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Homology and cohomology of Rees semigroup algebras

by

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Abstract. Let S be a Rees semigroup, and let $\ell^1(S)$ be its convolution semigroup algebra. Using Morita equivalence we show that bounded Hochschild homology and cohomology of $\ell^1(S)$ are isomorphic to those of the underlying discrete group algebra.

1. Introduction. In this paper we calculate the simplicial cohomology of the ℓ^1-algebra of Rees semigroups, motivated by the explicit computations of the first-order [2] and second-order cohomology groups [4] of several Banach algebras. These papers contain a number of results showing that the simplicial cohomology groups (in dimensions 1 and 2 respectively) of ℓ^1-semigroup algebras are trivial for many of the fundamental examples of semigroups. The first paper [2] considers the first-order simplicial and cyclic cohomology of Rees semigroup algebras, the bicyclic semigroup algebra and the free semigroup algebra. The second paper [4] shows that the second simplicial cohomology vanishes for the semigroup $\mathbb{Z}_+$, semilattice semigroups and Clifford semigroups. It should be noted that in each of these papers the arguments were mostly ad hoc. Subsequently the papers [10] and [9], co-authored by two of the present authors, cover the case of the third cohomology groups. In these papers there is some attempt at systematic methods which might be adapted to cover the case of all higher cohomology groups, but the authors were not able to do this at that time. Finally, the papers [6]–[8] show how to calculate the higher order simplicial cohomology groups of some of these algebras. These latter papers do not use ad hoc calculations, but general homological machinery, such as the Connes–Tsygan long exact sequence and topological simplicial homology to deduce their results. The present paper belongs to the latter family of papers, in that it uses general homological tools.

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Our primary concern is that of Rees semigroups. These semigroups have played an important rôle in the understanding of homological properties of semigroup algebras. In particular amenability of semigroup algebras can be described in terms of a principal series for the semigroup with factors being Rees semigroups (see [5, Theorem 10.12]).

The proofs in this paper are based firmly on the Morita equivalence methods developed by the second-named author [11], [12]. These methods apply to the class of so-called self-induced Banach algebras. Bounded Hochschild homology and cohomology are Morita invariant within this class, provided that coefficients are chosen appropriately.

We briefly describe our general approach. For a semigroup, T, with an absorbing zero, $\emptyset$, there are two Banach algebras that naturally arise, the discrete convolution algebra $\ell^1(T)$, and the reduced algebra $\mathcal{A}(T)$ (to be defined below) associated with the inclusion $\emptyset \hookrightarrow T$. In the case of a Rees semigroup, S, with underlying group, G, we prove that $\mathcal{A}(S)$ is Morita equivalent to $\ell^1(G)$. This may be interpreted as a manifestation of a basic fact from algebraic topology that for a path connected space X the fundamental groupoid $\pi(X)$ and the based homotopy group $\pi_1(X,a)$ are equivalent categories.

Our results then hinge on Theorem 3.1 in which natural isomorphisms between the homology and cohomology of $\mathcal{A}(S)$ and those of the underlying group algebra are established by means of Morita equivalence. Using an excision result from [17] we prove that the homology and cohomology of $\ell^1(S)$ and of $\mathcal{A}(S)$ are isomorphic for a large class of coefficient modules. Since $\ell^1(S)$ and $\mathcal{A}(S)$ are both H-unital (cf. [19]), we further obtain homology and cohomology results for the forced unitizations $\ell^1(S)^\sharp$ and $\mathcal{A}(S)^\sharp$. From this we derive our main result that for a Rees semigroup, S, the simplicial cohomology $\mathcal{H}^n(\ell^1(S),\ell^1(S)^*)$ $(n \ge 1)$ of the semigroup algebra is isomorphic to the simplicial cohomology $\mathcal{H}^n(\ell^1(G),\ell^1(G)^*)$ $(n \ge 1)$ of the underlying group algebra.

2. Basics. A *completely* 0*-simple semigroup* is a semigroup which has a 0, has no proper ideal other than $\{0\}$, and has a *primitive idempotent*, that is, a minimal idempotent e in the set of non-zero idempotents (or, precisely, an idempotent $e \neq 0$ such that, if $ef = fe = f \neq 0$ for an idempotent f, then $e = f$). Such semigroups arise naturally and notably in classification theory as quotients of semigroups by ideals. It is an important result that any completely 0-simple semigroup is isomorphic to what we call a Rees semigroup (and conversely). For this result and more background, see [14, Chapter 3 and Theorem 3.2.3]. We now give the definition of a Rees semigroup and some special cases of Rees semigroups to illustrate that many natural semigroups are of this form.

The data which are required to define a Rees semigroup are given by two index sets I and Λ, a group G, and a *sandwich matrix* P. We introduce two zero elements: the first, $\mathbf{o}$, is an absorbing element adjoined to G to make the semigroup denoted by $G^{\mathbf{o}}$. The second element, $\emptyset$, is an absorbing zero for the Rees semigroup itself. The sandwich matrix $P = (p_{\lambda i})$ is a set of elements of $G^{\mathbf{o}}$ indexed by $\Lambda \times I$ such that each row and column of P has at least one non-zero entry. The *Rees semigroup* S is then the set $(I \times G \times \Lambda) \cup \{\emptyset\}$, where $\emptyset$ is an absorbing zero for the semigroup, and the other products are defined by the rule

$$(i,g,\lambda)(j,h,\mu) = \begin{cases} (i, g p_{\lambda j} h, \mu) & (p_{\lambda j} \neq \mathbf{o}), \\ \emptyset & (p_{\lambda j} = \mathbf{o}). \end{cases}$$

2.1. Examples. There are two extreme, degenerate cases which provide good intuition for the logic of calculations.

The first is the case where I and Λ are just singletons and the sandwich matrix consists of the identity of G. In this case S is just $G^{\mathbf{o}}$. The reduced semigroup algebra $\mathcal{A}(S)$ defined below will give us $\ell^1(G)$ in this case.

The second case has the group, G, being trivial and the index sets being both equal to the set of the first n natural numbers $\{1,\dots,n\}$. The sandwich matrix is diagonal with the group identity repeated along the diagonal. If we identify G with $\{1\} \subseteq \mathbb{C}$ the Rees semigroup becomes the system of matrix units together with the zero-matrix, $S = \{E_{ij} : 1 \le i,j \le n\} \cup \{0\}$, and $\ell^1(S)$ is the algebra of complex $(n+1) \times (n+1)$ matrices which are 2×2 diagonal block matrices consisting of an upper $n \times n$ block and a lower 1×1 block. The reduced semigroup algebra, $\mathcal{A}(S)$, does not have this deficiency and is exactly the matrix algebra, $M_n(\mathbb{C})$.

Our third example is from homotopy theory and is almost generic for the concept of Rees semigroups. Recall that a small category in which every morphism is invertible is called a *groupoid*. It is *connected* if there is a morphism between each pair of objects. The canonical example is $\pi(X)$, the *fundamental groupoid* of a path connected topological space X. The objects are the points of X and, for each $x, y \in X$, the morphism set $\pi(X)(x,y)$ is the set of homotopy classes relative to x, y of paths from x to y with composition derived from products of paths. In particular $\pi(X)(x,x) = \pi_1(X,x)$ for each $x \in X$, so that the fundamental group of X at x is identified with the full subcategory of $\pi(X)$ with just one object x. Since X is path connected, $\pi_1(X,x) \cong \pi_1(X,y)$ for each $x, y \in X$. For details see [3].

Fix $a \in X$ and set $G = \pi_1(X,a)$. We associate Rees semigroups to $\pi(X)$ in the following way. For each $x \in X$ choose $s_x \in \pi(X)(a,x)$ and $t_x \in \pi(X)(x,a)$. This gives bijections $\psi_{x,y} \colon \pi(X)(x,y) \to G$ defined as

$$\psi_{x,y}(\gamma) = s_x \gamma t_y, \quad x, y \in X,\ \gamma \in \pi(X)(x,y).$$

With this we get

$$\psi_{x,z}(\gamma\gamma') = \psi_{x,y}(\gamma)(s_y t_y)^{-1}\psi_{y,z}(\gamma')$$

for $x, y, z \in X$, $\gamma \in \pi(X)(x, y)$, $\gamma' \in \pi(X)(y, z)$. Let I, Λ be sets and consider maps $\alpha\colon I \to X$, $\beta\colon \Lambda \to X$. We define multiplication on $(I \times G \times \Lambda) \cup \{\emptyset\}$ as follows. For $i, j \in I$, $g, h \in G$, $\lambda, \mu \in \Lambda$, put $\gamma = \psi^{-1}_{\alpha(i),\beta(\lambda)}(g)$, $\gamma' = \psi^{-1}_{\alpha(j),\beta(\mu)}(h)$ and set

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, \psi_{\alpha(i),\beta(\mu)}(\gamma\gamma'), \mu) & (\alpha(j) = \beta(\lambda)), \\ \emptyset & (\alpha(j) \neq \beta(\lambda)), \end{cases}$$

so that the product is simply given by products of paths, when defined. The sandwich matrix is

$$p_{\lambda i} = \begin{cases} (s_{\alpha(i)} t_{\beta(\lambda)})^{-1} & (\alpha(i) = \beta(\lambda)), \\ \mathbf{o} & (\alpha(i) \neq \beta(\lambda)). \end{cases}$$

The condition that the sandwich matrix $(p_{\lambda i})$ has a non-zero entry in each row and each column is $\alpha(I) = \beta(\Lambda)$.

It is a simple fact, but crucial to the use of groupoids in algebraic topology (cf. [3, Chapter 8]), that the fundamental groupoid $\pi(X)$ and the based homotopy group $\pi_1(X, a)$ are equivalent categories. Our main result on Morita equivalence is a manifestation of this fact in the setting of convolution Banach algebras.

2.2. Background on homological algebra. The main result of this paper concerns bounded homology and cohomology of Banach algebras, so we will give a brief description of the theory of homology and cohomology of Banach algebras, as we fix the notation we will use for this paper. For further details we refer to [13].

For a Banach algebra $\mathcal{A}$ we denote the categories of left (right) Banach $\mathcal{A}$-modules and bounded module homomorphisms by $\mathcal{A}$**-mod** (respectively **mod-**$\mathcal{A}$). If $\mathcal{B}$ is also a Banach algebra, the category of Banach $\mathcal{A}$-$\mathcal{B}$-bimodules and bounded homomorphisms is $\mathcal{A}$**-mod-**$\mathcal{B}$. A *full subcategory* is a subcategory $\mathfrak{C}$ which includes all morphisms between objects of $\mathfrak{C}$.

Let $\mathcal{A}$ be a Banach algebra, let $\mathcal{A}^{\#}$ be its forced unitization, and let $X \in \mathcal{A}$**-mod**. The *bar resolution* of X is

$$\begin{aligned} \mathscr{B}(\mathcal{A}, X)\colon 0 \leftarrow X \leftarrow \mathcal{A}^{\#} \widehat{\otimes} X \leftarrow \mathcal{A}^{\#} \widehat{\otimes} \mathcal{A} \widehat{\otimes} X \leftarrow \cdots \\ \leftarrow \mathcal{A}^{\#} \widehat{\otimes} \mathcal{A} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{A} \widehat{\otimes} X \leftarrow \cdots \end{aligned}$$

in which the arrows denote *boundary maps*

$$\begin{aligned} b(a_1 \otimes x) &= a_1 x, \\ b(a_1 \otimes \cdots \otimes a_n \otimes x) &= \sum_{k=1}^{n-1} (-1)^{k-1} a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes x \\ &\quad + (-1)^{n-1} a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n x. \end{aligned}$$

Similarly we define the bar resolution for $X \in \mathbf{mod}\text{-}\mathcal{A}$. It is a standard fact that $\mathscr{B}(\mathcal{A}, X)$ is a complex, i.e. the compositions of two consecutive boundary maps are trivial, and that this complex is contractible, i.e. there are bounded linear maps

$$s\colon X \to \mathcal{A}^{\#} \mathbin{\widehat{\otimes}} X, \quad s\colon \mathcal{A}^{\#} \mathbin{\widehat{\otimes}} \mathcal{A}^{\widehat{\otimes} n} \mathbin{\widehat{\otimes}} X \to \mathcal{A}^{\#} \mathbin{\widehat{\otimes}} \mathcal{A}^{\widehat{\otimes}(n+1)} \mathbin{\widehat{\otimes}} X$$

such that $sb + bs = \mathrm{id}$. For further details, cf. [13].

The *simplicial complex* of $\mathcal{A}$ is the subcomplex of $\mathscr{B}(\mathcal{A}, \mathcal{A})$ with the first tensor factor in $\mathcal{A}$ rather than $\mathcal{A}^{\#}$:

$$\mathscr{S}(\mathcal{A})\colon 0 \leftarrow \mathcal{A} \leftarrow \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} \leftarrow \cdots \leftarrow \mathcal{A}^{\widehat{\otimes} n} \leftarrow \cdots .$$

A module $X \in \mathcal{A}\text{-}\mathbf{mod}\text{-}\mathcal{B}$ is *induced* if the multiplication

$$\mathcal{A} \mathbin{\widehat{\otimes}}_{\mathcal{A}} X \mathbin{\widehat{\otimes}}_{\mathcal{B}} \mathcal{B} \to X\colon a \otimes_{\mathcal{A}} x \otimes_{\mathcal{B}} b \mapsto axb$$

is an isomorphism. If $\mathcal{A}$ is induced as a module in $\mathcal{A}\text{-}\mathbf{mod}\text{-}\mathcal{A}$, then $\mathcal{A}$ is *self-induced*.

A bounded linear map $L\colon E \to F$ between Banach spaces is *admissible* if $\ker L$ and $\operatorname{im} L$ are complemented as Banach spaces in E respectively F.

A module $P \in \mathcal{A}\text{-}\mathbf{mod}$ is *(left) projective* if, for every admissible epimorphism $q\colon Y \to Z$, all lifting problems in $\mathcal{A}\text{-}\mathbf{mod}$

$$\begin{tikzcd} & P \arrow[dl, dashed, "?"'] \arrow[d, "\phi"] & \\ Y \arrow[r, "q"'] & Z \arrow[r] & 0 \end{tikzcd}$$

can be solved. If all, not just admissible, lifting problems can be solved, then P is *strictly projective*. The module P is *(left) flat* if for every admissible short exact sequence in $\mathbf{mod}\text{-}\mathcal{A}$

$$0 \to X \to Y \to Z \to 0$$

the sequence

$$0 \to X \mathbin{\widehat{\otimes}}_{\mathcal{A}} P \to Y \mathbin{\widehat{\otimes}}_{\mathcal{A}} P \to Z \mathbin{\widehat{\otimes}}_{\mathcal{A}} P \to 0$$

is exact, and *strictly flat* if the requirement of admissibility can be omitted.

The fundamental concept of our approach is Morita equivalence.

Definition 2.1. Two self-induced Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* if there are induced modules $P \in \mathcal{B}$**-mod-**$\mathcal{A}$ and $Q \in \mathcal{A}$**-mod-**$\mathcal{B}$ so that

$$P \mathbin{\widehat{\otimes}}_{\mathcal{A}} Q \cong \mathcal{B} \quad \text{and} \quad Q \mathbin{\widehat{\otimes}}_{\mathcal{B}} P \cong \mathcal{A},$$

where the isomorphisms are implemented by bounded bilinear balanced module maps $[\cdot,\cdot]\colon P \times Q \to \mathcal{B}$ and $(\cdot,\cdot)\colon Q \times P \to \mathcal{A}$ satisfying

$$[p,q]\cdot p' = p \cdot (q,p'), \qquad q \cdot [p,q'] = (q,p)\cdot q', \qquad p,p' \in P,\ q,q' \in Q.$$

Our objective is to describe bounded Hochschild homology and cohomology of Rees semigroup algebras in terms of the homology and cohomology of the algebra of the underlying group. First we define homology.

Definition 2.2. For $X \in \mathcal{A}$**-mod-**$\mathcal{A}$ the *Hochschild complex* is

$$0 \leftarrow X \leftarrow X \mathbin{\widehat{\otimes}} \mathcal{A} \leftarrow \cdots \leftarrow X \mathbin{\widehat{\otimes}} \mathcal{A}^{\widehat{\otimes} n} \leftarrow \cdots,$$

with boundary maps given as

$$\begin{aligned}\delta(x \otimes a_1 \otimes \cdots \otimes a_n) = {} & x a_1 \otimes a_2 \otimes \cdots \otimes a_n \\ & + x \otimes \sum_{k=1}^{n-1} (-1)^k a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n \\ & + (-1)^n a_n x \otimes a_1 \otimes \cdots \otimes a_{n-1}.\end{aligned}$$

The homology of this complex is the *bounded Hochschild homology* of $\mathcal{A}$ with coefficients in X, $\mathcal{H}_n(A,X)$, $n = 0,1,\ldots$. The *bounded Hochschild cohomology* of $\mathcal{A}$ with coefficients in the dual module X^*, $\mathcal{H}^n(A,X^*)$, $n = 0,1,\ldots$, is the homology of the dual complex

$$0 \to X^* \to (\mathcal{A} \mathbin{\widehat{\otimes}} X)^* \to \cdots \to (\mathcal{A}^{\widehat{\otimes} n} \mathbin{\widehat{\otimes}} X)^* \to \cdots.$$

The important concept of H-unitality [19] can be expressed in terms of bounded Hochschild homology. This is essentially the content of [19, Remark (3)]. We use it as a definition.

Definition 2.3. A Banach algebra $\mathcal{A}$ is *H-unital* if $\mathcal{H}_n(\mathcal{A},X) = \{0\}$, $n \geq 0$, for all trivial modules $X \in \mathcal{A}$**-mod-**$\mathcal{A}$, i.e. modules with $\mathcal{A}X = X\mathcal{A} = \{0\}$.

Our main result hinges on the fact that bounded Hochschild homology and cohomology under certain conditions are Morita invariant (cf. [12]). We state the version that we need in the paper.

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be self-induced Morita equivalent Banach algebras with implementing modules $P \in \mathcal{B}$*-**mod**-*$\mathcal{A}$ and $Q \in \mathcal{A}$*-**mod**-*$\mathcal{B}$. If P is right flat as a module in* **mod**-*$\mathcal{A}$ and left flat as a module in $\mathcal{B}$*-**mod**, *then there are natural isomorphisms*

$$\mathcal{H}_n(\mathcal{A}, X) \cong \mathcal{H}_n(\mathcal{B}, P \widehat{\otimes}_{\mathcal{A}} X \widehat{\otimes}_{\mathcal{A}} Q) \text{ and } \mathcal{H}^n(\mathcal{A}, X^*) \cong \mathcal{H}^n(\mathcal{B}, (P \widehat{\otimes}_{\mathcal{A}} X \widehat{\otimes}_{\mathcal{A}} Q)^*)$$

for all induced modules $X \in \mathcal{A}$-**mod**-$\mathcal{A}$.

Proof. To establish the homology statement we briefly recall the Waldhausen first quadrant double complex. For details we refer to [12, pp. 132–133]. Set $N = X \widehat{\otimes}_{\mathcal{A}} Q$. Then $N \in \mathcal{A}$-**mod**-$\mathcal{B}$ and $N \widehat{\otimes}_{\mathcal{B}} P \cong X$, so we must prove that $\mathcal{H}_n(\mathcal{A}, N \widehat{\otimes}_{\mathcal{B}} P) \cong \mathcal{H}_n(\mathcal{B}, P \widehat{\otimes}_{\mathcal{A}} N)$ and similarly for cohomology. We set out by noting that $N \widehat{\otimes} E \in \mathcal{A}$-**mod**-$\mathcal{B}$ for all Banach spaces E with the module operations inherited from N. In particular we regard $N \widehat{\otimes} \mathcal{A}^{\widehat{\otimes}(m-1)}$ and $N \widehat{\otimes} \mathcal{B}^{\widehat{\otimes}(n-1)}$ as modules in $\mathcal{A}$-**mod**-$\mathcal{B}$ for all $m, n \in \mathbb{N}$ in this way. For each $m \in \mathbb{N}$ the right bar resolution $\mathscr{B}(\mathcal{B}, N \widehat{\otimes} \mathcal{A}^{\widehat{\otimes}(m-1)})$ is a complex in **mod**-$\mathcal{B}$, so we may form the complexes $\mathscr{B}(\mathcal{B}, N \widehat{\otimes} \mathcal{A}^{\widehat{\otimes}(m-1)}) \widehat{\otimes}_{\mathcal{B}} P$, $m \in \mathbb{N}$. Similarly we may form the complexes $P \widehat{\otimes}_{\mathcal{A}} \mathscr{B}(\mathcal{A}, \mathcal{B}^{\widehat{\otimes}(n-1)} \widehat{\otimes} N)$, $n \in \mathbb{N}$. We combine these complexes into a double complex in the first quadrant by setting the mth column to be $\mathscr{B}(\mathcal{B}, \mathcal{A}^{\widehat{\otimes}(m-1)} \widehat{\otimes} N) \widehat{\otimes}_{\mathcal{B}} P$ and the nth row to be $P \widehat{\otimes}_{\mathcal{A}} \mathscr{B}(\mathcal{A}, N \widehat{\otimes} \mathcal{B}^{\widehat{\otimes}(n-1)})$. On the axes $n = 0, m = 0$ we use the Hochschild boundary maps. This is possible since the (m, n)-entry up to a permutation of tensor factors will be $P \widehat{\otimes} N \widehat{\otimes} \mathcal{A}^{\widehat{\otimes}(m-1)} \widehat{\otimes} \mathcal{B}^{\widehat{\otimes}(n-1)}$ for $m, n \geq 1$ regardless of which of the two bar resolutions stipulates it.

$$\begin{tikzcd}
\mathcal{B}^{\widehat{\otimes}(n-1)} \widehat{\otimes} P \widehat{\otimes}_{\mathcal{A}} N \arrow[d] & \cdots \arrow[l] & \cdots & P \widehat{\otimes} N \widehat{\otimes} \mathcal{A}^{\widehat{\otimes}(m-1)} \widehat{\otimes} \mathcal{B}^{\widehat{\otimes}(n-1)} \arrow[l] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \\
P \widehat{\otimes}_{\mathcal{A}} N \arrow[d] & P \widehat{\otimes} N \arrow[l] \arrow[d] & \cdots \arrow[l] & \vdots \arrow[d] \\
0 & N \widehat{\otimes}_{\mathcal{B}} P \arrow[l] & \cdots \arrow[l] & N \widehat{\otimes}_{\mathcal{B}} P \widehat{\otimes} \mathcal{A}^{\widehat{\otimes}(m-1)} \arrow[l]
\end{tikzcd}$$

It is easy to show that the diagram is commutative. By [11, Lemma 6.1], to prove our statement it suffices to show that columns and rows are acyclic for $m, n \geq 1$. As the bar complexes are contractible, this follows from the assumed flatness properties of P.

Dualizing the Waldhausen double complex we obtain the cohomology statement. ∎

2.3. Semigroup algebras. Given a semigroup T, the semigroup algebra is the Banach space $\ell^1(T)$, equipped with the product which extends the product defined on the natural basis from T, by bilinearity, to the whole of $\ell^1(T)$. Throughout, we denote an element of the natural basis for $\ell^1(T)$, corresponding to $t \in T$, by t itself.

An *absorbing element* for a semigroup T is an element $\emptyset \in T$ so that $t\emptyset = \emptyset t = \emptyset$ for all $t \in T$. Obviously there is at most one absorbing element in T. If $\emptyset$ is absorbing, then $\mathbb{C}\emptyset$ is a 1-dimensional 2-sided ideal of $\ell^1(T)$. Our calculations are more easily done modulo this ideal.

DEFINITION 2.5. Let T be a semigroup with absorbing element $\emptyset$. The *reduced semigroup algebra* is

$$\mathcal{A}(T) = \ell^1(T)/\mathbb{C}\emptyset.$$

As a Banach space, $\mathcal{A}(T)$ is isometrically isomorphic to $\ell^1(T \setminus \{\emptyset\})$ and the multiplication is given by

$$st = \begin{cases} st & \text{if } st \neq \emptyset \text{ in } T, \\ 0 & \text{if } st = \emptyset \text{ in } T. \end{cases}$$

For $X \in \ell^1(T)$**-mod-**$\ell^1(T)$ the *reduced module* is

$$\widetilde{X} = \frac{X}{\mathrm{cl}(\emptyset X + X\emptyset)},$$

$\mathrm{cl}(\ldots)$ denoting closure. The reduced module is canonically a module in $\mathcal{A}(T)$**-mod-**$\mathcal{A}(T)$.

For later use we note that, if the semigroup satisfies $T^2 = T$, then the multiplication maps

$$\ell^1(T) \mathbin{\widehat{\otimes}} \ell^1(T) \to \ell^1(T) \quad \text{and} \quad \mathcal{A}(T) \mathbin{\widehat{\otimes}} \mathcal{A}(T) \to \mathcal{A}(T)$$

are both surjective.

EXAMPLE 2.6. The concept of reduced semigroup algebra fits in the more general context of extensions of semigroups. If I is a semigroup ideal of a semigroup T, then the equivalence class I of the Rees factor semigroup T/I is an absorbing zero and we get the corresponding admissible extension of Banach algebras

$$0 \to \ell^1(I) \to \ell^1(T) \to \mathcal{A}(T/I) \to 0.$$

Note that every module in $X \in \mathcal{A}(T)$**-mod-**$\mathcal{A}(T)$ can be obtained as a reduced module from a module in $\ell^1(T)$**-mod-**$\ell^1(T)$, simply by extending to an action of T by $\emptyset X = X\emptyset = \{0\}$, so that in this case $\widetilde{X} = X$.

From [17] we get the following proposition relating the Hochschild homology and cohomology of $\ell^1(T)$ and $\mathcal{A}(T)$.

PROPOSITION 2.7. *Let* $X \in \ell^1(T)$**-mod-**$\ell^1(T)$ *be such that* $\emptyset X = X\emptyset$. *Then*

$$\mathcal{H}_n(\ell^1(T), X) \cong \mathcal{H}_n(\mathcal{A}(T), \widetilde{X}) \quad \textit{and} \quad \mathcal{H}^n(\ell^1(T), X^*) \cong \mathcal{H}^n(\mathcal{A}(T), (\widetilde{X})^*)$$

for $n = 0, 1, \ldots$.

Proof. By [17, Theorem 4.5] we have the long exact sequence

$$\cdots \to \mathcal{H}_n(\mathbb{C}\emptyset, \emptyset X) \to \mathcal{H}_n(\ell^1(T), X) \to \mathcal{H}_n(\mathcal{A}(T), \widetilde{X}) \to \mathcal{H}_{n+1}(\mathbb{C}\emptyset, \emptyset X) \to \cdots.$$

As $\mathbb{C}\emptyset \cong \mathbb{C}$ we have $\mathcal{H}_n(\mathbb{C}\emptyset, \emptyset X) = \{0\}$ for all $n \geq 0$, yielding the claim about homology. A similar application of [17, Theorem 4.5] to cohomology gives the other statement. ∎

As a consequence, since our concern is to determine Hochschild homology and cohomology, we shall work with reduced semigroup algebras in the following.

2.4. The Rees semigroup algebra. For the remainder of the paper we fix a Rees semigroup, S, with index sets I and Λ over a group G. We set

$${}_iS_\lambda = \{i\} \times G \times \{\lambda\}, \quad i \in I,\ \lambda \in \Lambda.$$

Then

$$\mathcal{A}(S) = \bigoplus_{\substack{i \in I \\ \lambda \in \Lambda}} \ell^1({}_iS_\lambda)$$

is a decomposition of $\mathcal{A}(S)$ into an ℓ^1-direct sum of subalgebras such that for all $i, j \in I$ and $\lambda, \mu \in \Lambda$,

$$\begin{aligned} &\ell^1({}_iS_\lambda) \cong \ell^1(G) && \text{if } p_{\lambda i} \neq \mathbf{o}, \\ &\ell^1({}_iS_\lambda) \cdot \ell^1({}_jS_\mu) = \{0\} && \text{if } p_{\lambda j} = \mathbf{o}. \end{aligned}$$

The isomorphism above is implemented by the semigroup isomorphism

$$G \to_i S_\lambda \colon g \mapsto (i, g p_{\lambda i}^{-1}, \lambda).$$

Since $\ell^1({}_iS_\lambda) \cdot \ell^1({}_jS_\mu) \subseteq \ell^1({}_iS_\mu)$ for all $i, j \in I$ and $\lambda, \mu \in \Lambda$, this decomposition is organized as a rectangular band. We further put

$$\begin{aligned} {}_iS &= \{i\} \times G \times \Lambda, \quad i \in I, \\ S_\lambda &= I \times G \times \{\lambda\}, \quad \lambda \in \Lambda, \end{aligned}$$

so that for each $i \in I$ the subspace $\ell^1({}_iS) = \bigoplus_{\lambda \in \Lambda} \ell^1({}_iS_\lambda)$ is a closed right ideal of $\mathcal{A}(S)$ and for each $\lambda \in \Lambda$ the subspace $\ell^1(S_\lambda) = \bigoplus_{i \in I} \ell^1({}_iS_\lambda)$ is a closed left ideal of $\mathcal{A}(S)$.

With our fixed Rees semigroup we establish a number of generic properties of Rees semigroup algebras.

Proposition 2.8. *Each $\ell^1({}_iS)$, $i \in I$, is a closed right ideal with a left identity, and each $\ell^1(S_\lambda)$, $\lambda \in \Lambda$, is a closed left ideal with a right identity.*

Proof. Given an element of the indexing set $i \in I$ there is, by the property of the sandwich matrix P, a non-zero entry $p_{\mu i}$, for some $\mu \in \Lambda$ (as each row and column has a non-zero entry). We define $e_i = (i, p_{\mu i}^{-1}, \mu)$. The element e_i acts as a left identity for $s \in {}_iS$ as

$$e_i s = (i, p_{\mu i}^{-1}, \mu)(i, g, \lambda) = (i, p_{\mu i}^{-1} p_{\mu i} g, \lambda) = (i, g, \lambda) = s.$$

In particular e_i is idempotent and is clearly a left identity for $\ell^1({}_iS)$. Similarly, on the right we have a non-zero element $p_{\lambda j}$ of P, which gives the required element as $(j, p_{\lambda j}^{-1}, \lambda)$. ∎

Note that the idempotent e_i is not necessarily unique. Such an idempotent can be written down using any index from Λ which gives a non-zero entry in P. However, in what follows it will be useful to have a fixed family of left and right idempotents in mind.

Definition 2.9. We fix a family of left (and right) idempotents denoted $\{e_i\}_{i\in I}$ (and $\{f_\lambda\}_{\lambda\in\Lambda}$), which are left (respectively right) units for the $\ell^1({}_iS)$'s (respectively the $\ell^1(S_\lambda)$'s).

Proposition 2.10. *For each $i \in I$, the right ideal $\ell^1({}_iS)$ is strictly projective in* **mod-**$\mathcal{A}(S)$ *and, for each $\lambda \in \Lambda$, the left ideal $\ell^1(S_\lambda)$ is strictly projective in* $\mathcal{A}(S)$**-mod**.

Proof. We give the proof for $\ell^1({}_iS)$ as the other is completely analogous. Let $q\colon Y \to Z$ be an epimorphism in **mod-**$\mathcal{A}(S)$ and consider the lifting problem

$$\begin{tikzcd} & \ell^1({}_iS) \arrow[dl, dashed, "?"'] \arrow[d, "\phi"] & \\ Y \arrow[r, "q"] & Z \arrow[r] & 0 \end{tikzcd}$$

Choose $y_i \in Y$ so that $q(y_i) = \phi(e_i)$ and define for $s \in I \times G \times \Lambda$

$$\tilde{\phi}(e_i s) = y_i e_i s.$$

Since ${}_iS = e_i(I \times G \times \Lambda)$ in $\mathcal{A}(S)$, the universal property of ℓ^1-spaces provides a bounded linear map $\tilde{\phi}\colon \ell^1({}_iS) \to Y$ such that $q \circ \tilde{\phi} = \phi$. Clearly $\tilde{\phi}$ is a right module map. ∎

Corollary 2.11. *The Banach algebra $\mathcal{A}(S)$ is strictly projective in* **mod-**$\mathcal{A}(S)$ *and in* $\mathcal{A}(S)$**-mod**.

Proof. We utilize the direct sum decomposition $\mathcal{A}(S) = \bigoplus_{i\in I} \ell^1({}_iS)$ in **mod-**$\mathcal{A}(S)$. Consider the lifting problems

$$\begin{tikzcd}
 & \ell^1({}_iS) \arrow[d, "\kappa_i"] \arrow[ddl, "\tilde{\phi}_i"'] & \\
 & \mathcal{A}(S) \arrow[d, "\phi"] \arrow[dl, dashed] & \\
Y \arrow[r, "q"] & Z \arrow[r] & 0
\end{tikzcd}$$

where κ_i, $i \in I$, are the natural inclusions and $\tilde{\phi}_i$, $i \in I$, are the lifts of $\phi \circ \kappa_i$ constructed in the proof of 2.10. By the open mapping theorem applied to q, we can choose the elements y_i such that $\|y_i\| \le C$ for some constant C, and therefore such that $\|\tilde{\phi}_i\| \le C$ for all $i \in I$. Thus there is a unique module map $\tilde{\phi}\colon \mathcal{A}(S) \to Y$ with $\tilde{\phi} \circ \kappa_i = \tilde{\phi}_i$ for all $i \in I$. Since both $q \circ \tilde{\phi}$ and ϕ complete the direct sum diagram for the maps $\phi \circ \kappa_i$, it follows by uniqueness of universal elements that $q \circ \tilde{\phi} = \phi$.

The case of left projectivity is completely analogous. ∎

Corollary 2.12. *The Banach algebras $\mathcal{A}(S)$ and $\ell^1(S)$ are H-unital. In particular they are self-induced.*

Proof. The statement about $\mathcal{A}(S)$ is a general fact about one-sided projective Banach algebras with surjective multiplication $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} \to \mathcal{A}$. Let $\rho\colon \mathcal{A} \to \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}$ be a splitting of multiplication provided by right projectivity of $\mathcal{A}$. Then $\rho \otimes \mathbf{1}\colon \mathcal{A}^{\widehat{\otimes} n} \to \mathcal{A}^{\widehat{\otimes}(n+1)}$ is a contracting homotopy of the simplicial complex:

$$\begin{tikzcd}
0 & \mathcal{A} \arrow[l] \arrow[d, "\mathbf{1}"'] \arrow[dr, "\rho"] & \mathcal{A}^{\widehat{\otimes}2} \arrow[l, "b"] \arrow[d, "\mathbf{1}"'] \arrow[dr, "\rho\otimes\mathbf{1}"] & \cdots \arrow[l, "b"] & \\
0 & \mathcal{A} \arrow[l] & \mathcal{A}^{\widehat{\otimes}2} \arrow[l, "b"] & \mathcal{A}^{\widehat{\otimes}3} \arrow[l, "b"] & \cdots \arrow[l]
\end{tikzcd}$$

To see this, let $a_1 \otimes \cdots \otimes a_n \in \mathcal{A}^{\widehat{\otimes} n}$. Then

$$\begin{aligned}
\rho \otimes \mathbf{1}(b(a_1 \otimes \cdots \otimes a_n)) &= \rho(a_1 a_2) \otimes a_3 \otimes \cdots \otimes a_n - \rho(a_1) \otimes b(a_2 \otimes \cdots \otimes a_n) \\
&= \rho(a_1) a_2 \otimes \cdots \otimes a_n - \rho(a_1) \otimes b(a_2 \otimes \cdots \otimes a_n),
\end{aligned}$$

and

$$\begin{aligned}
b(\rho \otimes \mathbf{1}(a_1 \otimes \cdots \otimes a_n)) &= a_1 \otimes \cdots \otimes a_n \\
&\quad - \rho(a_1) a_2 \otimes \cdots \otimes a_n + \rho(a_1) \otimes b(a_2 \otimes \cdots \otimes a_n),
\end{aligned}$$

so that $(\rho \otimes \mathbf{1}) b + b(\rho \otimes \mathbf{1}) = \mathrm{id}$.

For a trivial module X the Hochschild complex is $-X \mathbin{\widehat{\otimes}} \mathscr{S}(\mathcal{A})$. Since $\mathscr{S}(\mathcal{A})$ is contractible, it follows that $\mathcal{H}_n(\mathcal{A}, X) = \{0\}$, $n \ge 0$, for trivial modules, i.e. $\mathcal{A}$ satisfies the definition of H-unitality (Definition 2.3).

As $\widetilde{X} = X$ for any trivial module, the statements about $\ell^1(S)$ follow from Proposition 2.7. ■

2.5. Morita equivalence. With S our fixed Rees semigroup, fix one of the idempotents $e = (i, p_{\lambda i}^{-1}, \lambda)$ and put

$$P = e\mathcal{A}(S) \quad \text{and} \quad Q = \mathcal{A}(S)e,$$

so that P is the closed right ideal $\ell^1({}_iS)$ and Q is the closed left ideal $\ell^1(S_\lambda)$. Further, let

$$\mathcal{B} = e\mathcal{A}(S)e,$$

so that $\mathcal{B} = \ell^1({}_iS_\lambda) \cong \ell^1(G)$.

For brevity in this section, let $\mathcal{A} = \mathcal{A}(S)$. Then $P \in$ $\mathcal{B}$**-mod-**$\mathcal{A}$ and $Q \in$ $\mathcal{A}$**-mod-**$\mathcal{B}$. Our main result is

THEOREM 2.13. *The modules P and Q give a Morita equivalence of $\mathcal{A}$ and $\mathcal{B}$, that is, multiplication gives bimodule isomorphisms*

$$\mathcal{A} \cong Q \mathbin{\widehat{\otimes}}_{\mathcal{B}} P \quad \textit{and} \quad \mathcal{B} \cong P \mathbin{\widehat{\otimes}}_{\mathcal{A}} Q.$$

Proof. Clearly multiplication $P \mathbin{\widehat{\otimes}} Q \to \mathcal{B}$ is surjective: see the note right before Example 2.6. Now suppose that

$$\sum_n e a_n b_n e = 0 \quad \text{for} \quad \sum_n \|e a_n\|\, \|b_n e\| < \infty,\ a_n, b_n \in \mathcal{A}.$$

Then

$$\sum_n e a_n \otimes_{\mathcal{A}} b_n e = \sum_n e a_n b_n e \otimes_{\mathcal{A}} e = 0$$

so that multiplication $P \mathbin{\widehat{\otimes}}_{\mathcal{A}} Q \to \mathcal{B}$ is injective. It follows that $P \mathbin{\widehat{\otimes}}_{\mathcal{A}} Q \cong \mathcal{B}$.

For the reversed tensor product first note that

$$(j, g, \mu) = (j, g, \lambda)(i, p_{\lambda i}^{-1}, \mu)$$

for all $j \in I$, $g \in G$, $\mu \in \Lambda$, so that the multiplication $Q \mathbin{\widehat{\otimes}}_{\mathcal{B}} P \to \mathcal{A}$ is surjective.

Identifying $Q \mathbin{\widehat{\otimes}} P$ with $\ell^1(Se \times eS)$ a generic element in $Q \mathbin{\widehat{\otimes}} P$ has the form

$$t = \sum_{j,g,h,\mu} \alpha_{jgh\mu} (j, g, \lambda) \otimes (i, h, \mu).$$

Assume that multiplication on t is 0, i.e.

$$\sum_{j,g,h,\mu} \alpha_{jgh\mu} (j, g, \lambda)(i, h, \mu) = 0.$$

This means that

$$\sum_{\substack{g,h \\ g p_{\lambda i} h = \gamma}} \alpha_{jgh\mu} = 0$$

for each $j \in I$, $\gamma \in G$, $\mu \in \Lambda$. Now

$$\begin{aligned}(j,g,\lambda) \otimes_{\mathcal{B}} (i,h,\mu) &= (j,g,\lambda) \otimes_{\mathcal{B}} (i,h,\lambda)(i,p_{\lambda i}^{-1},\mu) \\ &= (j,g,\lambda)(i,h,\lambda) \otimes_{\mathcal{B}} (i,p_{\lambda i}^{-1},\mu) \\ &= (j,gp_{\lambda i}h,\lambda) \otimes_{\mathcal{B}} (i,p_{\lambda i}^{-1},\mu),\end{aligned}$$

so

$$\sum_{j,g,h,\mu} \alpha_{jgh\mu}(j,g,\lambda) \otimes_{\mathcal{B}} (i,h,\mu) = \sum_{\gamma,j,\mu} \Big[\sum_{\substack{g,h \\ gp_{\lambda i}h=\gamma}} \alpha_{jgh\mu} \Big] (j,\gamma,\lambda) \otimes_{\mathcal{B}} (i,p_{\lambda i}^{-1},\mu) = 0.$$

It follows that multiplication $Q \widehat{\otimes}_{\mathcal{B}} P \to \mathcal{A}$ is injective so that $Q \widehat{\otimes}_{\mathcal{B}} P \cong \mathcal{A}$. Altogether, $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent. ■

We want to establish Morita invariance of Hochschild homology. We have already noted that P is strictly projective in **mod-**$\mathcal{A}$. We now prove

THEOREM 2.14. *The module $P = e\mathcal{A}$ is strictly projective in $\mathcal{B}$-***mod**.

Proof. Consider the direct sum decomposition in $\mathcal{B}$-**mod** $P = \bigoplus_\mu \ell^1({}_iS_\mu)$. Let

$$\begin{tikzcd} & \ell^1({}_iS_\mu) \arrow[dl, dashed] \arrow[d, "\phi_\mu"] & \\ Y \arrow[r, "q"] & Z \arrow[r] & 0 \end{tikzcd}$$

be the lifting problem, where ϕ_μ is the restriction of $\phi : P \to Z$. From the open mapping theorem applied to q, there exists y_μ with

$$\|y_\mu\| \le C \quad \text{and} \quad q(y_\mu) = \phi_\mu((i,p_{\lambda i}^{-1},\mu)),$$

for some constant C not depending on μ. Define $\tilde{\phi}_\mu : \ell^1({}_iS_\mu) \to Y$ by

$$\tilde{\phi}_\mu((i,g,\mu)) = (i,g,\lambda)y_\mu.$$

Then $\tilde{\phi}_\mu \in \mathcal{B}$-**mod** and $\|\tilde{\phi}_\mu\| \le C$. Since

$$\begin{aligned} q(\tilde{\phi}_\mu((i,g,\mu)) &= q((i,g,\lambda)y_\mu) = (i,g,\lambda)q(y_\mu) \\ &= (i,g,\lambda)\phi_\mu((i,p_{\lambda i}^{-1},\mu)) = \phi_\mu((i,g,\lambda)(i,p_{\lambda i}^{-1},\mu)) \\ &= \phi_\mu((i,g,\mu)), \end{aligned}$$

we have solved the lifting problem. Proceeding as in the proof of Corollary 2.11 we conclude that P, being a direct sum of strictly projective modules, is strictly projective in $\mathcal{B}$-**mod**. ■

3. Applications to homological properties. With S our fixed Rees semigroup and P, Q, $\mathcal{A} = \mathcal{A}(S)$, and $\mathcal{B} = e\mathcal{A}(S)e$ as in the previous section we have functors

$$\mathcal{A}(S)\textbf{-mod-}\mathcal{A}(S) \to \mathcal{B}\textbf{-mod-}\mathcal{B}\colon X \mapsto P \mathbin{\widehat{\otimes}_{\mathcal{A}}} X \mathbin{\widehat{\otimes}_{\mathcal{A}}} Q,$$

$$\mathcal{B}\textbf{-mod-}\mathcal{B} \to \mathcal{A}(S)\textbf{-mod-}\mathcal{A}(S)\colon Y \mapsto Q \mathbin{\widehat{\otimes}_{\mathcal{B}}} Y \mathbin{\widehat{\otimes}_{\mathcal{B}}} P.$$

Replacing $\mathcal{B}$ by the isomorphic $\ell^1(G)$ we get functors

$$\Phi\colon \mathcal{A}(S)\textbf{-mod-}\mathcal{A}(S) \to \ell^1(G)\textbf{-mod-}\ell^1(G),$$
$$\Gamma\colon \ell^1(G)\textbf{-mod-}\ell^1(G) \to \mathcal{A}(S)\textbf{-mod-}\mathcal{A}(S).$$

We collect our findings in

Theorem 3.1. *The functors Φ and Γ constitute an equivalence of the full subcategories of induced bimodules over $\mathcal{A}(S)$ and $\ell^1(G)$ and there are natural isomorphisms of homology and cohomology functors*

$$\mathcal{H}_n(\mathcal{A}(S), X) \cong \mathcal{H}_n(\ell^1(G), \Phi(X)),$$
$$\mathcal{H}^n(\mathcal{A}(S), X^*) \cong \mathcal{H}^n(\ell^1(G), \Phi(X)^*)$$

for all $n \geq 0$.

Proof. The equivalence follows from the natural isomorphisms

$$Q \mathbin{\widehat{\otimes}_{\mathcal{B}}} (P \mathbin{\widehat{\otimes}_{\mathcal{A}}} X \mathbin{\widehat{\otimes}_{\mathcal{A}}} Q) \mathbin{\widehat{\otimes}_{\mathcal{B}}} P \cong \mathcal{A} \mathbin{\widehat{\otimes}_{\mathcal{A}}} X \mathbin{\widehat{\otimes}_{\mathcal{A}}} \mathcal{A} \cong X,$$

$$P \mathbin{\widehat{\otimes}_{\mathcal{A}}} (Q \mathbin{\widehat{\otimes}_{\mathcal{B}}} Y \mathbin{\widehat{\otimes}_{\mathcal{B}}} P) \mathbin{\widehat{\otimes}_{\mathcal{A}}} Q \cong \mathcal{B} \mathbin{\widehat{\otimes}_{\mathcal{B}}} Y \mathbin{\widehat{\otimes}_{\mathcal{B}}} \mathcal{B} \cong Y$$

for induced modules $X \in \mathcal{A}(S)\textbf{-mod-}\mathcal{A}(S)$ and $Y \in \mathcal{B}\textbf{-mod-}\mathcal{B}$. As P is strictly projective in $\mathcal{B}$**-mod** and in **mod-**$\mathcal{A}(S)$, the statements about homology and cohomology groups follow from Theorem 2.4. ∎

We recall that S is a Rees semigroup with underlying group G. We note a number of consequences.

Corollary 3.2. *There are isomorphisms*

$$\begin{aligned}\mathcal{H}_n(\mathcal{A}(S), \mathcal{A}(S)) &\cong \mathcal{H}_n(\ell^1(S), \ell^1(S)) \\ &\cong \mathcal{H}_n(\mathcal{A}(S)^{\#}, \mathcal{A}(S)^{\#}) \cong \mathcal{H}_n(\ell^1(S)^{\#}, \ell^1(S)^{\#}) \\ &\cong \mathcal{H}_n(\ell^1(G), \ell^1(G))\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}^n(\mathcal{A}(S), \mathcal{A}(S)^*) &\cong \mathcal{H}^n(\ell^1(S), \ell^1(S)^*) \\ &\cong \mathcal{H}^n(\mathcal{A}(S)^{\#}, (\mathcal{A}(S)^{\#})^*) \cong \mathcal{H}^n(\ell^1(S)^{\#}, (\ell^1(S)^{\#})^*) \\ &\cong \mathcal{H}^n(\ell^1(G), \ell^1(G)^*)\end{aligned}$$

for all $n \geq 0$.

Proof. The proofs for homology and cohomology are identical. Since the reduced module of $\ell^1(S)$ is $\mathcal{A}(S)$, in both cases the first isomorphism follows from Proposition 2.7. The next two isomorphisms follow from H-unitality (cf.

Corollary 2.12). Finally the last isomorphism is a consequence of Theorem 3.1 since $\Phi(\mathcal{A}(S)) = \ell^1(G)$. ∎

A Banach algebra $\mathcal{A}$ is *weakly amenable* if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. It is an open question to determine exactly which semigroups give weakly amenable semigroup algebras. Our corollary below has in the instance $\ell^1(S)$ been obtained in [1, Corollary 5.3] by a different approach.

Corollary 3.3. *The algebras $\ell^1(S)^{\#}$, $\ell^1(S)$, $\mathcal{A}(S)^{\#}$, and $\mathcal{A}(S)$ are all weakly amenable.*

Proof. The Banach algebra $\ell^1(G)$ is weakly amenable [16]. ∎

A Banach algebra $\mathcal{A}$ is *biprojective* if multiplication $\Pi\colon \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ has a right inverse in $\mathcal{A}$**-mod-**$\mathcal{A}$, and is *biflat* if the dual of multiplication $\Pi^*\colon \mathcal{A}^* \to (\mathcal{A} \widehat{\otimes} \mathcal{A})^*$ has a left inverse in $\mathcal{A}$**-mod-**$\mathcal{A}$.

Corollary 3.4. *$\mathcal{A}(S)$ is biflat if and only if G is amenable.*

Proof. By [18, Theorem 5.8(i)] a Banach algebra $\mathcal{A}$ is biflat if and only if it is self-induced and $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for all induced modules X. As $\ell^1(G)$, being unital, is biflat if and only if it is amenable, the result follows from [15]. ∎

The corresponding result for biprojectivity is not immediate from Morita theory, as a description of biprojectivity in terms of Hochschild cohomology involves non-induced modules. But we can give a direct proof of

Theorem 3.5. *$\mathcal{A}(S)$ is biprojective if and only if G is finite.*

Proof. Assume that $|G| < \infty$. Choose an idempotent $e = (i, p_{\lambda i}^{-1}, \lambda)$ and define $\rho\colon \mathcal{A}(S) \to \mathcal{A}(S) \widehat{\otimes} \mathcal{A}(S)$ by

$$\rho((j, g, \mu)) = \frac{1}{|G|} \sum_{h \in G} (j, ghp_{\lambda i}^{-1}, \lambda) \otimes (i, h^{-1}, \mu), \quad (j, g, \mu) \in I \times G \times \Lambda.$$

Then clearly $\Pi \circ \rho = \mathbf{1}$. One checks, as in the proof of biprojectivity of group algebras over finite groups, that

$$(j, g, \mu)\rho((j', g', \mu')) = \rho((j, g, \mu)(j', g', \mu')) = \rho((j, g, \mu))(j', g', \mu')$$

for all $(j, g, \mu), (j', g', \mu') \in I \times G \times \Lambda$, so that ρ is a bimodule homomorphism.

Conversely, suppose that $\mathcal{A}(S)$ is biprojective. For a splitting of multiplication $\mathcal{A}(S) \to \mathcal{A}(S) \widehat{\otimes} \mathcal{A}(S)$ we consider its restriction $\rho\colon e\mathcal{A}(S)e \to e\mathcal{A}(S) \widehat{\otimes} \mathcal{A}(S)e$. Since ρ is a bimodule homomorphism we have

$$\rho(eae) = eae\rho(e) = \rho(e)eae, \quad a \in \mathcal{A}(S).$$

Using the decomposition

$$e\mathcal{A}(S) \widehat{\otimes} \mathcal{A}(S)e = \bigoplus_{j,\mu} \ell^1({}_iS_\mu \times {}_jS_\lambda)$$

as a direct sum of $e\mathcal{A}(S)e$-bimodules, we may write

$$\rho(e) = \sum_{j,\mu} \tau_{\mu j}$$

with $\tau_{\mu j} \in \ell^1({}_iS_\mu \times {}_jS_\lambda)$, $j \in I$, $\mu \in \Lambda$. It follows that

$$eae\tau_{\mu j} = \tau_{\mu j}eae$$

for all $a \in \mathcal{A}(S)$, $j \in I$, $\mu \in \Lambda$.

In the remainder of the proof it will be convenient to use the multiplication on a projective tensor product of Banach algebras given by $a\otimes b\cdot a'\otimes b' := aa' \otimes b'b$.

For each $j \in I,\ \mu \in \Lambda$ choose $f_{\mu j} \in {}_jS_\lambda$ and $e_{\mu j} \in {}_iS_\mu$ so that

$$\Pi(\tau_{\mu j} \cdot f_{\mu j} \otimes e_{\mu j}) = \Pi(\tau_{\mu j}).$$

This is clearly possible: If $p_{\mu j} = \mathbf{o}$ choose $e_{\mu j}$ and $f_{\mu j}$ arbitrarily. If $p_{\mu j} \neq \mathbf{o}$ choose $f_{\mu j} = (j, p_{\mu j}^{-1}, \lambda)$ and $e_{\mu j} = (i, p_{\lambda i}^{-1}, \mu)$.

Now put

$$\Delta = \sum_{j,\mu} \tau_{\mu j} \cdot f_{\mu j} \otimes e_{\mu j}.$$

Then $\Delta \in e\mathcal{A}(S)e \mathbin{\widehat{\otimes}} e\mathcal{A}(S)e$ and

$$\begin{aligned} &\Pi(\Delta) = e,\\ &eae\Delta = \Delta eae, \quad a \in \mathcal{A}(S), \end{aligned}$$

so that $e\mathcal{A}(S)e$ has a diagonal and therefore is biprojective. Since $e\mathcal{A}(S)e \cong \ell^1(G)$, the group G must be finite. ∎

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References

[1] T. D. Blackmore, *Weak amenability of discrete semigroup algebras*, Semigroup Forum 55 (1997), 196–205.

[2] S. Bowling and J. Duncan, *First order cohomology of Banach semigroup algebras*, Semigroup Forum 56 (1998), 130–145.

[3] R. Brown, *Elements of Modern Topology*, McGraw-Hill, London, 1968.

[4] H. G. Dales and J. Duncan, *Second order cohomology in groups of some semigroup algebras*, in: Banach Algebras '97, de Gruyter, Berlin, 1998, 101–117.

[5] H. G. Dales, A. T.-M. Lau and D. Strauss, *Banach algebras on semigroups and on their compactifications*, Mem. Amer. Math. Soc. 205 (2010), no. 966.

[6] F. Gourdeau, B. E. Johnson and M. C. White, *The cyclic and simplicial cohomology of* $l^1(\mathbb{N})$, Trans. Amer. Math. Soc. 357 (2005), 5097–5113.

[7] F. Gourdeau, Z. A. Lykova and M. C. White, *A Künneth formula in topological homology and its applications to the simplicial cohomology of* $l^1(\mathbb{Z}_+^k)$, Studia Math. 166 (2005), 29–54.

[8] —, —, —, *The simplicial homology and cohomology of* $L^1(\mathbb{R}_+^k)$, in: Banach Algebras and Their Applications, Contemp. Math. 363, Amer. Math. Soc., Providence, RI, 2004, 95–109.

[9] F. Gourdeau, A. Pourabbas and M. C. White, *Simplicial cohomology of some semigroup algebras*, Canad. Math. Bull. 50 (2007), 56–70.

[10] F. Gourdeau and M. C. White, *Vanishing of the third simplicial cohomology group of* $l^1(\mathbb{Z}_+)$, Trans. Amer. Math. Soc. 357 (2005), 5097–5113.

[11] N. Grønbæk, *Morita equivalence for Banach algebras*, J. Pure Appl. Algebra 99 (1995), 183–219.

[12] —, *Morita equivalence for self-induced Banach algebras*, Houston J. Math. 22 (1996), 109–140.

[13] A. Ya. Helemskiĭ, *The Homology of Banach and Topological Algebras*, Kluwer, Dordrecht, 1989.

[14] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford Univ. Press, Oxford, 1995.

[15] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. 127 (1972).

[16] —, *Weak amenability of group algebras*, Bull. London Math. Soc. 23 (1991), 281–284.

[17] Z. Lykova and M. C. White, *Excision in the cohomology of Banach algebras with coefficients in dual modules*, in: Banach Algebras '97, de Gruyter, Berlin, 1998, 341–361.

[18] Yu. Selivanov, *Cohomological Characterizations of biprojective and biflat Banach algebras*, Monatsh. Math. 128 (1999), 35–60.

[19] M. Wodzicki, *The long exact sequence in cyclic homology associated with an extension of algebras*, C. R. Acad. Sci. Paris Sér. I Math. 306 (1988), 399–403.

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