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An alternative polynomial Daugavet property

by

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Abstract. We introduce a weaker version of the polynomial Daugavet property: a Banach space X has the alternative polynomial Daugavet property (APDP) if every weakly compact polynomial $P : X \to X$ satisfies

$$\max_{\omega \in \mathbb{T}} \|\mathrm{Id} + \omega P\| = 1 + \|P\|.$$

We study the stability of the APDP by c_0-, ℓ_∞- and ℓ_1-sums of Banach spaces. As a consequence, we obtain examples of Banach spaces with the APDP, namely $L_\infty(\mu, X)$ and $C(K, X)$, where X has the APDP.

1. Introduction. Let X and Y be Banach spaces over $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. By B_X we denote the closed unit ball and by S_X the unit sphere of X. If $k \in \mathbb{N}$, then a mapping $P : X \to Y$ is said to be a *k-homogeneous polynomial* if there exists a k-linear mapping $A : X^k \to Y$ such that $P(x) = A(x, \ldots, x)$ for every $x \in X$. We denote by $\mathcal{P}(^k X, Y)$ the vector space of all continuous k-homogeneous polynomials from X into Y. For convenience we denote by $\mathcal{P}(^0 X, Y)$ the vector space of all constant mappings from X into Y. A mapping $P : X \to Y$ is said to be a *polynomial* if it is a finite sum of homogeneous polynomials. We denote by $\mathcal{P}(X, Y)$ the vector space of all continuous polynomials from X into Y. It is a normed space for the norm

$$\|P\| = \sup_{x \in B_X} \|P(x)\|.$$

Each $\mathcal{P}(^k X, Y)$ is a Banach space for the norm induced from $\mathcal{P}(X, Y)$. When $Y = \mathbb{K}$ we write $\mathcal{P}(^k X)$ and $\mathcal{P}(X)$ instead of $\mathcal{P}(^k X, \mathbb{K})$ and $\mathcal{P}(X, \mathbb{K})$, respectively. Finally we denote by $\ell_\infty(B_X, X)$ the Banach space of all bounded functions from B_X into Y, with the supremum norm.

This paper is devoted to the study of the so-called Daugavet equation and alternative Daugavet equation for polynomials. In 1963, I. K. Dauga-

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vet [4] proved that every compact linear operator T on $C[0,1]$ satisfies

$$\|\mathrm{Id} + T\| = 1 + \|T\|,$$

a norm equality which has become known as the *Daugavet equation*. Several authors have shown that various classes of linear operators on different Banach spaces satisfy the Daugavet equation: we mention C. Foiaş and I. Singer [6], for weakly compact linear operators on $C[0,1]$; G. Ya. Lozanovskiĭ [10], for compact linear operators on $L_1[0,1]$; H. Kamowitz [8], for compact linear operators on $C(K)$, where K is a compact Hausdorff space without isolated points; J. R. Holub [7], for weakly compact linear operators on $L_1(\mu)$, where μ is an atomless σ-finite measure; and T. Oikhberg [14], for weakly compact linear operators on a non-atomic C*-algebra. We say that a Banach space X has the *Daugavet property* (DP) if every rank-one operator on X satisfies the Daugavet equation. This is the case of the Banach spaces $C(K)$ when K is a compact Hausdorff space without isolated points, and $L_1(\mu)$ when μ is an atomless σ-finite measure.

Also a weaker version of the Daugavet equation has been studied by several authors. In 1970, J. Duncan et al. [5] showed that, if T is a bounded linear operator on $C(K)$, where K is a compact Hausdorff space, or if T is a bounded linear operator on $L_1(\mu)$, where μ is a σ-finite measure, then T satisfies the equation

$$\max_{\omega\in\mathbb{T}} \|\mathrm{Id} + \omega T\| = 1 + \|T\|,$$

known as the *alternative Daugavet equation*. We say that a Banach space X has the *alternative Daugavet property* (ADP) if every rank-one operator on X satisfies the alternative Daugavet equation.

In 2007 the study of the Daugavet equation and the alternative Daugavet equation was extended to bounded functions from the unit ball of a Banach space into that space [2] and, in particular, to polynomials. Let X denote a real or complex Banach space. A function $\Phi \in \ell_\infty(B_X, X)$ satisfies the *Daugavet equation* if

$$\text{(DE)} \qquad \|\mathrm{Id} + \Phi\| = 1 + \|\Phi\|,$$

and $\Phi \in \ell_\infty(B_X, X)$ satisfies the *alternative Daugavet equation* if

$$\text{(ADE)} \qquad \max_{\omega\in\mathbb{T}} \|\mathrm{Id} + \omega\Phi\| = 1 + \|\Phi\|.$$

We say that a Banach space X has the *polynomial Daugavet property* (PDP) if every weakly compact polynomial on X satisfies (DE). Analogously, X has the *alternative polynomial Daugavet property* (APDP) if every weakly compact polynomial on X satisfies (ADE).

Given a compact Hausdorff space K, we denote by $C(K,X)$ (resp. $C_w(K,X)$) the Banach space of all continuous functions (resp. weakly continuous functions) from K into X, and by $C_{w^*}(K,X^*)$ the Banach space of all weakly* continuous functions from K into X^*. For a locally compact Hausdorff space L, we denote by $C_0(L,X)$ the Banach space of all continuous functions from L into X vanishing at infinity. Finally, for a completely regular space Ω, we write $C_b(\Omega,X)$ for the Banach space of all bounded continuous functions from Ω into X. Also, given a σ-finite measure space (Ω,Σ,μ), we denote by $L_\infty(\mu,X)$ the Banach space of all (equivalence classes of) essentially bounded Bochner-measurable functions from Ω into X with the essential supremum norm, and by $L_1(\mu,X)$ the Banach space of all (equivalence classes of) Bochner-integrable functions from Ω into X with the norm

$$\|f\| = \int_\Omega \|f(t)\|\,d\mu.$$

The main examples of Banach spaces having the PDP are: $C_b(\Omega,X)$ when the completely regular space Ω is perfect, $L_\infty(\mu,X)$ and $L_1(\mu,X)$ when the measure μ is atomless, $C_w(K,X)$ and $C_{w^*}(K,X^*)$ when the compact space K is perfect. We refer the reader to [2, 3, 11] for more information and background.

Let us remark that the PDP and APDP may be characterized in terms of scalar-valued polynomials. We state these results here for completeness.

LEMMA 1.1 ([2, Corollary 2.2]). *Let X be a Banach space. The following are equivalent:*

(i) *For every $p\in\mathcal{P}(X)$ with $\|p\|=1$, every $x_0\in S_X$, and every $\varepsilon>0$, there exist $\omega\in\mathbb{T}$ and $y\in B_X$ such that*
$$\operatorname{Re}\omega p(y) > 1-\varepsilon \quad \text{and} \quad \|x_0+\omega y\| > 2-\varepsilon.$$
(ii) *Every weakly compact $P\in\mathcal{P}(X,X)$ satisfies* (DE)*, i.e., X has the* PDP.

LEMMA 1.2 ([2, Corollary 1.2]). *Let X be a Banach space. The following are equivalent:*

(i) *For every $p\in\mathcal{P}(X)$ with $\|p\|=1$, every $x_0\in S_X$, and every $\varepsilon>0$, there exist $\omega_1,\omega_2\in\mathbb{T}$ and $y\in B_X$ such that*
$$\operatorname{Re}\omega_1 p(y) > 1-\varepsilon \quad \text{and} \quad \|x_0+\omega_2 y\| > 2-\varepsilon.$$
(ii) *Every weakly compact $P\in\mathcal{P}(X,X)$ satisfies* (ADE)*, i.e., X has the* APDP.

Moreover, the (DE) and (ADE) are related to numerical ranges. To explain this, let us give some definitions. For a Banach space X, we write

$$\Pi(X) = \{(x,x^*)\in X\times X^* : x\in S_X,\ x^*\in S_{X^*},\ x^*(x)=1\}.$$

Given a bounded function $\Phi: S_X\to X$, the *numerical range* of Φ is

$$V(\Phi) = \{x^*(\Phi(x)) : (x,x^*)\in\Pi(X)\},$$

and the *numerical radius* of Φ is

$$v(\Phi) = \sup\{|\lambda| : \lambda \in V(\Phi)\}.$$

The following characterizations of the (DE) and (ADE) can be stated.

PROPOSITION 1.3 ([2, Proposition 1.3]). *Let X be a Banach space and let $\Phi : B_X \to X$ be a uniformly continuous function. Then:*

(a) *Φ satisfies* (DE) *if and only if $\|\Phi\| = \sup \operatorname{Re} V(\Phi)$;*
(b) *Φ satisfies* (ADE) *if and only if $\|\Phi\| = v(\Phi)$.*

The outline of the paper is the following. In Section 2 we study the stability of the alternative polynomial Daugavet property by c_0-, ℓ_∞- and ℓ_1-sums. Given a sequence of Banach spaces $(X_j)_{j=1}^\infty$, we show that $[\bigoplus_{j=1}^\infty X_j]_{\ell_\infty}$ (or $[\bigoplus_{j=1}^\infty X_j]_{c_0}$) has the APDP if and only if every X_j has the APDP. We also show that if $[\bigoplus_{j=1}^\infty X_j]_{\ell_1}$ has the PDP (resp. the APDP), then every X_j has the PDP (resp. the APDP). In Section 3 we obtain examples of vector-valued function spaces that have the alternative polynomial Daugavet property using the results of Section 2. For a σ-finite measure μ, a compact Hausdorff space K and a Banach space X, we prove the following assertions: $L_\infty(\mu, X)$ has the APDP if and only if μ is atomless or X has the APDP; for a complex Banach space X, $C(K, X)$ has the APDP if and only if K is perfect or X has the APDP; if $L_1(\mu, X)$ has the PDP (resp. APDP), then μ is atomless or X has the PDP (resp. APDP).

2. Stability of the alternative polynomial Daugavet property. According to M. Martín & T. Oikhberg [12] and Y. S. Choi et al. [3], the alternative Daugavet property and the polynomial Daugavet property are stable by c_0- and ℓ_∞-sums. More precisely, given a sequence $(X_j)_{j=1}^\infty$ of Banach spaces, then:

(i) $[\bigoplus_{j=1}^\infty X_j]_{\ell_\infty}$ (or $[\bigoplus_{j=1}^\infty X_j]_{c_0}$) has the alternative Daugavet property if and only if every X_j has the alternative Daugavet property.
(ii) $[\bigoplus_{j=1}^\infty X_j]_{\ell_\infty}$ (or $[\bigoplus_{j=1}^\infty X_j]_{c_0}$) has the polynomial Daugavet property if and only if every X_j has the polynomial Daugavet property.

The first goal of this section is to show that the alternative polynomial Daugavet property is also stable by c_0- and ℓ_∞-sums. The proof of the following proposition is based on the proof of [3, Proposition 6.7].

PROPOSITION 2.1. *Let $(X_j)_{j=1}^\infty$ be a sequence of Banach spaces. Then $[\bigoplus_{j=1}^\infty X_j]_{\ell_\infty}$ or $[\bigoplus_{j=1}^\infty X_j]_{c_0}$ has the* APDP *if and only if every X_j has the* APDP.

Proof. Let $X = [\bigoplus_{j=1}^\infty X_j]_{\ell_\infty}$. Suppose that X has the APDP and fix $j_0 \in \mathbb{N}$. Given a non-null weakly compact polynomial $P : X_{j_0} \to X_{j_0}$, define the polynomial $Q : X \to X$ by

$$Q((x_j)_{j=1}^{\infty}) = i_{j_0}(P(x_{j_0})),$$

where i_{j_0} is the natural inclusion of X_{j_0} into X. It is not difficult to see that Q is weakly compact and $\|Q\| = \|P\|$. It follows that Q satisfies (ADE). Then

$$\begin{aligned}1 < 1 + \|P\| = 1 + \|Q\| &= \max_{\omega\in\mathbb{T}} \|\mathrm{Id}_X + \omega Q\| \\ &= \max_{\omega\in\mathbb{T}} \Big\{ \max\Big\{ \sup_{\|x_{j_0}\|\le 1} \|x_{j_0} + \omega P(x_{j_0})\|, \sup_{\|x_j\|\le 1} \{\|x_j\| : j \neq j_0\} \Big\} \Big\} \\ &= \max_{\omega\in\mathbb{T}} \Big\{ \sup_{\|x_{j_0}\|\le 1} \|x_{j_0} + \omega P(x_{j_0})\| \Big\} \\ &= \max_{\omega\in\mathbb{T}} \|\mathrm{Id}_{X_{j_0}} + \omega P\|,\end{aligned}$$

that is, P satisfies (ADE). Thus, X_{j_0} has the APDP.

Conversely, suppose that every X_j has the APDP. Let $p \in \mathcal{P}(X)$ with $\|p\| = 1$, $y = (y_j)_{j=1}^{\infty} \in S_X$ and $0 < \varepsilon < 1$. Since $\|y\| = 1$ there exists $j_0 \in \mathbb{N}$ such that $\|y_{j_0}\| > 1 - \varepsilon/2$. Take $z = (z_j)_{j=1}^{\infty} \in B_X$ such that

$$|p(z)| > \frac{1-\varepsilon}{1-\varepsilon/2},$$

and define the polynomial $q \in \mathcal{P}(X_{j_0})$ by

$$q(x_{j_0}) = p(z + i_{j_0}(x_{j_0} - z_{j_0})).$$

It follows that

$$1 = \|p\| \ge \|q\| \ge |q(z_{j_0})| = |p(z)| > \frac{1-\varepsilon}{1-\varepsilon/2}.$$

Since X_{j_0} has the APDP, we can apply Lemma 1.2 with $q/\|q\|$, $y_{j_0}/\|y_{j_0}\|$ and $\varepsilon/2$ to obtain $\omega_1, \omega_2 \in \mathbb{T}$ and $x_{j_0}^0 \in B_{X_{j_0}}$ such that

$$\mathrm{Re}\,\omega_1 \frac{q}{\|q\|}(x_{j_0}^0) > 1 - \frac{\varepsilon}{2} \quad \text{and} \quad \left\| \frac{y_{j_0}}{\|y_{j_0}\|} + \omega_2 x_{j_0}^0 \right\| > 2 - \frac{\varepsilon}{2}.$$

Hence, defining $x_0 = z + i_{j_0}(x_{j_0}^0 - z_{j_0}) \in B_X$, we have

$$\mathrm{Re}\,\omega_1 p(x_0) = \mathrm{Re}\,\omega_1 q(x_{j_0}^0) > \left(1 - \frac{\varepsilon}{2}\right)\|q\| > 1 - \varepsilon,$$

and

$$\begin{aligned}\|y + \omega_2 x_0\| \ge \|y_{j_0} + \omega_2 x_{j_0}^0\| &\ge \left\| \frac{y_{j_0}}{\|y_{j_0}\|} + \omega_2 x_{j_0}^0 \right\| - \left\| \frac{y_{j_0}}{\|y_{j_0}\|} - y_{j_0} \right\| \\ &> 2 - \frac{\varepsilon}{2} - (1 - \|y_{j_0}\|) > 2 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 2 - \varepsilon.\end{aligned}$$

Therefore, X has the APDP, by Lemma 1.2. The argument for the c_0-sum is the same. ∎

This proposition implies that, for a Banach space X, $c_0(X)$ and $\ell_\infty(X)$ have the alternative polynomial Daugavet property if and only if X has the alternative polynomial Daugavet property.

The alternative Daugavet property is also stable by ℓ_1-sums, according to the following proposition.

PROPOSITION 2.2 ([12, Proposition 3.1]). *Let $(X_j)_{j=1}^\infty$ be a sequence of Banach spaces. Then $[\bigoplus_{j=1}^\infty X_j]_{\ell_1}$ has the alternative Daugavet property if and only if every X_j has the alternative Daugavet property.*

Unfortunately, this proposition cannot be extended to the alternative polynomial Daugavet property, as we will show later. However, we have the following result for the polynomial Daugavet property and the alternative polynomial Daugavet property.

PROPOSITION 2.3. *Let $(X_j)_{j=1}^\infty$ be a sequence of Banach spaces. If $[\bigoplus_{j=1}^\infty X_j]_{\ell_1}$ has the* PDP (*resp. the* APDP), *then every X_j has the* PDP (*resp. the* APDP).

Proof. Let $X = [\bigoplus_{j=1}^\infty X_j]_{\ell_1}$. Suppose that X has the PDP and fix $j_0 \in \mathbb{N}$. Given a non-null weakly compact polynomial $P : X_{j_0} \to X_{j_0}$, define the polynomial $Q : X \to X$ by

$$Q((x_j)_{j=1}^\infty) = i_{j_0}(P(x_{j_0})),$$

where i_{j_0} is the natural inclusion of X_{j_0} into X. Clearly Q is a non-null weakly compact polynomial and $\|Q\| = \|P\|$. It follows that Q satisfies (DE). Then, given $\varepsilon > 0$, there exist $(x_j)_{j=1}^\infty \in S_X$ and $(x_j^*)_{j=1}^\infty \in S_{X^*} = S_{[\bigoplus_{j=1}^\infty X_j^*]_{\ell_\infty}}$ such that $\sum_{j=1}^\infty x_j^*(x_j) = 1$ and

$$\|Q\| - \varepsilon \le \operatorname{Re}(x_j^*)[Q((x_j)_{j=1}^\infty)],$$

by Proposition 1.3. Since $\sum_{j=1}^\infty x_j^*(x_j) = 1$, $(x_j)_{j=1}^\infty \in S_X$ and $(x_j^*)_{j=1}^\infty \in S_{X^*}$, we have

$$\sum_{j=1}^\infty x_j^*(x_j) = \sum_{j=1}^\infty \operatorname{Re} x_j^*(x_j) \le \sum_{j=1}^\infty |x_j^*(x_j)| \le \sum_{j=1}^\infty \|x_j^*\|\,\|x_j\| \le \sum_{j=1}^\infty \|x_j\| = 1.$$

Thus, $\operatorname{Re} x_{j_0}^*(x_{j_0}) = \|x_{j_0}^*\|\,\|x_{j_0}\|$. Otherwise, we would obtain $\operatorname{Re} x_{j_0}^*(x_{j_0}) < \|x_{j_0}^*\|\,\|x_{j_0}\|$, with would imply the contradiction

$$\sum_{j=1}^\infty \operatorname{Re} x_j^*(x_j) < \sum_{j=1}^\infty \|x_j^*\|\,\|x_j\|,$$

because $\sum_{j\neq j_0} \operatorname{Re} x_j^*(x_j) \le \sum_{j\neq j_0} \|x_j^*\|\,\|x_j\|$. Since

$$\|x_{j_0}^*\|\,\|x_{j_0}\| = \operatorname{Re} x_{j_0}^*(x_{j_0}) \le |x_{j_0}^*(x_{j_0})| \le \|x_{j_0}^*\|\,\|x_{j_0}\|,$$

we have $x_{j_0}^*(x_{j_0}) = \operatorname{Re} x_{j_0}^*(x_{j_0}) = \|x_{j_0}^*\|\,\|x_{j_0}\|$. Write $P = P_0 + P_1 + \cdots + P_n$, where $P_k \in \mathcal{P}(^k X_{j_0}, X_{j_0})$. Then

$$\begin{aligned}
\|P\| - \varepsilon = \|Q\| - \varepsilon &\le \operatorname{Re}(x_j^*)[Q((x_j)_{j=1}^\infty)] = \operatorname{Re} x_{j_0}^*(P(x_{j_0}))\\
&= \operatorname{Re} x_{j_0}^*(P_0(x_{j_0})) + \operatorname{Re} x_{j_0}^*(P_1(x_{j_0})) + \cdots + \operatorname{Re} x_{j_0}^*(P_n(x_{j_0}))\\
&\le \frac{\operatorname{Re} x_{j_0}^*(P_0(x_{j_0}))}{\|x_{j_0}^*\|} + \frac{\operatorname{Re} x_{j_0}^*(P_1(x_{j_0}))}{\|x_{j_0}^*\|\|x_{j_0}\|} + \cdots + \frac{\operatorname{Re} x_{j_0}^*(P_n(x_{j_0}))}{\|x_{j_0}^*\|\|x_{j_0}\|^n}\\
&= \operatorname{Re} \frac{x_{j_0}^*}{\|x_{j_0}^*\|}\biggl(P_0\biggl(\frac{x_{j_0}}{\|x_{j_0}\|}\biggr)\biggr) + \operatorname{Re} \frac{x_{j_0}^*}{\|x_{j_0}^*\|}\biggl(P_1\biggl(\frac{x_{j_0}}{\|x_{j_0}\|}\biggr)\biggr)\\
&\quad + \cdots + \operatorname{Re} \frac{x_{j_0}^*}{\|x_{j_0}^*\|}\biggl(P_n\biggl(\frac{x_{j_0}}{\|x_{j_0}\|}\biggr)\biggr)\\
&= \operatorname{Re} \frac{x_{j_0}^*}{\|x_{j_0}^*\|}\biggl(P\biggl(\frac{x_{j_0}}{\|x_{j_0}\|}\biggr)\biggr) \le \sup \operatorname{Re} V(P),
\end{aligned}$$

because $\|x_{j_0}^*\| \le 1$, $\|x_{j_0}\| \le 1$ and

$$\frac{x_{j_0}^*}{\|x_{j_0}^*\|}\biggl(\frac{x_{j_0}}{\|x_{j_0}\|}\biggr) = 1.$$

In other words, P satisfies (DE), by Proposition 1.3. Hence, X_{j_0} has the PDP.

Now suppose that X has the APDP and fix $j_0 \in \mathbb{N}$. Given a non-null weakly compact polynomial $P : X_{j_0} \to X_{j_0}$, define $Q : X \to X$ by

$$Q((x_j)_{j=1}^\infty) = i_{j_0}(P(x_{j_0})),$$

as in the PDP case. Then Q is a non-null weakly compact polynomial and $\|Q\| = \|P\|$. Hence Q satisfies (ADE), that is, there exists $\omega \in \mathbb{T}$ such that ωQ satisfies (DE). By the first part of the proof, we can conclude that ωP also satisfies (DE). Thus, P satisfies (ADE). Therefore, X_{j_0} has the APDP. ∎

The proof of the last proposition made use of the ideas of [1, Proposition 2.8] and [13, Proposition 1].

Remark 2.4. In the case of complex Banach spaces, it is not true that $[\bigoplus_{j=1}^\infty X_j]_{\ell_1}$ has the APDP if every X_j has the APDP. Indeed, $\mathbb{C}$ has the APDP, by [2, Example 2.1.b], while $\ell_1(\mathbb{C})$ does not have the APDP, by [2, Example 3.12].

3. Spaces with the alternative polynomial Daugavet property.

Characterizations of the alternative Daugavet property and the polynomial Daugavet property are known for vector-valued essentially bounded function spaces and for continuous vector-valued function spaces: see [12, Theorem 3.4] and [3, Corollary 6.9 and Proposition 6.10] for more information.

We now present a characterization of the alternative polynomial Daugavet property in these spaces. The proofs of these results are based on the proofs of [3, Corollary 6.9 and Proposition 6.10] and [13, Remark 6].

Making use of Proposition 2.1 we obtain a characterization of the alternative polynomial Daugavet property for vector-valued essentially bounded function spaces.

PROPOSITION 3.1. *Let (Ω, Σ, μ) be a σ-finite measure space and let X be a Banach space. Then $L_\infty(\mu, X)$ has the* APDP *if and only if μ is atomless or X has the* APDP.

Proof. By [3, Theorem 6.5], we know that if μ is an atomless σ-finite measure then $L_\infty(\mu, X)$ has the PDP and, in particular, the APDP. Now, if μ is a σ-finite measure with an atom, then μ has at most countably many atoms. Hence, there exist a non-empty countable set J and an atomless σ-finite measure ν such that

$$L_\infty(\mu, X) = L_\infty(\nu, X) \oplus_\infty \Big[\bigoplus_{j \in J} X\Big]_{\ell_\infty}.$$

In this case, by Proposition 2.1, $L_\infty(\mu, X)$ has the APDP if and only if X has the APDP. ■

We also obtain a characterization of the alternative polynomial Daugavet property for continuous vector-valued function spaces. To prove this result we make use of the following lemma.

LEMMA 3.2 ([9], Lemma 1). *Let K be a compact Hausdorff space and let X be a Banach space. For every $f \in C_w(K, X)$, the set*

$$\{t \in K : f \text{ is norm continuous at } t\}$$

is dense in K.

PROPOSITION 3.3. *Let X be a complex Banach space, K a compact Hausdorff space, L a locally compact Hausdorff space and Ω a completely regular Hausdorff space. The following statements hold:*

(a) *$C(K, X)$ has the* APDP *if and only if K is perfect or X has the* APDP.
(b) *$C_w(K, X)$ has the* APDP *if and only if K is perfect or X has the* APDP.
(c) *$C_0(L, X)$ has the* APDP *if and only if L is perfect or X has the* APDP.
(d) *$C_b(\Omega, X)$ has the* APDP *if and only if Ω is perfect or X has the* APDP.

Proof. We will only prove statement (b), because the proofs of the others are analogous. Suppose firstly that $C_w(K,X)$ has the APDP and that K has an isolated point. Then there exists a Banach space Z such that $C_w(K,X) = X \oplus_\infty Z$. Hence, by Proposition 2.1 we find that X has the APDP.

Now, suppose that K is perfect. In this case, [3, Theorem 6.5] ensures that $C_w(K,X)$ has the PDP and, in particular, the APDP.

Finally, suppose that X has the APDP. Given a weakly compact polynomial $P \in \mathcal{P}(C_w(K,X), C_w(K,X))$ with $\|P\| = 1$ and $\varepsilon > 0$, by Lemma 3.2 there exist $f_0 \in S_{C_w(K,X)}$ and $t_0 \in K$ such that f_0 is norm continuous at t_0 and

$$\|P(f_0)(t_0)\| > 1 - \frac{\varepsilon}{2}. \tag{3.1}$$

Since P is continuous at f_0, there exists $\delta > 0$ such that

$$\|P(f_0) - P(g)\| < \frac{\varepsilon}{2} \quad \text{if } \|f_0 - g\| < \delta. \tag{3.2}$$

Moreover, since f_0 is norm continuous at t_0, we infer that

$$W = \overline{\{t \in K : \|f_0(t) - f_0(t_0)\| \geq \delta\}}$$

does not contain t_0. Thus, by Urysohn's lemma, there exists a continuous function $\varphi : K \to [0,1]$ such that

$$\varphi(t) = \begin{cases} 1, & t = t_0, \\ 0, & t \in W. \end{cases}$$

Fix $x_0 \in S_X$ such that $f_0(t_0) = \|f_0(t_0)\| x_0$ and define $\Psi : \mathbb{C} \to C_w(K,X)$ by

$$\Psi(z) = (1 - \varphi) f_0 + \varphi x_0 z.$$

Then

$$\begin{aligned} \big\|f_0 - \Psi(\|f_0(t_0)\|)\big\| &= \sup_{t \in K} \big\|f_0(t) - \big((1-\varphi(t)) f_0(t) + \varphi(t) f_0(t_0)\big)\big\| \\ &= \sup_{t \in K} \varphi(t) \|f_0(t) - f_0(t_0)\| < \delta, \end{aligned}$$

because $\varphi(W) = \{0\}$. Hence, by (3.2),

$$\big\|P(f_0) - P\big(\Psi(\|f_0(t_0)\|)\big)\big\| < \frac{\varepsilon}{2},$$

which implies

$$\big\|P(f_0)(t_0) - P\big(\Psi(\|f_0(t_0)\|)\big)(t_0)\big\| < \frac{\varepsilon}{2}.$$

It follows from (3.1) that

$$\big\|P\big(\Psi(\|f_0(t_0)\|)\big)(t_0)\big\| > \|P(f_0)(t_0)\| - \frac{\varepsilon}{2} > 1 - \varepsilon.$$

Thus, by the Hahn–Banach theorem, we may find $x_0^* \in S_{X^*}$ such that

$$x_0^*\big(\big[P\big(\Psi(\|f_0(t_0)\|)\big)\big](t_0)\big) > 1-\varepsilon.$$

Since the function

$$z \mapsto x_0^*\big([P(\Psi(z))](t_0)\big)$$

is holomorphic, the maximum modulus theorem ensures the existence of $z_0 \in \mathbb{T}$ such that

$$\begin{aligned}\|P(\Psi(z_0))(t_0)\| &\geq \big|x_0^*\big([P(\Psi(z_0))](t_0)\big)\big| \\ &\geq x_0^*\big([P(\Psi(\|f_0(t_0)\|))](t_0)\big) > 1-\varepsilon.\end{aligned}$$

Now, define $x_1 = z_0 x_0 \in S_X$, consider $x_1^* \in S_{X^*}$ such that $x_1^*(x_1) = 1$, and define $\Phi : X \to C_w(K, X)$ by

$$\Phi(x) = x_1^*(x)(1-\varphi)f_0 + \varphi x.$$

Observe that $\|\Phi(x)\| \leq 1$ for all $x \in B_X$ and that $\Phi(x_1) = \Psi(z_0)$. Then

$$\|P(\Phi(x_1))(t_0)\| > 1-\varepsilon.$$

Finally, define the polynomial $Q : X \to X$ by

$$Q(x) = [P(\Phi(x))](t_0),$$

which is weakly compact and satisfies

$$\|Q\| = \sup_{x\in B_X} \|Q(x)\| \geq \|Q(x_1)\| = \|[P(\Phi(x_1))](t_0)\| > 1-\varepsilon.$$

Since X has the APDP, Q satisfies (ADE). So,

$$\begin{aligned}\max_{\omega\in\mathbb{T}} &\|\mathrm{Id}_{C_w(K,X)} + \omega P\| \\ &\geq \max_{\omega\in\mathbb{T}} \sup_{x\in B_X} \|\Phi(x) + \omega P(\Phi(x))\| \\ &\geq \max_{\omega\in\mathbb{T}} \sup_{x\in B_X} \|\Phi(x)(t_0) + \omega[P(\Phi(x))](t_0)\| \\ &= \max_{\omega\in\mathbb{T}} \sup_{x\in B_X} \|x_1^*(x)(1-\varphi(t_0))f_0(t_0) + \varphi(t_0)x + \omega Q(x)\| \\ &= \max_{\omega\in\mathbb{T}} \sup_{x\in B_X} \|x + \omega Q(x)\| = \max_{\omega\in\mathbb{T}} \|\mathrm{Id}_X + \omega Q\| \\ &= 1 + \|Q\| > 2-\varepsilon.\end{aligned}$$

Therefore, $C_w(K, X)$ has the APDP. ∎

For spaces of Bochner integrable functions, only characterizations of the Daugavet property and the alternative Daugavet property are known. See [13, Remark 9] and [12, Theorem 3.4] for more information.

As a consequence of Proposition 2.3 we can partially generalize these characterizations to the polynomial Daugavet property and the alternative polynomial Daugavet property.

PROPOSITION 3.4. *Let (Ω, Σ, μ) be a σ-finite measure space and let X be a Banach space. If $L_1(\mu, X)$ has the* PDP (*resp.* APDP)*, then μ is atomless or X has the* PDP (*resp.* APDP).

Proof. Suppose that μ has an atom; then it has at most countably many atoms. Hence, there exist a non-empty countable set J and an atomless σ-finite measure ν such that

$$L_1(\mu, X) = L_1(\nu, X) \oplus_1 \Big[\bigoplus_{j \in J} X\Big]_{\ell_1}.$$

Thus, by Proposition 2.3, if $L_1(\mu, X)$ has the PDP (resp. APDP), then X has the PDP (resp. APDP). ■

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