functional $f_t(\theta) := f(t+\theta)$, $t \in \mathbb{T}$, $\theta \le 0$, the kth Fourier coefficient in t is then given by $\hat{f}_t(k) = e^{ik\theta}\hat{f}(k)$.

Let $f \in L^p(\mathbb{T}, X)$. Then by Fejér's theorem, one has

$$f = \lim_{n\to\infty} \sigma_n(f), \tag{2.1}$$

where

$$\sigma_n(f) := \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \hat{f}(k)$$

with $e_k(t) := e^{ikt}$.

We now introduce the UMD spaces. Since we will just use some results from the literature, it is enough for us to present a simple definition. A Banach space X is said to be UMD if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1, \infty)$.

Next we recall the basic concepts necessary to obtain our results. Let X and Y be Banach spaces. We denote by $\mathscr{L}(X, Y)$ the Banach space of bounded linear operators from X into Y, and write $\mathscr{L}(X)$ in the case $X = Y$.

For $j \in \mathbb{N}$, denote by r_j the jth Rademacher function on $[0, 1]$, i.e. $r_j(t) = \operatorname{sgn}(\sin(2^j \pi t))$. For $x \in X$ denote by $r_j \otimes x$ the vector-valued function $t \mapsto r_j(t)x$.

Definition 2.1. Let X and Y be Banach spaces, and let $p \in [1, \infty)$. A family $\mathcal{T} \subset \mathscr{L}(X, Y)$ of operators is called *R-bounded* if there is a constant $C > 0$ such that

$$\Big\| \sum_{j=1}^{N} r_j \otimes T_j x_j \Big\|_{L^p(0,1,Y)} \le C \Big\| \sum_{j=1}^{N} r_j \otimes x_j \Big\|_{L^p(0,1,X)}$$

for all $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$. The smallest such C is called the *R-bound* of $\mathcal{T}$ and denoted by $R_p(\mathcal{T})$.

We remark that large classes of classical operators are R-bounded (see [GW] and references therein). Hence, this assumption is not too restrictive for the applications that we consider in this article.

Remark 2.2. Several properties of R-bounded families can be found in the recent monograph of Denk et al. [DHR]. For the reader's convenience, we summarize some results here.

(a) If $\mathcal{T} \subset \mathscr{L}(X, Y)$ is R-bounded, then it is uniformly bounded with

$$\sup\{\|T\| : T \in \mathcal{T}\} \le R_p(\mathcal{T}).$$

(b) The definition of R-boundedness is independent of $p \in [1, \infty)$.

(c) When X and Y are Hilbert spaces, $\mathcal{T} \subset \mathscr{L}(X, Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.

(d) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathscr{L}(X, Y)$ be R-bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T},\ S \in \mathcal{S}\}$$

is R-bounded as well, and $R_p(\mathcal{T} + \mathcal{S}) \le R_p(\mathcal{T}) + R_p(\mathcal{S})$.

(e) Let X, Y, Z be Banach spaces, and $\mathcal{T} \subset \mathscr{L}(X, Y)$ and $\mathcal{S} \subset \mathscr{L}(Y, Z)$ be R-bounded. Then

$$\mathcal{S}\mathcal{T} = \{ST : T \in \mathcal{T},\ S \in \mathcal{S}\}$$

is R-bounded, and $R_p(\mathcal{S}\mathcal{T}) \le R_p(\mathcal{S}) R_p(\mathcal{T})$.

(f) In particular, each subset $M \subset \mathscr{L}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R-bounded whenever $\Omega \subset \mathbb{C}$ is bounded. This follows from Kahane's contraction principle (see [AB1, Lemma 1.7]).

Now we briefly recall the definition of periodic Besov and Triebel–Lizorkin spaces in the vector-valued case, introduced in [AB2] and [BK]. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$, and $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorm $\|f\|_\alpha = \sup_{x\in\mathbb{T}} |f^{(\alpha)}(x)|$ for $\alpha \in \mathbb{N}$. Let $\mathcal{D}'(\mathbb{T}, X) := \mathscr{L}(\mathcal{D}(\mathbb{T}), X)$ be the space of all bounded linear operators from $\mathcal{D}(\mathbb{T})$ to X. In order to define Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : |t| \le 2\}, \qquad I_k = \{t \in \mathbb{R} : 2^{k-1} < |t| \le 2^{k+1}\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ satisfying $\operatorname{supp}(\phi_k) \subset \bar{I}_k$ for each $k \in \mathbb{N}$,

$$\sum_{k\in\mathbb{N}} \phi_k(x) = 1 \quad \text{ for } x \in \mathbb{R},$$

and for each $\alpha \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R},\, k\in\mathbb{N}} 2^{k\alpha} |\phi_k^{(\alpha)}(x)| < \infty.$$

Let $\phi = (\phi_k)_{k\in\mathbb{N}} \in \phi(\mathbb{R})$ be fixed. For $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, the *X-valued periodic Besov space* is defined by

$$B^s_{p,q}(\mathbb{T}, X) = \Big\{ f \in \mathcal{D}'(\mathbb{T}, X) : \\ \|f\|_{B^s_{p,q}} := \Big(\sum_{j\ge 0} 2^{sjq} \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \hat{f}(k) \Big\|_p^q \Big)^{1/q} < \infty \Big\},$$

and for $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$, the *X-valued periodic Triebel–Lizorkin space* is defined by

$$F^s_{p,q}(\mathbb{T}, X) = \Big\{ f \in \mathcal{D}'(\mathbb{T}, X) : \\ \|f\|_{F^s_{p,q}} := \Big\| \Big(\sum_{j\ge 0} 2^{sjq} \Big| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \hat{f}(k) \Big|^q \Big)^{1/q} \Big\|_p < \infty \Big\}$$

with the usual modification if $q = \infty$. The spaces $B^s_{p,q}(\mathbb{T}, X)$ and $F^s_{p,q}(\mathbb{T}, X)$ are independent of the choice of ϕ, and different choices of ϕ lead to equivalent norms $\|\cdot\|_{B^s_{p,q}}$, $\|\cdot\|_{F^s_{p,q}}$ on $B^s_{p,q}(\mathbb{T}, X)$ and $F^s_{p,q}(\mathbb{T}, X)$ respectively. All these spaces are Banach spaces. See [AB2, BK] for more information.

The theory of Fourier multipliers plays an important role in the whole paper. Here we collect the definitions and some basic results of this theory.

Definition 2.3. Let X and Y be Banach spaces and let $\Gamma(\mathbb{T}, X)$ be one of the following X-valued function spaces: $L^p(\mathbb{T}, X)$ $(1 \le p < \infty)$, $B^s_{p,q}(\mathbb{T}, X)$ $(1 \le p, q \le \infty,\ s \in \mathbb{R})$, $F^s_{p,q}(\mathbb{T}, X)$ $(1 \le p < \infty,\ 1 \le q \le \infty,\ s \in \mathbb{R})$. We say

$\{M_k\}_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ is a *Γ-multiplier* if for each $f \in \Gamma(\mathbb{T},X)$ there exists $u \in \Gamma(\mathbb{T},X)$ such that $\hat{u}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$.

From the uniqueness theorem for Fourier series, it follows that u is uniquely determined by f.

REMARK 2.4. It is clear from the definition that if $\{M_k\}_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ and $\{N_k\}_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ are Fourier multipliers, then $\{M_kN_k\}_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ is a Fourier multiplier as well.

The following theorems are due to Arendt and Bu [AB1, Theorem 1.3], Arendt and Bu [AB2, Theorem 4.5] and Bu and Kim [BK, Theorem 3.2] respectively.

THEOREM 2.5. *Let X, Y be UMD spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.*

THEOREM 2.6. *Let X, Y be Banach spaces, let $1 \le p,q \le \infty$ and $s \in \mathbb{R}$, and let $\{M_k\}_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$. Assume the so-called Marcinkiewicz condition of order two holds:*

$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k(M_{k+1}-M_k)\|) < \infty, \tag{2.2}$$

$$\sup_{k\in\mathbb{Z}}\|k^2(M_{k+2}-2M_{k+1}+M_k)\| < \infty. \tag{2.3}$$

Then $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Moreover, if X and Y are B-convex, then the first order condition (2.2) *is sufficient for $\{M_k\}_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-multiplier.*

THEOREM 2.7. *Let X, Y be Banach spaces, let $1 \le p < \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$, and let $\{M_k\}_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$. Assume that the Marcinkiewicz condition of order three holds:*

$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k(M_{k+1}-M_k)\| + \|k^2(M_{k+2}-2M_{k+1}+M_k)\|) < \infty, \tag{2.4}$$

$$\sup_{k\in\mathbb{Z}}\|k^3(M_{k+3}-3M_{k+2}+3M_{k+1}-M_k)\| < \infty. \tag{2.5}$$

Then $\{M_k\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier. Moreover, if $1 < p < \infty$ and $1 < q \le \infty$, then the first condition (2.4) *is sufficient for $\{M_k\}_{k\in\mathbb{Z}}$ to be an $F^s_{p,q}$-multiplier.*

Recall that a Banach space X has *Fourier type* p, with $1 \le p \le 2$, if the Fourier transform defines a bounded linear operator from $L^p(\mathbb{R},X)$ to $L^q(\mathbb{R},X)$, where q is the conjugate index of p. As examples, the space $L^p(\Omega)$

with $1 \le p \le 2$ has Fourier type p; a Banach space X has Fourier type 2 if and only if X is isomorphic to a Hilbert space; X has Fourier type p if and only if X^* has Fourier type p. Every Banach space has Fourier type 1; X is B-convex if it has Fourier type p for some $p > 1$. Every uniformly convex space is B-convex.

In this article, we employ an axiomatic definition of the phase space $\mathscr{B}$, that is, $\mathscr{B}$ will be a linear space of functions from $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathscr{B}}$ and satisfying the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathscr{B}$, then for every $t \in [\sigma, \sigma + a)$ the following hold:

- (i) x_t is in $\mathscr{B}$;
- (ii) $\|x(t)\| \le H\|x_t\|_{\mathscr{B}}$;
- (iii) $\|x_t\|_{\mathscr{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$. Here $H \ge 0$ is a constant, $K, M : [0,\infty) \to [1,\infty)$, $K(\cdot)$ is continuous and $M(\cdot)$ is locally bounded, and H, $K(\cdot)$, $M(\cdot)$ are independent of $x(t)$.

(A_1) For the function $x(\cdot)$ in (A), x_t is a $\mathscr{B}$-valued continuous function on $[\sigma, \sigma + a]$.

(B) The space $\mathscr{B}$ is complete.

We will also need an additional property of $\mathscr{B}$. For this, let $C_{00}(X)$ denote the space of continuous functions from $(-\infty, 0]$ to X with compact support. We assume that the following axiom holds for the phase space $\mathscr{B}$:

(C_2) If a uniformly bounded sequence $\{\varphi^n\}$ in $C_{00}(X)$ converges to a function φ uniformly on every compact set on $(-\infty, 0]$, then $\varphi \in \mathscr{B}$ and $\lim_{n\to\infty} \|\varphi^n - \varphi\|_{\mathscr{B}} = 0$.

It is known from [HMN] that if axiom (C_2) holds, then $C_b((-\infty, 0], X)$, the space of all bounded continuous functions from $(-\infty, 0]$ to X, is continuously imbedded in $\mathscr{B}$. Set

$$\|\varphi\|_\infty = \sup\{\|\varphi(\theta)\| : \theta \le 0\}.$$

Then there exists a constant $Q > 0$ such that

$$\|\varphi\|_{\mathscr{B}} \le Q\|\varphi\|_\infty, \quad \varphi \in C_b((-\infty, 0], X). \tag{2.6}$$

Here we present two examples of phase spaces satisfying the above axioms (A), (A_1), (B), and (C_2).

Example 2.8 (The phase space $C_g(X)$). Let g be a positive continuous function on $(-\infty, 0]$. Let $\mathscr{B} = C_g(X)$ consist of all continuous functions $\varphi : (-\infty, 0] \to X$ such that $\|\varphi(\theta)\|/g(\theta)$ is bounded on $(-\infty, 0]$. We assume that g satisfies conditions (g-1) and (g-2) of [HMN]:

(g-1) The function $G(t) = \sup_{\theta \le -t} \frac{g(t+\theta)}{g(\theta)}$ is locally bounded for $t \ge 0$.
(g-2) $g(\theta) \to \infty$ as $\theta \to -\infty$.

With the seminorm defined by

$$\|\varphi\|_g = \sup_{\theta \le 0} \frac{\|\varphi(\theta)\|}{g(\theta)},$$

$\mathscr{B}$ is a phase space that satisfies axioms (A), (B) and (A_1) [HMN, Theorem 1.3.2].

Example 2.9 (The phase space $C_r \times L^p(\rho, X)$). Let $r \ge 0$ and $1 \le p < \infty$. Let $\mathscr{B} = C_r \times L^p(\rho, X)$ consist of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue measurable, and $\rho\|\varphi(\cdot)\|^p$ is Lebesgue integrable on $(-\infty, -r)$, where $\rho : (-\infty, -r) \to \mathbb{R}$ is a nonnegative Lebesgue integrable function which satisfies conditions (g-5), (g-6) of [HMN]. Briefly, this means that ρ is locally integrable and there exists a nonnegative, locally bounded function γ on $(-\infty, 0)$ such that $\rho(\xi + \theta) \le \gamma(\xi)\rho(\theta)$ for all $\xi \le 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subset (-\infty, r)$ is a set of Lebesgue measure zero. The seminorm in $\mathscr{B}$ is defined by

$$\|\varphi\|_{\mathscr{B}} = \sup\{\varphi(\theta) : -r \le \theta \le 0\} + \Big(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p \, d\theta\Big)^{1/p}.$$

The space $\mathscr{B}$ satisfies axioms (A), (B) and (A_1). Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + (\int_{-\infty}^{-r} \rho(\theta)\, d\theta)^{1/2}$ for $t \ge 0$. See [HMN, Theorem 1.3.8] for details.

In particular, if the function g satisfies conditions (g-1) and (g-2), then the space $C_g(X)$ defined in Example 2.8 satisfies axiom (C_2) [HMN, Theorem 1.3.2]. In a similar way, if $\int_{-\infty}^{-r} \rho(\theta)\, d\theta < \infty$, then $C_r \times L^p(\rho, X)$ also satisfies axiom (C_2).

3. Maximal regularity in L^p space. In this section, we establish some maximal regularity results for equation (1.1). For this, we assume that A and B are closed linear operators with $D(A) \cap D(B) \ne \emptyset$. Denote the domain of $A + B$ by $[D(A) \cap D(B)]$, endowed with the graph norm $\|x\|_{[D(A)\cap D(B)]} = \|x\| + \|Ax\| + \|Bx\|$, so that it becomes a Banach space. In this section we always suppose that the pair (A, B) is *coercive*, that is, for all $t > 0$, the operator $A + tB$, with domain $[D(A) \cap D(B)]$, is closed, and there is a constant $M > 0$ such that

$$\|Ax\| + t\|Bx\| \le M\|Ax + tBx\|$$

for all $x \in [D(A) \cap D(B)]$. The hypothesis of coercivity originates from [LP2] and has also been used in [Pr] and [So], among others.

To discuss the existence of periodic solutions for equation (1.1) we adopt the following notation:

$$\begin{aligned} &H^{1,p}(\mathbb{T},X) \\ &\quad = \{u \in L^p(\mathbb{T},X) : \exists v \in L^p(\mathbb{T},X),\ \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\} \\ &\quad = \{u \in L^p(\mathbb{T},X) : u \text{ is differentiable a.e., } u' \in L^p(\mathbb{T},X), \\ &\qquad\qquad \text{and } u(0) = u(2\pi)\}, \\ &H^{2,p}(\mathbb{T},X) \\ &\quad = \{u \in L^p(\mathbb{T},X) : \exists v \in L^p(\mathbb{T},X),\ \hat{v}(k) = -k^2\hat{u}(k) \text{ for all } k \in \mathbb{Z}\} \\ &\quad = \{u \in L^p(\mathbb{T},X) : u \text{ is twice differentiable a.e., } u', u'' \in L^p(\mathbb{T},X), \\ &\qquad\qquad \text{and } u(0) = u(2\pi),\ u'(0) = u'(2\pi)\}. \end{aligned}$$

If $u \in H^{1,p}(\mathbb{T},X)$, it follows from [AB1, Lemma 2.1] that u has a unique continuous representative, and we always identify u with this continuous function. Thus

$$u(t) = u(0) + \int_0^t v(s)\,ds \qquad (t \in \mathbb{T})$$

and $u(0) = u(2\pi)$ for all $u \in H^{1,p}(\mathbb{T},X)$.

Now we present the concept of strong L^p-solution for (1.1). As mentioned in Section 2, any function $u \in L^p(\mathbb{T},X)$ is extended 2π-periodically to the interval $(-\infty,0]$ so that $u_t(\cdot)$ makes sense. Moreover, if $u \in H^{2,p}(\mathbb{T},X)$, we immediately get $u \in C^1(\mathbb{T},X)$, so $u_t(\cdot), u_t'(\cdot) \in C_b((-\infty,0],X) \subset \mathscr{B}$, and hence the definition below is meaningful.

Definition 3.1. Let X be a Banach space, and A, B be closed linear operators on X.

(i) A function $u \in H^{2,p}(\mathbb{T},X)$ is defined to be a *strong L^p-solution* of (1.1) if $u \in D(A) \cap D(B)$ and (1.1) holds for a.e. $t \in [0,2\pi)$, and $u'', Au + Bu', Fu_t, Gu_t' \in L^p(\mathbb{T},X)$.

(ii) Equation (1.1) is said to have *L^p-maximal regularity* if for every $f \in L^p(\mathbb{T},X)$, there exists a unique strong L^p-solution of (1.1).

Denote $e_\lambda(\theta) := e^{i\lambda\theta}$ for all $\lambda \in \mathbb{R}$, $\theta \le 0$, and define the operators $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ and $\{G_\lambda\}_{\lambda\in\mathbb{R}}$ by

$$F_\lambda x := F(e_\lambda x) \quad \text{and} \quad G_\lambda x := G(e_\lambda x), \qquad \text{for all } \lambda \in \mathbb{R},\ x \in X.$$

Since $e^{i\lambda\theta}x \in C_b((-\infty,0],X) \subset \mathscr{B}$, by (2.6) one has

$$\|F_\lambda x\| = \|F(e_\lambda x)\| \le \|F\|\,\|e^{i\lambda\theta}x\|_{\mathscr{B}} \le Q\|F\|\,\|x\|,$$

from which we deduce that $\{F_\lambda\}_{\lambda\in\mathbb{R}} \subset \mathscr{L}(X)$, and similarly $\{G_\lambda\}_{\lambda\in\mathbb{R}} \subset \mathscr{L}(X)$.

We denote

$$\sigma_{\mathbb{Z}}(\triangle) = \{k \in \mathbb{Z} : -k^2 I + ikB + A - ikG_k - F_k \text{ is not invertible from } D(A) \cap D(B) \text{ to } X\}$$

and

$$N_k = (-k^2 I + ikB + A - ikG_k - F_k)^{-1}, \quad M_k = -k^2 N_k, \quad k \in \mathbb{Z}.$$

We need the following preparation.

Lemma 3.2. *Let A and B be closed linear operators defined on a UMD space X, let $1 < p < \infty$, and suppose $\mathscr{B}$ satisfies (A)–(C_2). Moreover, suppose that $\sigma_{\mathbb{Z}}(\triangle) = \emptyset$. Then the following assertions are equivalent:*

(i) *$\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.*
(ii) *$\{M_k\}_{k\in\mathbb{Z}}$ is R-bounded and $\{ikBN_k\}_{k\in\mathbb{Z}}$ is uniformly bounded.*

Proof. From [AB1, Proposition 1.11] it follows that (i) implies $\{M_k\}_{k\in\mathbb{Z}}$ is R-bounded, so $N_k = \big(-\frac{1}{k^2}\big)M_k$ and $ikN_k = \frac{1}{ik}M_k$ are R-bounded as well, since $\big\{-\frac{1}{k^2}I\big\}_{k\in\mathbb{Z}}$ and $\big\{\frac{1}{ik}I\big\}_{k\in\mathbb{Z}}$ are R-bounded by Remark 2.2. Moreover, $\big\{-\frac{1}{k^2}I\big\}_{k\in\mathbb{Z}}$ and $\big\{\frac{1}{ik}I\big\}_{k\in\mathbb{Z}}$ are L^p-multipliers by [AB1, Lemma 1.3]. From the definition of N_k, it is obvious that

$$(A + ikB)N_k = k^2 N_k + ikG_k N_k + F_k N_k + I.$$

Thus $\{(A + ikB)N_k\}_{k\in\mathbb{Z}}$ is R-bounded, and hence uniformly bounded. Because (A, ikB) is a coercive pair, there is a constant $K > 0$ such that $\|AN_k x\| + \|ikBN_k x\| \le K\|(A + ikB)N_k x\|$ for all $x \in X$. This proves (ii).

Conversely, in view of Theorem 2.5 it is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. We first prove $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ are R-bounded. For any given $x_j \in X$, from Definition 2.1 one has

$$\begin{aligned}
\Big\|\sum_{j=1}^{N} r_j \otimes F_j x_j\Big\|^p_{L^p(0,1,X)} &= \int_0^1 \Big\|\sum_{j=1}^{N} r_j(t) F(e_j x_j)\Big\|^p_X dt \\
&= \int_0^1 \Big\|F\Big(\sum_{j=1}^{N} r_j(t) e_j x_j\Big)\Big\|^p_X dt \le \|F\|^p \int_0^1 \Big\|\sum_{j=1}^{N} r_j(t) e_j x_j\Big\|^p_{\mathscr{B}} dt \\
&\le Q^p \|F\|^p \int_0^1 \Big\|\sum_{j=1}^{N} r_j(t) x_j\Big\|^p_X dt = Q^p \|F\|^p \Big\|\sum_{j=1}^{N} r_j \otimes x_j\Big\|^p_{L^p(0,1,X)},
\end{aligned}$$

which shows

$$R_p(\{F_k\}_{k\in\mathbb{Z}}) \le Q\|F\|,$$

i.e. $\{F_k\}_{k\in\mathbb{Z}}$ is R-bounded. Similarly, $\{G_k\}_{k\in\mathbb{Z}}$ is R-bounded too.

Furthermore, since $M_k = -k^2 N_k$, $k \in \mathbb{Z}$, it is easy to calculate that

$$k(M_{k+1} - M_k) = -k^3(N_{k+1} - N_k) - 2k^2 N_{k+1} - kN_{k+1}.$$

We already know that $M_k = k^2 N_k$ and $kN_k = -(k^2 N_k)/ik$ are also R-bounded, so it suffices to verify that $\{k^3(N_{k+1} - N_k)\}_{k\in\mathbb{Z}}$ is R-bounded. Indeed,

$$\begin{aligned} k^3(N_{k+1} - N_k) &= k^3 N_{k+1}(N_k^{-1} - N_{k+1}^{-1})N_k \\ &= k^3 N_{k+1}[(1+2k)I + i(k+1)G_{k+1} - ikG_k + F_{k+1} - F_k - iB]N_k \\ &= kN_{k+1}(k^2 N_k) + k^2 N_{k+1}[2I + i(G_{k+1} - G_k)](k^2 N_k) + ikN_{k+1}G_{k+1}(k^2 N_k) \\ &\quad + kN_{k+1}(F_{k+1} - F_k)(k^2 N_k) - k^2 N_{k+1}(ikBN_k). \end{aligned}$$

Since products and sums of R-bounded sequences are still R-bounded (cf. Remark 2.2), we get assertion (i) by Theorem 2.5. ■

Remark 3.3. We note that the coercivity condition on the pair (A, B) was used only in the implication (i)⇒(ii).

The main result of this section is the following theorem.

Theorem 3.4. *Let A and B be closed linear operators defined on a UMD space X and $1 < p < \infty$. Suppose $\mathscr{B}$ satisfies (A)–(C_2). Then the following assertions are equivalent:*

(i) *Equation (1.1) has L^p-maximal regularity.*
(ii) *$\sigma_{\mathbb{Z}}(\triangle) = \emptyset$, and $\{M_k\}_{k\in\mathbb{Z}}$ is R-bounded and $\{ikBN_k\}_{k\in\mathbb{Z}}$ is uniformly bounded.*

Proof. (i)⇒(ii). Let $k \in \mathbb{Z}$ and $y \in X$. Define $f(t) = e_k(t)y$. Since $f \in L^p(\mathbb{T}, X)$, by hypothesis there exists $u \in H^{2,p}(\mathbb{T}, X)$ with $u \in D(A) \cap D(B)$ such that

$$u''(t) + Bu'(t) + Au(t) = Gu'_t + Fu_t + f(t), \quad t \in \mathbb{T}. \tag{3.1}$$

By the linearity of F and G we have

$$(Fu_t)^\wedge(k) = F(e^{ik\theta}\hat{u}(k)) = F_k\hat{u}(k)$$

and

$$(Gu'_t)^\wedge(k) = G(e^{ik\theta}\hat{u'}(k)) = G(e^{ik\theta}ik\hat{u}(k)) = ikG_k\hat{u}(k).$$

Then taking the Fourier transform on both sides of (3.1) and noting $\hat{u}(k) \in D(A) \cap D(B)$, we obtain

$$\begin{aligned} -k^2\hat{u}(k) + ikB\hat{u}(k) + A\hat{u}(k) &= ikG_k\hat{u}(k) + F_k\hat{u}(k) + \hat{f}(k) \\ &= ikG_k\hat{u}(k) + F_k\hat{u}(k) + y. \end{aligned} \tag{3.2}$$

So $-k^2 I + ikB + A - ikG_k - F_k$ is surjective for all $k \in \mathbb{Z}$.

Let $x \in D(A) \cap D(B)$. If $(-k^2 I + ikB + A - ikG_k - F_k)x = 0$, that is, $(-k^2 I + ikB + A)x = (ikG_k + F_k)x$, then it is easy to check that $u(t) = e^{ikt}x$ defines a periodic solution of (1.1) with $f \equiv 0$. In fact, since $u_t(\theta) = e^{ik\theta}u(t)$,

$u_t'(\theta) = e^{ik\theta}u'(t) = ike^{ik\theta}u(t)$, we see that $F(u_t) = F(e_k u(t)) = F_k(u(t))$, and $G(u_t') = ikG(e_k u(t)) = ikG_k(u(t))$. Hence

$$\begin{aligned} u''(t) + Bu'(t) + Au(t) &= (-k^2 I + ikB + A)e^{ikt}x \\ &= (ikG_k + F_k)e^{ikt}x = Gu_t' + Fu_t, \end{aligned}$$

which implies $u \equiv 0$ by the assumption of uniqueness, and hence $x = 0$. These arguments show that $\sigma_{\mathbb{Z}}(\triangle) = \emptyset$.

To prove the second part of (ii), by Lemma 3.2, it is sufficient to show that the set $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. Let $f \in L^p(\mathbb{T}, X)$ and u be the corresponding unique strong solution. Then (3.2) easily yields

$$-k^2\hat{u}(k) = -k^2(-k^2 I + ikB + A - ikG_k - F_k)^{-1}\hat{f}(k).$$

On the other hand, since $u \in H^{2,p}(\mathbb{T}, X)$, there exists $v \in L^p(\mathbb{T}, X)$ such that $\hat{v}(k) = -k^2\hat{u}(k)$. This proves the claim.

(ii)⇒(i). Let $f \in L^p(\mathbb{T}, X)$. From Lemma 3.2, the family $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier, so there exists $v \in L^p(\mathbb{T}, X)$ such that

$$\hat{v}(k) = M_k\hat{f}(k). \tag{3.3}$$

By Remark 2.4, $N_k = (-1/k^2)M_k$ is also an L^p-multiplier, thus there exists $u \in L^p(\mathbb{T}, X)$ such that

$$\hat{u}(k) = N_k\hat{f}(k). \tag{3.4}$$

Combining (3.3) and (3.4) gives

$$\hat{v}(k) = -k^2\hat{u}(k) = -k^2 N_k\hat{f}(k). \tag{3.5}$$

This implies $u \in H^{2,p}(\mathbb{T}, X)$ and clearly $\hat{u}(k) \in D(A)\cap D(B)$. It follows from [AB1, Lemma 2.1] that $u(\cdot)$ is twice differentiable and $u''(\cdot), u'(\cdot) \in L^p(\mathbb{T}, X)$.

We claim that the families $\{F_k N_k\}_{k\in\mathbb{Z}}$ and $\{ikG_k N_k\}_{k\in\mathbb{Z}}$ are both L^p-multipliers. In fact, it is clear that $\{F_k N_k\}_{k\in\mathbb{Z}}$ and $\{ikG_k N_k\}_{k\in\mathbb{Z}}$ are R-bounded. On the other hand, since $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ are R-bounded (cf. the proof of Lemma 3.2), the identities

$$k(F_{k+1}N_{k+1} - F_k N_k) = F_{k+1}(kN_{k+1}) - F_k(kN_k) \tag{3.6}$$

and

$$k[i(k+1)G_{k+1}N_{k+1} - ikG_k N_k] = iG_{k+1}(k^2 N_{k+1}) - iG_k(k^2 N_k) + ikG_{k+1}N_{k+1}$$

show that $\{k(F_{k+1}N_{k+1} - F_k N_k)\}_{k\in\mathbb{Z}}$ and $\{k[i(k+1)G_{k+1}N_{k+1} - ikG_k N_k]\}_{k\in\mathbb{Z}}$ are R-bounded as well. Hence from Theorem 2.5 we infer immediately that $\{F_k N_k\}_{k\in\mathbb{Z}}, \{ikG_k N_k\}_{k\in\mathbb{Z}}$ are both L^p-multipliers. Also note that by (3.4),

$$ikG_k\hat{u}(k) = ikG_k N_k\hat{f}(k), \quad F_k\hat{u}(k) = F_k N_k\hat{f}(k), \quad k \in \mathbb{Z},$$

from which we get $Gu_t', Fu_t \in L^p(\mathbb{T}, X)$.

Now we show that $A\hat{u}(k) + ikB\hat{u}(k) \in L^p(\mathbb{T}, X)$. As above, we have $\hat{u}(k) \in D(A)\cap D(B)$, $k \in \mathbb{Z}$. Observe that $\{k^2 N_k + ikG_k N_k + F_k N_k + I\}_{k\in\mathbb{Z}}$

is an L^p-multiplier and

$$A\hat{u}(k) + ikB\hat{u}(k) = (k^2N_k + ikG_kN_k + F_kN_k + I)\hat{f}(k). \tag{3.7}$$

Then using the fact $A+ikB$ is closed we conclude that $Au+Bu' \in L^p(\mathbb{T}, X)$ (cf. [AB1, Lemma 3.1]).

Since for any $\theta \in (-\infty, 0]$, $u.(\theta) \in L^p(\mathbb{T}, X)$, by Fejér's theorem (cf. (2.1)) we have

$$u_t(\theta) = u(t+\theta) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} e^{ik\theta} \hat{u}(k),$$

$$u_t'(\theta) = u'(t+\theta) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} e^{ik\theta} \hat{u'}(k),$$

which implies that

$$u_t = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} e_k(\theta) \hat{u}(k),$$

$$u_t' = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} e_k(\theta) ik\hat{u}(k).$$

Then, since F and G are linear and bounded,

$$\begin{aligned} Fu_t &= \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} F(e_k \hat{u}(k)) \\ &= \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} F_k \hat{u}(k) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} Gu_t' &= \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} G(e_k \hat{u'}(k)) \\ &= \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} ikG_k \hat{u}(k). \end{aligned} \tag{3.9}$$

From (3.4) it is easy to get

$$-k^2\hat{u}(k) + ikB\hat{u}(k) + A\hat{u}(k) = ikG_k\hat{u}(k) + F_k\hat{u}(k) + \hat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

Therefore, from (3.8) and (3.9), and uniqueness of Fourier coefficients, it follows that (1.1) is valid for a.e. $t \in \mathbb{T}$.

Finally, to show uniqueness, let $u \in H^{2,p}(\mathbb{T}, X)$ with $u(t) \in D(A) \cap D(B)$ be such that

$$u''(t) + Bu'(t) + Au(t) = Gu_t' + Fu_t, \qquad t \in \mathbb{T}.$$

Then $\hat{u}(k) \in D(A) \cap D(B)$ and $(-k^2 I + ikB + A - ikG_k - F_k)\hat{u}(k) = 0$. Since $\sigma_{\mathbb{Z}}(\triangle) = \emptyset$, this implies $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$. ∎

Corollary 3.5. *In the setting of Theorem* 3.4, *if condition* (ii) *is fulfilled, we have* $u'', Au + Bu', Gu'_\cdot, Fu_\cdot \in L^p(\mathbb{T}, X)$. *Moreover, there exists a constant* $C > 0$ *independent of* $f \in L^p(\mathbb{T}, X)$ *such that*

$$\|u''\|_{L^p(\mathbb{T},X)} + \|Au + Bu'\|_{L^p(\mathbb{T},X)} + \|Gu'_\cdot\|_{L^p(\mathbb{T},X)} + \|Fu_\cdot\|_{L^p(\mathbb{T},X)} \le C\|f\|_{L^p(\mathbb{T},X)}. \tag{3.10}$$

4. Maximal regularity on Besov and Triebel–Lizorkin spaces. In this section we consider the $B^s_{p,q}$- and $F^s_{p,q}$-maximal regularity for the equation

$$u''(t) + \alpha u'(t) + Au(t) = Fu_t + Gu'_t + f(t), \quad t \in \mathbb{T}, \tag{4.1}$$

where A is a closed linear operator in X, $\alpha \in \mathbb{C}$, $f \in B^s_{p,q}(\mathbb{T}, X)$ (or $f \in F^s_{p,q}(\mathbb{T}, X)$) is given, and the history $u_t(\cdot) : (-\infty, 0] \to X$, given by $u_t(\theta) = u(t + \theta)$ for $\theta \le 0$, also belongs to some abstract phase space $\mathscr{B}$ defined axiomatically in Section 2. Assume $F, G : \mathscr{B} \to X$ are bounded linear operators.

In a Besov space, the definition of strong solution is given below.

Definition 4.1. Let X be a Banach space, A be a closed linear operator on X, and let $1 \le p, q \le \infty$ and $s > 0$.

(i) Given $f \in B^s_{p,q}(\mathbb{T}, X)$, a function $u \in B^{s+2}_{p,q}(\mathbb{T}, X)$ is called a *strong* $B^s_{p,q}$-*solution* of (4.1) if $u \in D(A)$ and (4.1) holds for a.e. $t \in \mathbb{T}$, $Au \in B^s_{p,q}(\mathbb{T}, X)$ and the functions Fu_t and Gu'_t also belong to $B^s_{p,q}(\mathbb{T}, X)$.

(ii) Equation (4.1) is said to have $B^s_{p,q}$-*maximal regularity* if for every $f \in B^s_{p,q}(\mathbb{T}, X)$, there exists a unique strong $B^s_{p,q}$-solution of (4.1).

Let $k \in \mathbb{Z}$. As in Section 3, the operators F_k and G_k are given by $F_k x := F(e_k x)$ and $G_k x := G(e_k x)$ for all $x \in X$. Then $F_k, G_k \in \mathscr{L}(X)$. We define the spectrum of (4.1) by

$$\sigma_{\mathbb{Z}}(\triangle) = \{k \in \mathbb{Z} : -k^2 I + ik\alpha I - ikG_k - F_k - A \text{ is not invertible from } D(A) \text{ to } X\}.$$

Since A is closed, if $k \in \mathbb{Z} \setminus \sigma_{\mathbb{Z}}(\triangle)$, then $(-k^2 I + ik\alpha I - ikG_k - F_k - A)^{-1}$ is a bounded linear operator on X. This is an easy consequence of the closed graph theorem. We will use the following notation: for $k \in \mathbb{Z}$,

$$D_k = ikG_k, \quad N_k := (-k^2 I + ik\alpha I - ikG_k - F_k - A)^{-1}, \quad M_k := -k^2 N_k.$$

We first establish the following lemma as a preparation to proving that $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier.

LEMMA 4.2. *Assume that $\sigma_{\mathbb{Z}}(\triangle) = \emptyset$, and $\{M_k\}_{k\in\mathbb{Z}}$ and $\{D_k\}_{k\in\mathbb{Z}}$ are uniformly bounded. Then*

$$\sup_{k\in\mathbb{Z}} \|k^3(N_{k+1} - N_k)\| < \infty, \tag{4.2}$$

$$\sup_{k\in\mathbb{Z}} \|k^4(N_{k+2} - 2N_{k+1} + N_k)\| < \infty. \tag{4.3}$$

Proof. For $k \in \mathbb{Z}$,

$$\begin{aligned} N_{k+1} - N_k &= N_{k+1}(N_k^{-1} - N_{k+1}^{-1})N_k \\ &= N_{k+1}[(1 + 2k - i\alpha)I + D_{k+1} - D_k + F_{k+1} - F_k]N_k. \end{aligned}$$

Thus it is easy to check that $\sup_{k\in\mathbb{Z}} \|k^3(N_{k+1} - N_k)\| < \infty$ by assumption. Further, note that for $k \in \mathbb{Z}$,

$$\begin{aligned} N_{k+2} - 2N_{k+1} + N_k &= (N_{k+2} - N_{k+1}) - (N_{k+1} - N_k) \\ &= N_{k+2}[(3 + 2k - i\alpha)I + D_{k+2} - D_{k+1} + F_{k+2} - F_{k+1}]N_{k+1} \\ &\quad - N_{k+1}[(1 + 2k - i\alpha)I + D_{k+1} - D_k + F_{k+1} - F_k]N_k \\ &= (N_{k+2} - N_k)[(3 + 2k - i\alpha)I + D_{k+2} - D_{k+1} + F_{k+2} - F_{k+1}]N_{k+1} \\ &\quad + N_{k+1}[2I + D_{k+2} - 2D_{k+1} + D_k + F_{k+2} - 2F_{k+1} + F_k]N_k. \end{aligned}$$

Hence $\sup_{k\in\mathbb{Z}} \|k^4(N_{k+2} - 2N_{k+1} + N_k)\| < \infty$. ∎

It should be seen that without the boundedness of $\{D_k\}_{k\in\mathbb{Z}}$ one can also obtain (4.2). Making use of Lemma 4.2 we can prove the following theorem.

THEOREM 4.3. *Let $A : X \supset D(A) \to X$ be a closed linear operator defined on a Banach space X. Suppose that $\sigma_{\mathbb{Z}}(\triangle) = \emptyset$ and $\{D_k\}_{k\in\mathbb{Z}}$ is uniformly bounded. Then the following assertions are equivalent:*

(i) *$\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier for all* (*or equivalently for some*) *$1 \le p, q \le \infty$ and $s \in \mathbb{R}$.*
(ii) *$\{M_k\}_{k\in\mathbb{Z}}$ is uniformly bounded.*

Proof. (i)⇒(ii) is trivially true (see [AB2]). Conversely, assume that the sequence $\{M_k\}_{k\in\mathbb{Z}}$ is uniformly bounded; we are going to show that $\{M_k\}_{k\in\mathbb{Z}}$ satisfies the Marcinkiewicz conditions (2.2) and (2.3). By the definition of $\{M_k\}_{k\in\mathbb{Z}}$ we get, for $k \in \mathbb{Z}$,

$$k(M_{k+1} - M_k) = -k^3(N_{k+1} - N_k) - 2k^2N_{k+1} - kN_{k+1}.$$

Thus $k(M_{k+1} - M_k)$ is uniformly bounded by assumption and Lemma 4.2. This shows that $\{M_k\}_{k\in\mathbb{Z}}$ satisfies (2.2). To show that $\{M_k\}_{k\in\mathbb{Z}}$ also satisfies (2.3), we see that, for $k \in \mathbb{Z}$,

$$\begin{aligned}
&k^2(M_{k+2}-2M_{k+1}+M_k)\\
&\quad = k^2[-(k+2)^2N_{k+2}+2(k+1)^2N_{k+1}-k^2N_k]\\
&\quad = -k^4(N_{k+2}-2N_{k+1}+N_k)-4k^3(N_{k+2}-N_{k+1})-4k^2N_{k+2}+2k^2N_{k+1},
\end{aligned}$$

which, by Lemma 4.2, implies (2.3). Then the result follows readily from Theorem 2.6. ∎

We now address the $B^s_{p,q}$-maximal regularity for equation (4.1).

THEOREM 4.4. *Let A be a closed linear operator defined in a Banach space X, and let $1 \le p,q \le \infty$ and $s > 0$. Assume that $\{D_k\}_{k\in\mathbb{Z}}$ is uniformly bounded. Then the following assertions are equivalent:*

(i) *Equation* (4.1) *has $B^s_{p,q}$-maximal regularity.*
(ii) *$\sigma_{\mathbb{Z}}(\triangle) = \emptyset$ and $\{M_k\}_{k\in\mathbb{Z}}$ is uniformly bounded.*

Proof. Note that for $s > 0$, we have $B^s_{p,q}(\mathbb{T},X) \subset L^p(\mathbb{T},X)$, hence the implication (i)⇒(ii) follows from the same arguments used in the proof of Theorem 3.4; we omit the details.

To show (ii)⇒(i), assume that $\sigma_{\mathbb{Z}}(\triangle) = \emptyset$ and $\{M_k\}_{k\in\mathbb{Z}}$ is uniformly bounded. We claim that $\{F_kN_k\}_{k\in\mathbb{Z}}$ and $\{D_kN_k\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-multipliers. Indeed, it is obvious from the assumption and (3.6) that $\{F_kN_k\}_{k\in\mathbb{Z}}$ and $\{k(F_{k+1}N_{k+1}-F_kN_k)\}_{k\in\mathbb{Z}}$ are both uniformly bounded. It remains to show that $\sup_{k\in\mathbb{Z}}\|k^2(F_{k+2}N_{k+2}-2F_{k+1}N_{k+1}+F_kN_k)\| < \infty$; this, however, is implied by the identity

$$\begin{aligned}
&F_{k+2}N_{k+2}-2F_{k+1}N_{k+1}+F_kN_k\\
&\qquad = [F_{k+2}(N_{k+2}-N_{k+1})-F_k(N_{k+1}-N_k)]+(F_{k+2}-2F_{k+1}+F_k)N_{k+1}.
\end{aligned}$$

Thus we deduce that $\{F_kN_k\}_{k\in\mathbb{Z}}$ satisfies (2.2) and (2.3), and from Theorem 2.6 it is a $B^s_{p,q}$-multiplier.

In addition, as $\{D_k\}_{k\in\mathbb{Z}}$ is uniformly bounded, similarly $\{D_kN_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier too. The rest of the proof follows the same lines as that of Theorem 3.4. ∎

When the underlying Banach space X is B-convex and $1 \le p,q \le \infty$ and $s \in \mathbb{R}$, the first order condition (2.2) is sufficient for the sequence $\{M_k\}_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-multiplier by Theorem 2.6. From this fact and the proof of Lemma 4.2, we easily deduce the following result on $B^s_{p,q}$-maximal regularity of the problem (4.1) when X is B-convex.

COROLLARY 4.5. *Let A be a closed linear operator defined in a B-convex space X, and let $1 \le p,q \le \infty$ and $s > 0$. Then the following assertions are equivalent:*

(i) *Equation* (4.1) *has $B^s_{p,q}$-maximal regularity.*
(ii) *$\sigma_{\mathbb{Z}}(\triangle) = \emptyset$ and $\{M_k\}_{k\in\mathbb{Z}}$ is uniformly bounded.*

The periodic Hölder continuous function space is a particular case of the periodic Besov space $B^s_{p,q}(\mathbb{T},X)$, Actually, from [AB2, Theorem 3.1], we have $B^s_{\infty,\infty}(\mathbb{T},X) = C^\alpha(\mathbb{T},X)$ whenever $0 < \alpha < 1$, where $C^\alpha(\mathbb{T},X)$ is the space of all X-valued functions f defined on $\mathbb{T}$ satisfying

$$f(0) = f(2\pi) \quad \text{and} \quad \sup_{x\neq y} \frac{\|f(x) - f(y)\|}{|x-y|^\alpha} < \infty.$$

Moreover,

$$\|f\|_{C^\alpha} := \max_{t\in\mathbb{T}} \|f(t)\| + \sup_{x\neq y} \frac{\|f(x) - f(y)\|}{|x-y|^\alpha}$$

on $C^\alpha(\mathbb{T},X)$ is an equivalent norm of $B^s_{\infty,\infty}(\mathbb{T},X)$. If $0 < s < 1$, we say that problem (4.1) has C^α-*maximal regularity.*

Finally, the $F^s_{p,q}$-maximal regularity for equation (4.1) can be discussed in an analogous way, the main tool being Theorem 2.7. Since the discussion is very similar to the above arguments, here we only state the $F^s_{p,q}$-maximal regularity results briefly for completeness.

Definition 4.6. Let X be a Banach space, A be a closed linear operator on X, and let $1 \le p < \infty, 1 \le q \le \infty$ and $s > 0$.

(i) Given $f \in F^s_{p,q}(\mathbb{T},X)$, a function $u \in F^{s+2}_{p,q}(\mathbb{T},X)$ is called a *strong* $F^s_{p,q}$-*solution* of (4.1) if $u(t) \in D(A)$ and (4.1) holds for a.e. $t \in \mathbb{T}$, $Au \in F^s_{p,q}(\mathbb{T},X)$ and the functions Fu_t, Gu'_t also belong to $F^s_{p,q}(\mathbb{T},X)$.

(ii) Equation (4.1) is said to have $F^s_{p,q}$-*maximal regularity* if for every $f \in F^s_{p,q}(\mathbb{T},X)$, there exists a unique strong $F^s_{p,q}$-solution of (4.1).

Theorem 4.7. *Let A be a closed linear operator defined in a Banach space X, and let $1 \le p < \infty$, $1 \le q \le \infty$ and $s > 0$. Assume that $\{kD_k\}_{k\in\mathbb{Z}}$ is uniformly bounded. Then the following assertions are equivalent:*

(i) *For every $f \in F^s_{p,q}(\mathbb{T},X)$, there exists a unique strong $F^s_{p,q}$-solution of* (4.1).

(ii) *$\sigma_{\mathbb{Z}}(\triangle) = \emptyset$ and $\{M_k\}_{k\in\mathbb{Z}}$ is uniformly bounded.*

Remark 4.8. When $1 < p < \infty, 1 \le q \le \infty$ and $s \in \mathbb{R}$, the Marcinkiewicz condition of order 2 is already sufficient for a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathscr{L}(X)$ to be an $F^s_{p,q}$-multiplier by Theorem 2.7. This fact together with the proof of Lemma 4.2 and Theorem 4.3 implies that under the weaker assumption that $\{D_k\}_{k\in\mathbb{Z}}$ is uniformly bounded, problem (4.1) has $F^s_{p,q}$-maximal regularity when $1 < p < \infty, 1 \le q \le \infty$ and $s > 0$ if and only if $\sigma_{\mathbb{Z}}(\triangle) = \emptyset$ and $\{M_k\}_{k\in\mathbb{Z}}$ is uniformly bounded.

5. An application. As an application of Theorem 3.4, we study the following integro-differential equation with infinite delay:

$$\begin{cases} u_{tt}(x,t)+\gamma u_t(x,t) \\ \qquad = u_{xx}(x,t)+\int_{-\infty}^{t} a(s-t)u(x,s)\,ds+f(x,t), & t\in\mathbb{T},\ 0\le x\le\pi, \\ u(0,t)=u(\pi,t)=0, & t\in\mathbb{T}, \end{cases} \tag{5.1}$$

where $a(\cdot)\in C(-\infty,0]$, $\gamma>0$ is a constant, $f(\cdot,\cdot)\in L^p(\mathbb{T},L^2[0,\pi])$. This equation arises in some thermal conduction models.

Let $X=L^2[0,\pi]$, and define the operator $(A,D(A))$ on X by

$$A\xi=\xi'',\qquad D(A)=\{\xi\in X\mid \xi',\xi''\in X,\ \xi(0)=\xi(\pi)=0\}.$$

Then it is well known that A is a closed operator and

$$\sigma(A)=\sigma_p(A)=\{-n^2\mid n\in\mathbb{N}^+\}.$$

Here we take the phase space $\mathscr{B}=C_g$ described in Section 2 (satisfying axioms (A), (A_1), (B), and (C_2)). Next, to rewrite (5.1) into the form of (1.1), we need to define the operator F on C_g as

$$F(\varphi)=\int_{-\infty}^{0} a(\theta)\varphi(\theta)\,d\theta\qquad\text{for any }\varphi\in C_g,$$

where $a(\cdot):(-\infty,0]\to\mathbb{C}$ is a continuous function with $\int_{-\infty}^0|a(\theta)|g(\theta)\,d\theta<\infty$. Since

$$\begin{aligned}\|F(\varphi)\|_X&=\Big\|\int_{-\infty}^{0} a(\theta)\varphi(\theta)\,d\theta\Big\|_X\\ &\le\Big\|\int_{-\infty}^{0}|a(\theta)|g(\theta)\frac{\|\varphi(\theta)\|}{g(\theta)}\,d\theta\Big\|_X\\ &\le\int_{-\infty}^{0}|a(\theta)|g(\theta)\,d\theta\,\|\varphi(\theta)\|_g,\end{aligned}$$

we see $F\in\mathscr{L}(\mathscr{B},X)$, and (5.1) can be rewritten in the form (1.1) with $B=\gamma I$ and $G=0$.

In what follows, we verify condition (ii) in Theorem 3.4 so that (5.1) has L^p-maximal regularity (existence of $\mathbb{T}$-periodic solutions on $\mathbb{R}$). As X is a Hilbert space, it suffices to show that the family $\{M_k\}_{k\in\mathbb{Z}}$ is uniformly bounded.

First it is known that, for any $\alpha,\beta\in\mathbb{R}$, $\beta\ne0$, one has $\alpha+i\beta\in\rho(A)$ and $0\in\rho(A)$. Moreover, $\sup_{\lambda\in\rho(A)}\|A(\lambda I-A)^{-1}\|<C$ $(<\infty)$ for some

$C > 0$. Then from the identity

$$-k^2I + ik\gamma I - A - F_k = [(-k^2 + ik\gamma)I - A][I - ((-k^2 + ik\gamma)I - A)^{-1}F_k],$$

it follows that $-k^2I + ik\gamma I - A - F_k$ is invertible whenever

$$\|((-k^2 + ik\gamma)I - A)^{-1}F_k\| < 1.$$

Observe that

$$\begin{aligned}\|((-k^2 + ik\gamma)I - A)^{-1}F_k\| &= \|A^{-1}A((-k^2 + ik\gamma)I - A)^{-1}F_k\| \\ &\le C\|F_k\|\ \|A^{-1}\| \le LC\|A^{-1}\|,\end{aligned}$$

where $L = Q\|F\|$ (see Section 3). Therefore, under the condition

$$L \le \frac{1}{C\|A^{-1}\|},$$

we infer that $\sigma_{\mathbb{Z}}(\triangle) = \emptyset$. Moreover, since $\|\lambda(\lambda I - A)^{-1}\| < 1$ for $\lambda \in \rho(A)$, we have

$$\begin{aligned}&\|-k^2(-k^2I + ik\gamma I - A - F_k)^{-1}\| \\ &\quad \le \|k^2((-k^2 + ik\gamma)I - A)^{-1}\|\ \|[I - ((-k^2 + ik\gamma)I - A)^{-1}F_k]^{-1}\| \\ &\quad = \left|\frac{k^2}{-k^2 + ik\gamma}\right|\ \|(-k^2 + ik\gamma)((-k^2 + ik\gamma)I - A)^{-1}\| \\ &\qquad \cdot \|[I - ((-k^2 + ik\gamma)I - A)^{-1}F_k]^{-1}\| \\ &\quad \le \frac{1}{1 - LC\|A^{-1}\|},\end{aligned}$$

which means condition (ii) in Theorem 3.4 is fulfilled and consequently equation (5.1) has L^p-maximal regularity.

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